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Revisiting the connection between the no-show paradox and monotonicity[☆]

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HIGHLIGHTS

- We study the relation between monotonicity and the no-show paradox in voting rules.
- We show that both notions are closer than commonly believed.
- Our results hold for both fixed and variable-size electorates.

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ABSTRACT

We investigate the relation between monotonicity and the no-show paradox in voting rules. Although the literature has established their logical independence, we show, by presenting logical dependency results, that the two conditions are closer than a general logical independency result would suggest. Our analysis is made both under variable and fixed-size electorates.

1. Introduction

Among the countless contributions of Hervé Moulin to our enlightenment on the collective decision making problem, his research on the axiomatic analysis of social choice rules presents a distinguished chapter which inspired generations of scholars. We view this issue of Mathematical Social Sciences dedicated to him as a nice opportunity to revisit the connection between participation and monotonicity, two conditions of social choice theory which have been much elaborated by the fine work of Hervé Moulin.

Moulin (1988, 1991) defines participation as the absence of the no-show paradox introduced by Fishburn and Brams (1983): a social choice rule exhibits the no-show paradox when the vote casted by an additional voter changes the outcome in a way which makes this new-comer worse off compared to the case he had not shown up. Thus, the paradox can be viewed as a way to manipulate social choice rules by abstaining from voting, such as Moulin (1991) who sees it as a particular case of manipulation by truncation of preferences as defined by Fishburn and Brams (1984).

Such views, however, necessitate some caution on how the new-comer/abstainer is interpreted. Here, two approaches come to the fore: One is the fixed-electorate approach where the number of voters are fixed and the abstainers are those voters who express full indifference over the set of alternatives. So a "new-comer" is an individual who is an incumbent member of the society who moves away from his full indifference position. The other approach necessitates a variable-electorate, as the new-comer is a voter who earlier was not a member of the society, hence "abstaining" means his altogether departing from the society to which he used to be part of.

Refusing to express an opinion should not be interpreted as leaving the electorate. As a result, the two interpretations have different meanings. However, for social choice rules which are "regular", i.e., ignore voters who show up without expressing a preference, one could expect that the choice of the interpretation would not matter. Theorem 14 somehow justifies this expectation by establishing an equivalence between variable and fixed electorate social choice rules regarding the satisfaction of PART.[1]

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[1] While most of the well-known social choice rules are regular, there are notable exceptions such as those who use a quorum or those who allow voters to vote for

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On the other hand, the choice of the interpretation has implications on the relationship between the no-show paradox and monotonicity—a fact that we discuss in the sequel. However, we wish to note right away that the literature on the paradox has almost always adopted the variable-electorate approach, including and perhaps following the seminal paper of Moulin (1988).[2]

We start, in Section 2, by considering the paradox under this standard variable-electorate interpretation and revisit its relation to a well-known monotonicity condition of social choice theory. Monotonicity, broadly speaking, requires that an "improvement" of the status of an alternative in the preferences of the electorate should result in a "raising" of the status of this alternative as the social outcome. It is clear that, different meanings can be attributed to "improvement" and "raising", each of which leading to a different definition of monotonicity. In fact, the literature exhibits a plethora of monotonicity conditions. As all of these can be connected to the (non)-manipulability of social choice rules, the logical relationship of participation to those monotonicity conditions stands out as an interesting question.

Among the various monotonicity conditions, perhaps the simplest and oldest known is the one we consider[3]:

MON: Raising an alternative x in voters' preferences while leaving the rankings otherwise unchanged can never result in x becoming the loser while x was initially the winner.

Although normatively appealing and simple, MON is violated by various well-known social choice rules, in particular by all point run-off systems (Smith, 1973).[4] As participation (PART) is also violated by an interesting class of social choice rules, namely those which are Condorcet consistent (Moulin, 1988), the logical relationship between MON and PART turns out to be of further interest.

In fact, the general logical independence between MON and PART is already established. The question is addressed by Nurmi (1999) in p. 62 who remarks that MON does not imply PART, as there exist Condorcet extensions, such as the Copeland rule, which satisfy MON but, by Moulin (1988), fail PART. Nurmi (1999) furthermore suggests the conjecture that PART implies MON which is falsified by Campbell and Kelly (2002) who give examples of social choice rules that satisfy PART but fail MON.

We present instances of logical dependencies between PART and MON. We show that in the particular case of two alternatives, PART implies MON. On the other hand, even with two alternatives, MON does not imply PART. Nevertheless, the failure of this implication merits some attention to the contextual difference regarding the definitions of the two conditions: while MON is a property that can be defined for fixed or variable electorate social choice rules, PART necessitates a variable electorate. As a result, PART requires a connection between how a social rule behaves in electorates of different sizes but MON does not. This renders the construction of a social choice rule which satisfies MON but fails PART very easy. In fact, the example we use in Proposition 1 to show that MON does not imply PART even with two alternatives exploits this ease.

A fairer question is whether MON implies PART under mild consistency requirements over the behavior of social choice rules in different electorates. One such condition is reinforcement, also known as consistency, which requires that alternatives which are separately chosen by both of two disjoint electorates must form the choice made by the union of these electorates (Smith, 1973; Young, 1974, 1975). A much milder version of reinforcement is homogeneity which requires that an alternative which is chosen by some given electorate must also be chosen when this electorate is replicated.[5] We show that in the two-alternative case, under the homogeneity assumption, MON implies PART.

With three or more alternatives, Moulin (1988), while establishing the logical independence between PART and reinforcement,uses a threshold scoring rule to exemplify the satisfaction of reinforcement and the failure of PART. As all threshold scoring rules satisfy MON, the example also shows that even when homogeneity is replaced by reinforcement, MON fails to imply PART.

Inspired by this example, we devote further attention to threshold scoring rules[6] and ask whether they always fail PART. The answer is almost affirmative: we show that, except one member, the class of threshold scoring rules fails PART.

We also consider a weaker version of participation (WPART) as the absence of a stronger version of the no-show paradox (Pérez, 2001) where a voter, by abstaining, can enforce his most preferred alternative as the social outcome. We show that MON, even when homogeneity is assumed, does not imply WPART. On the other hand, reinforcement, when combined with a weak unanimity condition, implies WPART.

Regarding the implication of monotonicity by PART, we show that PART implies a weaker version of monotonicity (WMON) which is nevertheless sufficiently strong to discriminate among social choice rules that fail MON: we observe that Campbell and Kelly (2002)'s examples that fail MON satisfy WMON while plurality with a runoff even fails WMON. In fact, we are able to extend this latter observation to the almost whole class of point runoff procedures which, except Borda, all fail WMON. Our Theorem 6 which states this failure not only strengthens the result of Smith (1973) on the failure of MON by point runoff procedures, but also paves the way to our Theorem 7 which announces that all point runoff procedures fail PART.

We close the section by giving a partial characterization of PART through a lower contour set intersection property which we call Condition λ. We also establish the relationship between Condition λ and MON, which brings another perspective to our previous findings.

Section 3 carries our analysis to the fixed-electorate interpretation of PART. We start by establishing an equivalence between fixed and variable electorate interpretations regarding the satisfaction of PART. Based on this equivalence, we are able to note that the general logical independence between PART and MON prevails in the fixed electorate setting. On the other hand, regarding the logical relationship between PART and MON, our findings differ from those obtained under the variable-electorate interpretation. We show that with two alternatives, PART and MON are logically equivalent. Moreover, when three or more alternatives are available MON implies WPART and PART implies WMON.

Section 4 makes some closing remarks.

2. The Variable population case

We consider a finite set of alternatives A with $\#A \geq 2$. $\mathbf{N}$ denotes the set of natural numbers. For each $n \in \mathbf{N}$, we define $N = \{1, \ldots, n\} \subset \mathbf{N}$ as the n-voter electorate, where each $v \in N$ is

"none of the above (NOTA) ". A specific analysis of these rules, though out of the scope of this paper, can contribute to our understanding of the notion of abstention. We thank the associate editor who draw our attention to this.

2 The paradox has also been considered in the framework of judgement aggregation (see Balinski and Laraki, 2010).

3 For discussion on monotonicity conditions in social choice theory, one can see Fishburn (1982), Moulin (1983), Brams and Fishburn (2002) and Sanver and Zwicker (2009).

4 Other interesting violations of MON are established by Fishburn (1977), Richelson (1980) and Fishburn and Brams (1983).

5 Without omitting to note some borderline counter examples in Fishburn (1977), we can nevertheless say that almost all social choice rules considered in the literature are homogeneous.

6 One can see Saari (1990) for an analysis of these rules.

a voter. Π stands for the set of linear orders over A. $P_v \in \Pi$ is the preference of $v \in N$ over A, where for any distinct $x, y \in A$, xP_vy indicates that voter v prefers x to y.[7] We write $P_N = \{P_v\}_{v\in N}$ for a preference profile over A. A social choice rule (SCR) is a mapping F that returns, for each $n \in \mathbf{N}$ and each $P_N \in \Pi^N$, a single alternative $F(P_N) \in A$. So the SCRs we consider are variable-electorate in the sense of being defined for every number of voters and they satisfy the full domain condition in the sense that given any electorate, they are defined for every possible preference profile.

For any two electorates $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$, we define the joint electorate $M \oplus N = \{1, \ldots, m+n\}$. Note that $\oplus$ is commutative. Now letting $m \leq n$, for any two profiles R_N, Q_M, we let $P_{N\oplus M} = (R_N, Q_M)$ stand for the profile of $M \oplus N$ where $P_v = R_v \; \forall v \in \{1, \ldots, n\}$ and $P_{n+v} = Q_v \; \forall v \in \{1, \ldots, m\}$. Note that when $m < n$, $P_{M\oplus N}$ is uniquely defined by setting the first n voters as the voters of N and the remaining m voters as the voters of M. Abusing notation, when $M = \{v\}$, we write (R_N, Q_v) to denote the profile obtained from R_N by adding the preference Q_v of voter v. Given any $n \in \mathbf{N}$ and $v \in N$, we let $N^{-v} = N \setminus \{v\}$.

Definition 1. A SCR F satisfies participation (PART) iff $\forall N$ with $n \geq 2$, $\forall v \in N$, $\forall P_{N^{-v}}$, $\forall P_v$,

$$F(P_{N^{-v}}, P_v) \neq F(P_{N^{-v}}) \Longrightarrow F(P_{N^{-v}}, P_v)P_vF(P_{N^{-v}}).$$

Definition 2. Given any N, any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$, we say that P_N is an improvement for x w.r.t. P'_N if

(1) $xP'_vy \Longrightarrow xP_vy$ for every $y \in A \setminus \{x\}$,
(2) $yP'_vz \Longleftrightarrow yP_vz$ for every $y, z \in A \setminus \{x\}$.

Definition 3. A SCR F is monotonic (MON) iff given $x \in A$, $P_N, P'_N \in \Pi^N$ such that P_N is an improvement for x w.r.t. P'_N

$$x = F(P'_N) \Longrightarrow x = F(P_N).$$[8]

2.1. The case of two alternatives

The logical independence between MON and PART vanishes when there are only two alternatives. In fact, in this case PART implies MON, as stated by the following theorem.

Theorem 1. *Let $\#A := 2$. If a SCR F satisfies PART, then it satisfies MON.*

Proof. Let $A := \{x, y\}$. Take some F which satisfies PART but fails MON. So $\exists N, v, P_{N^{-v}}, P_v, P'_v$ with xP_vy, yP'_vx while $F(P_{N^{-v}}, P_v) = y$ and $F(P_{N^{-v}}, P'_v) = x$. However, by PART, $F(P_{N^{-v}}, P_v) = y$ implies $F(P_{N^{-v}}) = y$ and $F(P_{N^{-v}}, P'_v) = x$ implies $F(P_{N^{-v}}) = x$, giving a contradiction. □

The reverse implication holds, when a mild homogeneity condition is assumed. For any positive integer m, and any profile P_N, we write mP_N for any of the profiles obtained from P_N by replacing each single voter v of P_N with m voters having the same preference as v.[9]

Definition 4. A SCR F satisfies homogeneity (HOM) if $\forall N$, $\forall P_N$, $\forall m \in \mathbf{N}$, $\forall mP_N$, $F(mP_N) = F(P_N)$.

As the following theorem shows, any homogeneous SCR which satisfies MON satisfies PART as well.

Theorem 2. *Let $\#A := 2$. If a SCR F satisfies HOM and MON, then it satisfies PART.*

Proof. Let $A := \{x, y\}$. Assume that F satisfies HOM and MON but fails PART. Since F fails PART, there is some N with $\#N := n \geq 2$, some profile P_N and some voter $v \in N$ with xP_vy while $F(P_{N^{-v}}) = x$ and $F(P_N) = y$. Let $\#\{v \in N \mid xP_vy\} := k$ and $\#\{v \in N \mid yP_vx\} := n - k$. Consider now the profiles $nP_{N^{-v}}$ and $(n-1)P_N$. Due to HOM, it follows that $F(nP_{N^{-v}}) = x$ and $F((n-1)P_N) = y$. Moreover, both profiles have the same number $n(n-1)$ of voters. Yet, they differ on the number of voters who prefer x to y: there are $nk - n$ voters who prefer x to y in $nP_{N^{-v}}$ and $nk - k$ in $(n-1)P_N$. Hence, there is some profile $(n-1)P_N$ which is an improvement for x w.r.t. some profile $nP_{N^{-v}}$. Since $F(nP_{N^{-v}}) = x$, applying MON it follows that $F((n-1)P_N) = x$, giving a contradiction. □

However, it should be noted that MON does not imply PART without HOM, as the following proposition shows.

Proposition 1. *Let $\#A := 2$. There exists some SCR F that satisfies MON and fails PART.*

Proof. Let $A := \{x, y\}$. We construct some F that satisfies MON but fails PART (and HOM, by Theorem 2). When $\#N$ is even, let $F(P_N) = x$ iff $\#\{v \in N \mid xP_vy\} \geq \#\{v \in N \mid yP_vx\}$. When $\#N$ is odd, let $F(P_N) = x$ iff $\#\{v \in N \mid xP_vy\} = \#N$. In other words, when $\#N$ is even, F is the majority rule biased towards x in the event of tie and when $\#N$ is odd F is the unanimity rule biased towards y in the absence of a unanimously agreed alternative. It is clear that F satisfies MON. Let P_N be a profile with two out of three voters who prefer x to y and one voter who prefers y to x. Thus, there is some $P_v \in \Pi$ with xP_vy. It follows that $F(P_{N^{-v}}) = x$ and $F(P_N) = y$. Since $F(P_{N^{-v}})P_vF(P_N)$, F violates PART as desired. To see that F fails HOM, note that $F(2(P_N)) = x$ whereas $F(P_N) = y$. □

2.2. The case of three or more alternatives

2.2.1. On the implication of PART by MON

We start by showing that PART is not implied by MON even under the following reinforcement condition.

Definition 5. A SCR F satisfies reinforcement (REIN) if for any pair of electorates M and N, for any P_M, P_N and for any $x \in A$,

$$F(P_M) = F(P_N) = x \Longrightarrow F(P_{M\oplus N}) = x.$$

Assuming anonymity and neutrality, the conjunction of REIN with a few mild conditions characterizes scoring rules (Smith, 1973; Young, 1974, 1975; Myerson, 1995). In fact, one can see REIN as a core condition for being a scoring rule. On the other hand, Saari (1990) presents a weaker version of REIN and shows that threshold scoring rules represent a class of SCRs that fail REIN but satisfy its weak version. For single-valued SCRs, the strong and the weak versions of the condition coincide. Since scoring rules satisfy PART, we now show our claim through a threshold scoring rule.

For any preference P_v and any alternative x, the rank of x in P_v equals $r(x; P_v) = 1 + \#\{y \in A \mid yP_vx\}$. A *score vector* $s = (s_1, \ldots, s_{\#A})$ is an $\#A$-dimensional vector with $s_1 \geq s_2 \geq \cdots \geq s_{\#A}$ and $s_1 > s_{\#A}$. Under a score vector s, the score of alternative x at the preference profile P_N equals $S(x; P_N; s) = \sum_{v\in N} s_{r(x;P_v)}$. For any electorate N, any set A of alternatives and any score vector s, we define a threshold $t(N, A, s) := \frac{\#N}{\#A} \sum_{i=1}^{\#A} s_i$.

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7 Since P_v is a linear order it is complete, asymmetric and transitive. So, by completeness, for any distinct $x, y \in A$, we have xP_vy or yP_vx. Moreover, since P_v is asymmetric, $\forall x, y \in A \; xP_vy \Longrightarrow y\neg P_vx$. Furthermore, if xP_vy and yP_vz then xP_vz by transitivity.

8 The definition of MON applies just to profiles that differ by a single voter's preference, since our focus is on its relation to PART which is defined with respect to the addition of a single voter. However, it should be noted that our definition is equivalent to the more common definition in the literature where MON applies also to profiles which possibly differ in a group of voters' preferences.

9 Note that mP_N is equivalent to $P_{N\oplus N\oplus\cdots\oplus N}$. As here, $\oplus$ is applied to sets of equal size, mP_N is not uniquely defined.

Definition 6. Every score vector s induces a threshold scoring rule F which is defined for every N and every P_N as $F(P_N) = \{x \in A \mid S(x; P_N; s) \geq t(N, A, s)\}$. When $F(P_N)$ is multi-valued, ties are broken according to an exogenous (alphabetical) linear order.

Threshold scoring rules are well-defined since, by the choice of $t(N, A, s)$, $F(P_N)$ is not empty for every P_N. The next lemma proves this point formally.

Lemma 1. *If F is a threshold scoring rule, $F(P_N)$ is not empty for every P_N.*

Proof. Assume by contradiction that there is some profile P_N for which $F(P_N)$ is empty. Thus, $S(x; P_N, s) < t(N, A, s)$ for any $x \in A$. It follows that $\sum_{x\in A} S(x; P_N, s) < \sum_{x\in A} t(N, A, s)$. However, the left part of the inequality equals $\#N \sum_{i=1}^{\#A} s_i$ whereas the right part equals $\#At(N, A, s)$ which contradicts the definition of $t(N, A, s)$. □

Proposition 2. *Let $\#A \geq 3$. There exists some SCR F that satisfies MON and REIN and fails PART.*

Proof. Let $A := \{x, y, z\}$ and consider F to be the threshold scoring rule with $s = (1, 1, 0)$. Hence, $t(N, A, s) = \frac{2}{3}n$. F satisfies MON and REIN. To see that F fails PART, let $N = \{1, \ldots, 6\}$ and take the preference profile P_N with $aP_v bP_v c$ for $v = 1, 2$, $cP_v bP_v a$ for $v = 3, 4$ and $aP_v cP_v b$ for $v = 5$, and $bP_v aP_v c$ for $v = 6$. It follows that for $v = 6$, $F(P_{N^{-v}}) = b$ and $F(P_N) = a$ while $F(P_{N^{-v}})P_v F(P_N)$, proving that PART fails.[10] □

We now ask whether all threshold scoring rules fail PART. The answer is almost affirmative as the theorem below shows.

Theorem 3. *Let $\#A \geq 3$. Let F be a threshold scoring rule induced by a score vector s. F satisfies PART if and only if*

$$s = \left(s_1, \frac{s_1 + s_{\#A}}{2}, \frac{s_1 + s_{\#A}}{2}, \ldots, \frac{s_1 + s_{\#A}}{2}, s_{\#A}\right).$$

Proof. Take some threshold scoring rule F with score vector s. Suppose PART fails so $F(P_{N^{-v}}) = x$ and $F(P_{N^{-v}}, P_v) = y$ with $xP_v y$ for some N, v, $P_{N^{-v}}$ and P_v. This can occur under one of the following two exhaustive cases.

Case 1: $S(x, P_{N^{-v}}, s) \geq t(N^{-v}, A, s)$ and $S(x, P_N, s) < t(N, A, s)$.

Case 2: $\exists y \neq x$ with $S(y, P_{N^{-v}}, s) < t(N^{-v}, A, s)$ and $S(y, P_N, s) \geq t(N, A, s)$.

Note that $t(N, A, s) - t(N^{-v}, A, s) = \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

A necessary and sufficient condition to avoid case 1 is that the lowest additional score that x receives with the arrival of v is at least $\frac{\sum_{i=1}^{\#A} s_i}{\#A}$. As $xP_v y$, hence $r(x; P_v) \geq \#A - 1$, this is ensured by setting $s_{\#A-1} \geq \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

A necessary and sufficient condition to avoid case 2, it is that the highest additional score that y gets with the arrival of v does not exceed $\frac{\sum_{i=1}^{\#A} s_i}{\#A}$. As $xP_v y$, hence $r(y; P_v) \leq 2$, this is ensured by setting $s_2 \leq \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

The two inequalities combined with $s_2 \leq s_3 \leq \cdots s_{\#A-1}$ imply $s_j = \frac{s_1 + s_{\#A}}{2}$ for all $j = 2, \ldots, \#A - 1$. □

Having shown that MON does not imply PART even under REIN, we now adopt a weaker version of participation, introduced by Pérez (2001), as the absence of a stronger version of the no-show paradox where a voter, by abstaining, can enforce his most preferred alternative as the social outcome.

[10] This example is equivalent to the one discussed by Moulin (1988) in p. 62 showing that REIN and PART are logically independent.

Definition 7. A SCR F satisfies weak participation (WPART) iff $\forall N$, $\forall v \in N$, $\forall P_{N^{-v}}$, $\forall P_v$,

$$F(P_{N^{-v}}, P_v) \neq F(P_{N^{-v}}) \Longrightarrow \exists x \in A \quad \text{s.t. } xP_v F(P_{N^{-v}}).$$

Note that WPART is equivalent to PART with just two alternatives whereas it is weaker with more than two alternatives: PART requires that when adding the vote of v, voter v prefers $F(P_{N^{-v}}, P_v)$ to $F(P_{N^{-v}})$, whereas WPART just requires the existence of some alternative that v prefers to $F(P_{N^{-v}})$.

We now show that MON does not imply WPART either, even when combined with REIN.

Proposition 3. *Let $\#A \geq 3$. There exists some SCR F that satisfies REIN and MON but fails WPART.*

Proof. Let $A := \{x, y, z\}$ and consider the threshold scoring rule with $s = (1, 1, 0)$. This rule satisfies REIN and MON. In order to see why it fails WPART, consider the example used in the proof of Proposition 2. Since $b = F(P_{N^{-v}})P_v F(P_{N^{-v}}, P_v) = a \text{with} bP_v aP_v c$, this shows that WPART fails. □

Proposition 3 shows that MON and REIN do not imply WPART. However, when combined with the following weak unanimity condition imposed over singleton electorates, REIN implies WPART by its own.

Definition 8. A SCR F is weakly unanimous iff $\forall v \in N$, $\forall P_v$,

$$xP_v y \forall y \neq x \Longrightarrow F(P_v) = x.$$

Theorem 4. *Let $\#A \geq 3$. If a weakly unanimous SCR F satisfies REIN, then it satisfies WPART.*

Proof. Take some weakly unanimous F that satisfies REIN. If it fails WPART, there exist $P_{N^{-v}}$, P_v such that $F(P_{N^{-v}}, P_v) = x \neq F(P_{N^{-v}}) = z$ with $zP_v y\, \forall y \in A\backslash\{z\}$. As z is first ranked in P_v, by weak unanimity, $F(P_v) = z$. By REIN, it follows that $F(P_{N^{-v}}, P_v) = z$, which leads to a contradiction. □

It is worth noting that WPART is also studied by Richelson (1980) under the name "voter adaptability" and Theorem 4 is a precise expression of his statement in p.464 that "voter adaptability is a much weakened version of Young's consistency", as the validity of this claim requires weak unanimity. In fact, the threshold scoring rule in our Proposition 3 is not weakly unanimous; satisfies REIN (i.e., Young's consistency) and fails WPART.

2.2.2. On the implication of MON by PART

We ask whether MON has reasonable weakenings implied by PART. We first strengthen the definition of an improvement, by asking that the lifted alternative must be raised from the bottom of the ranking.

Definition 9. Given any N, any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$, we say that P_N is a strong improvement for x w.r.t. P'_N if

(1) $yP'_v x$ for every $y \in A \setminus \{x\}$,
(2) $yP'_v z \Longleftrightarrow yP_v z$ for every $y, z \in A \setminus \{x\}$.

The following is a weakening of MON because the definition of improvement is strengthened but also because it allows alternatives above the lifted alternative to be chosen. As in the case of WPART and PART, it turns out that WMON and MON are equivalent with just two alternatives.

Definition 10. A SCR F is weakly monotonic (WMON) iff given $x \in A$, $P_N, P'_N \in \Pi^N$ such that P_N is a strong improvement for x w.r.t. P'_N:

$$x = F(P'_N) \quad \text{and} \quad F(P_N) \neq F(P'_N) \Longrightarrow F(P_N)P_v x.$$

Theorem 5. *Let* $\#A \geq 3$. *If a SCR F satisfies PART, then it satisfies WMON.*

Proof. Take some F that satisfies PART but violates WMON. Since WMON fails, there exist some N, $v \in N$, P_N, P'_N with $P'_v \neq P_v$ where (P_{N-v}, P_v) is a strong improvement for x with respect to (P_{N-v}, P'_v), while $F(P_{N-v}, P'_v) = x$ and $F(P_{N-v}, P_v) = y$ with $xP_v y$. Due to PART, $F(P_{N-v}, P'_v) = x$ implies $F(P_{N-v}) = x$, as otherwise $F(P_{N-v})P'_v x$ would hold, violating PART. Since $F(P_{N-v}) = x$, then by PART again, we have $F(P_{N-v}, P_v) \neq y$, giving a contradiction. □

It should be noted that WMON is not too weak: it is able to discriminate among the SCRs that fail MON. For instance, one can check that the examples described in Campbell and Kelly (2002) which fail MON do satisfy WMON, which is the case by Theorem 5, as they all satisfy PART. On the other hand, plurality with a runoff, well known to fail MON, fails WMON as well, as we illustrate through the example below.

Example 1. Let $A := \{x, y, z\}$ and consider two profiles P'_N, P_N with eight voters such that P_N is a strong improvement for z w.r.t. to P'_N:

$P_N : \#\{v \in N \mid xP_v yP_v z\} = 2, \quad \#\{v \in N \mid yP_v zP_v x\} = 2,$
$\#\{v \in N \mid zP_v xP_v y\} = 4,$
$P'_N : \#\{v \in N \mid xP'_v yP'_v z\} = 3, \quad \#\{v \in N \mid yP'_v zP'_v x\} = 2,$
$\#\{v \in N \mid zP'_v xP'_v y\} = 3.$

Under plurality with a runoff where ties are broken in favor of y, at P'_N, x and z which are both first ranked by three voters go for a runoff and, since there is a majority of voters who prefer z to x, we have $F(P'_N) = z$. At P_N, z is first ranked by four voters whereas both x and y are ranked first by two voters each. As ties are broken in favor of y, y and z go for a runoff where $F(P_N) = y$. However, this violates WMON since P_N is a strong improvement for z w.r.t. to P'_N and $F(P'_N) = z$.

In fact, the observation made by the example above reflects a more general fact: all point runoff procedures but one fail WMON. To see this, we first remark that when there are precisely three alternatives, every point runoff procedure eliminates the alternative with the lowest score according to some score vector $(1, \lambda, 0)$ with $0 \leq \lambda \leq 1$ (we assume that ties are broken alphabetically).

Theorem 6. *Let* $\#A \geq 3$. *Every point runoff procedure fails WMON, unless* $\lambda = \frac{1}{2}$.

Proof. Let $A := \{x, y, z\}$ and consider some profile P_N as follows, for some positive integers n_1, n_2:

– three groups of n_1 voters, each group having respective preferences $xP_v yP_v z$, $yP_v zP_v x$ and $zP_v xP_v y$,

– three groups of n_2 voters, each group having respective preferences $xP_v zP_v y$, $zP_v yP_v x$ and $yP_v xP_v z$ and

– four voters, each with respective preference: $xP_v yP_v z$, $xP_v yP_v z$, $yP_v zP_v x$ and $zP_v xP_v y$.

There are, hence, $3n_1 + 3n_2 + 4$ voters in the profile P_N. It follows that there are $n_1 - n_2 + 2$ more voters who prefer x to y than y to x and that there are $n_1 - n_2$ more voters who prefer z to x than x to z. We let $n_1 > n_2$.

For any runoff procedure with vector $(1, \lambda, 0)$, the score that each alternative receives from the first six groups is equal to $\eta = (n_1 + n_2)(1 + \lambda)$. It follows that the score for the alternatives at P_N equals $s(x) = \eta + 2 + \lambda$, $s(y) = \eta + 1 + 2\lambda$ and $s(z) = \eta + 1 + \lambda$ so that $s(x) > s(y) \geq s(z)$ as long as $\lambda < \frac{1}{2}$ and $s(x) \geq s(y) > s(z)$ when $\lambda > \frac{1}{2}$. Breaking ties alphabetically, x, y go for a runoff and x is the winner under any runoff procedure since x is majority preferred to y by the choice of n_1, n_2.

We now show that WMON fails as long as $\lambda \neq \frac{1}{2}$. We first analyze the case in which $\lambda < \frac{1}{2}$.

Consider the profile P'_N such that the only difference with P_N is that k of the n_1 voters with preference $yP_v zP_v x$, raise x to the top and switch to $xP'_v yP'_v z$. P'_N is a strong improvement for x w.r.t. P_N. Letting $s'(\cdot)$ be the scores of the alternatives under P'_N we have $\Delta(x) = s'(x) - s(x) = k$, $\Delta(y) = s'(y) - s(y) = k(1 - \lambda)$ and $\Delta(z) = s'(z) - s(z) = k\lambda$. Note that when $\Delta(y) - \Delta(z) > \lambda$, hence $k(1 - 2\lambda) > \lambda \iff k > \frac{\lambda}{1-2\lambda}$, x and z go for a runoff. It is possible to pick such a k and $n_1 - n_2$ arbitrarily large, so that z is majority winner against x at P'_N as well, showing that any runoff procedure with $\lambda < \frac{1}{2}$ fails WMON.

The case for $\lambda > \frac{1}{2}$ is similar to the previous one: it suffices to construct a profile P'_N such that the only difference with P_N is that k of the n_2 voters with preference $zP_v yP_v x$ in P_N switch to $xP'_v zP'_v y$ in P'_N. The above logic applies showing that any runoff procedure with $\lambda > \frac{1}{2}$ fails WMON as well. □

The proof of Theorem 6 suggests that runoff procedures under Borda scores satisfy WMON, which we present as a conjecture. Nevertheless, we use Theorem 6 to show that no point runoff procedure satisfies PART.

Theorem 7. *Let* $\#A \geq 3$. *Every point runoff procedure fails PART.*

Proof. Note that, when $\#A = 3$, any runoff procedure with $\lambda \neq \frac{1}{2}$ fails WMON (Theorem 6) and hence, due to Theorem 5, fails PART. Hence, the only runoff rule that might satisfy PART is the one with $\lambda = \frac{1}{2}$. To see that this rule also fails PART, take the profile P_N as defined in the proof of Theorem 6, where, letting $\lambda = \frac{1}{2}$, we have $s(x) = \eta + 2.5$, $s(y) = \eta + 2$ and $s(z) = \eta + 1.5$ so that x wins. Consider the profile (P_N, P_v) where $xP_v zP_v y$. Now, the scores are $s'(x) = \eta + 3.5$, $s'(y) = \eta + 2$ and $s'(z) = \eta + 2$. If the tie between y and z is broken in favor of z, x and z go for a runoff, which leads to the victory of z, showing the failure of PART. □

2.3. A partial characterization of PART

While we mainly focus on the connection between PART and MON, this section gives a partial characterization of PART through a lower contour set intersection property, which we call Condition λ.[11]

For any P_v and any alternative x, let $L(x; P_v) = \{y \in A \mid xP_v y\}$ denote the set of alternatives such that x is at least as good as any of them under P_v (the lower contour set of x under P_v).

Definition 11. A SCR F satisfies Condition λ if for any P_{N-v} the following holds:

$$\bigcap_{P_v} L(F(P_{N-v}; P_v), P_v) \neq \emptyset.$$

Building on this condition, we now provide two sets of results: the first one deals with the relation of Condition λ with PART whereas the second one focuses on its relation with MON.

To start with, we prove that PART and Condition λ are almost equivalent.

[11] We thank the associate editor for suggesting this nice extension.

Theorem 8. *If a SCR F satisfies PART, then F satisfies Condition* λ.

Proof. Take some F that satisfies PART. Therefore, $\forall v \in N$, $\forall P_{N-v}$, $\forall P_v$, $F(P_{N-v}, P_v) \neq F(P_{N-v}) \implies F(P_{N-v}, P_v) P_v F(P_{N-v})$. It follows that $F(P_{N-v}) \in L(F(P_{N-v}, P_v); P_v)$ for every profile P_{N-v}, thus Condition λ holds. □

It is hence clear that every scoring rule satisfies Condition λ since any such rule satisfies PART. Yet, the same property does not apply to Condorcet extensions: as we now show, no such rule satisfies Condition λ.

Now, in order to state our claim, we introduce *half-way monotonicity* as defined by Sanver and Zwicker (2009). For any linear order P_v over A, we let $rev(P_v)$ be the linear order obtained by reversing P_v so that xP_vy iff $y\, rev(P_v) x$ for any pair x, y of alternatives.

Definition 12. A SCR F is half-way monotonic (HMON), if for any N, any $v \in N$, any P_v, P_{N-v} and any $x, y \in A$:

$$x = F(P_{N-v}, P_v) \quad \text{and} \quad y = F(P_{N-v}, rev(P_v)) \Longrightarrow xP_vy.$$

HMON can be interpreted as follows: a rule that violates HMON can be manipulated by some voter who completely misrepresents his preference, in the sense of announcing a preference that reverses every possible pairwise comparison among alternatives.

Lemma 2. *If a SCR F satisfies Condition* λ*, then F satisfies HMON*.

Proof. Take some F that fails *HMON*. Therefore, there exists some P_{N-v} and some pair $x, y \in A$ with $x = F(P_{N-v}, P_v)$andy $= F(P_{N-v}, rev(P_v))$ with yP_vx. However, $L(x; P_v) \cap L(y; rev(P_v)) = \emptyset$ since $rev(P_v)$ is the reversal of P_v and yP_vx. Hence, F fails Condition λ, concluding the proof. □

Theorem 9. *No Condorcet extension satisfies Condition* λ.

Proof. Since no Condorcet extension satisfies HMON with four or more alternatives and sufficiently many voters (see Corollary 5.3 in Sanver and Zwicker, 2009) and Condition λ implies HMON, it follows that no Condorcet extension satisfies Condition λ. □

As a corollary to Theorems 8 and 9, we obtain Moulin (1988)'s result that no Condorcet extension satisfies PART.

The converse of Theorem 8 does not hold as proved by the next result.

Proposition 4. *Let* $\#A \geq 3$. *There exists some SCR F that satisfies Condition* λ *and fails WPART, hence PART.*

Proof. Fix a pair of alternatives x, y and some voter d. Consider the following SCR F: (1) if d takes part in the election, then F chooses d's most preferred alternative among $\{x, y\}$ whereas (2) when d does not take part in the election, F chooses the winner according to Plurality voting. One can check that F fails PART. In order to see why F satisfies Condition λ, take first any profile P_N where d takes part in the election. Since the outcome $F(P_N)$ is the most preferred one of d over $\{x, y\}$, Condition λ holds. Take now a profile P_N without d so that the outcome $F(P_N)$ is determined through Plurality rule. Since this rule satisfies PART, it also satisfies Condition λ as proved by Theorem 8. Therefore, F satisfies Condition λ. □

Now, in order to prove that the converse of Theorem 8 holds provided that some mild conditions are added, we introduce the reversal cancellation property. Reversal cancellation, as defined by Sanver and Zwicker (2009), is arguably quite mild: according to it, adding a linear order and its reversal should leave the outcome of the SCR unchanged.

Definition 13. A SCR F satisfies Reversal Cancellation (RC) if for any N, any $v \in N$, any P_v and any $P_N \in \Pi^N$:

$$F(P_N) = F(P_N, P_v, rev(P_v)).$$

The next proposition shows that, combined with HOM and RC, Condition λ implies PART.

Theorem 10. *Let* $\#A \geq 3$. *If a SCR F satisfies Condition* λ*, HOM and RC, then F satisfies PART.*

Proof. Take some SCR F that fails PART. Therefore, there exists some P_{N-v} and P_v with $x = F(P_{N-v}) P_v F(P_{N-v}, P_v) = y$ for some x, y. If F satisfies HOM and RC, then it follows that $F(P_{N-v}) = F(2P_{N-v}) = F(2P_{N-v}, P_v, rev(P_v))$ and $F(P_{N-v}, P_v) = F(2P_{N-v}, P_v, P_v)$. It follows that $F(2P_{N-v}, P_v, rev(P_v)) = x$ and $F(2P_{N-v}, P_v, P_v) = y$. However, $L(x; rev(P_v)) \cap L(y; P_v) = \emptyset$ since xP_vy by definition and $rev(P_v)$ is the reversal linear order of P_v. Hence, F fails Condition λ, which concludes the proof. □

To see why RC cannot be dropped with more than three alternatives, it suffices to consider the voting rule described by the proof of Proposition 4: this rule satisfies HOM and Condition λ but fails both RC and PART. With two alternatives, RC is not anymore needed. Indeed, Condition λ is equivalent to MON as will be shown by Theorem 12. Therefore, using Theorem 2, one can see that Condition λ jointly with HOM implies PART in this case.

Once we have shown the almost equivalence of PART and Condition λ, this final set of results studies the relationship between Condition λ and MON. We first show that both conditions are logically independent with at least three alternatives.

Theorem 11. *Let* $\#A \geq 3$. *MON and Condition* λ *are logically independent.*

Proof. In order to check that MON does not imply Condition λ, it suffices to see any Condorcet extension that satisfies MON, e.g. Copeland rule, must fail Condition λ, as proved by Theorem 9.

Now, in order to see that Condition λ does not imply MON, note that Theorem 8 shows that PART implies Condition λ. Hence, combining this observation with the examples provided by Campbell and Kelly (2002) that satisfy PART but fail MON, concludes the proof. □

Even though the conditions are logically independent, we can prove that they are equivalent with just two alternatives and that Condition λ generally implies WMON, the weaker notion of monotonicity.

Theorem 12. *Let* $\#A := 2$. *A SCR F satisfies Condition* λ *if and only if F satisfies MON.*

Proof. Let $A := \{x, y\}$. Take some F that satisfies Condition λ but fails MON. So, $\exists N, v, P_{N-v}, P_v, P_{v'}$ with xP_vy, $yP_{v'}x$ while $F(P_{N-v}, P_v) = y$ and $F(P_{N-v}, P_{v'}) = x$. However, this implies that $L(y; P_v) = \{y\}$ whereas $L(x; P_{v'}) = \{x\}$ so that Condition λ fails, a contradiction.

Take now some F that satisfies MON. So, $\exists N, v, P_{N-v}, P_v, P_{v'}$ with $L(F(P_{N-v}, P_v); P_v) \cap L(F(P_{N-v}, P_{v'}); P_{v'}) = \emptyset$. Assume w.l.o.g. that xP_vy and $yPvx$. Then if $F(P_{N-v}, P_v) = x$, $L(F(P_{N-v}, P_v); P_v) = \{x, y\}$ so that Condition λ holds. If $F(P_{N-v}, P_v) = y$, then $L(F(P_{N-v}, P_v); P_v) = \{y\}$. If, now $F(P_{N-v}, P_{v'}) = y$ then $L(F(P_{N-v}, P_{v'}); P_{v'}) = \{x, y\}$ so that Condition λ holds again. Therefore, it must be the case that $F(P_{N-v}, P_{v'}) = x$ to ensure that Condition λ holds. But this contradicts MON since xP_vy and $yPvx$, concluding the proof. □

Combining Theorem 12 with Theorem 1, Theorem 2 and Proposition 1, one can see that with just two alternatives, PART implies Condition λ; Condition λ does not imply PART and condition λ together with HOM imply PART.

Theorem 13. *If a SCR F satisfies Condition λ, then it satisfies WMON.*

Proof. Take some SCR F that fails WMON but satisfies Condition λ. Since F fails WMON, there must exist $P_{N^{-v}}, P_v, P'_v$ and x, y such that $F(P_{N^{-v}}, P_v) = x$ and $F(P_{N^{-v}}, P'_v) = y$ with x ranked last at P_v and y ranked below x at P'_v. Hence $L(x, P_v) = \{x\}$ whereas $x \notin L(y, P'_v)$ since y ranked below x. But this implies that F fails Condition λ since the previous equalities imply that $L(F(P_{N^{-v}}; P_v), P_v) \cap L(F(P_{N^{-v}}, P'_v); P'_v) = \emptyset$. □

3. The fixed electorate case

We now consider the case where the electorate N is of fixed size $n \geq 2$. A voter v is now allowed to have as preference a linear order $P_v \in \Pi$ or to abstain, i.e. have full indifference over the whole set of alternatives. This indifference is denoted by the null preference R_0 where xR_0y holds for any $x, y \in A$. We let $\overline{\Pi} := \Pi \cup \{R_0\}$. The profile $(P_{N^{-v}}, R_0) \in \overline{\Pi}^N$ is the profile in which voter v abstains and the rest of voters' preferences are as in $P_{N^{-v}}$.

Given any $n \in \mathbf{N}$, a fixed-size social choice rule (FSCR) is a mapping F_n that returns, for $P_N \in \overline{\Pi}^N$, a single alternative $F_n(P_N) \in A$. Note that the full domain assumption prevails, i.e. given the fixed electorate N, F_n is defined for every possible preference profile P_N.

We now define MON under the possibility of abstention in individual preferences.

Definition 14. Given any x and any P_N, P'_N with $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$,
If $P'_v \neq R_0$, then P_N is an improvement for x w.r.t. P'_N if

(1) $xP'_vy \Longrightarrow xP_vy$ for every $y \in A \setminus \{x\}$,
(2) $yP'_vz \Longleftrightarrow yP_vz$ for every $y, z \in A \setminus \{x\}$.

If $P'_v = R_0$, then P_N is an improvement for x w.r.t. P'_N if

(1) xP_vy for every $y \in A \setminus \{x\}$.

Definition 15. A FSCR F_n is monotonic (MON) iff given $x \in A$, $P_N, P'_N \in \overline{\Pi}^N$ such that P_N is an improvement for x w.r.t. P'_N,

$$x = F_n(P'_N) \Longrightarrow x = F_n(P_N).$$

We now define PART in this framework.

Definition 16. A FSCR F_n satisfies participation (PART) iff $\forall v \in N$, $\forall P_{N^{-v}} \in \overline{\Pi}^{N^{-v}}$, $\forall P_v \in \overline{\Pi}$,

$$F_n(P_{N^{-v}}, P_v) \neq F_n(P_{N^{-v}}, R_0) \Longrightarrow F_n(P_{N^{-v}}, P_v)P_vF_n(P_{N^{-v}}, R_0).$$

We now establish an equivalence between the fixed and variable electorate interpretations regarding the satisfaction of PART. We start by giving two definitions.

Definition 17. A family of FSCRs $\{F_n\}_{n\in\mathbf{N}}$ is equivalent to a variable electorate SCR F if and only if for any $n \in \mathbf{N}$ and any $P_N \in \Pi^N$, $F_n(P_N) = F(P_N)$.

Definition 18. A family of FSCRs $\{F_n\}_{n\in\mathbf{N}}$ is regular if for any $n \in \mathbf{N}$, for any $P_N \in \overline{\Pi}^N$ with $P_i = R_0$ for some $i \in \{1, \ldots, n\}$, $F_n(P_N) = F_n(P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$.

The meaning of Definition 17 is clear. The idea behind regularity is to ignore voters who take part in the election without expressing a preference; this is satisfied by many well-known rules such as scoring rules and the Condorcet principle. Nevertheless, the quorum rules (see Houy, 2009), where some minimal level of participation is required, fail to satisfy regularity.

Observe that for each variable electorate SCR F, there exists some equivalent family $\{F_n\}_{n\in\mathbf{N}}$ of FSCRs which is not unique. Yet, uniqueness is obtained when regularity is imposed over $\{F_n\}_{n\in\mathbf{N}}$.

Theorem 14. *Let F be a variable electorate SCR and $\{F_n\}_{n\in\mathbf{N}}$ the regular family of FSCRs which is equivalent to F. F satisfies PART if and only if F_n satisfies PART for each $n \in \mathbf{N}$.*

Proof. Take some SCR F and its equivalent regular family of FSCRs $\{F_n\}_{n\in\mathbf{N}}$.

Assume that F satisfies PART. Due to Definition 17, it follows that for any $n \geq 2$, $\forall v \in N$, $\forall P_{N^{-v}}$, $\forall P_v$,

$$F(P_{N^{-v}}, P_v) = F_n(P_{N^{-v}}, P_v) \quad \text{and} \quad F(P_{N^{-v}}) = F_{n-1}(P_{N^{-v}}).$$

Moreover, Definition 18 implies that $F_{n-1}(P_{N^{-v}}) = F_n(P_{N^{-v}}, R_0)$. Since F satisfies PART, it follows that $F(P_{N^{-v}}, P_v) \neq F(P_{N^{-v}}) \Longrightarrow F(P_{N^{-v}}, P_v)P_vF(P_{N^{-v}})$. Therefore, the previous equalities imply that, as long as $F_n(P_{N^{-v}}, P_v) \neq F_n(P_{N^{-v}}, R_0)$,

$$F_n(P_{N^{-v}}, P_v)P_vF_n(P_{N^{-v}}, R_0),$$

which proves that F_n satisfies PART for any $n \in \mathbf{N}$.

Assume now that each F_n satisfies PART. Again Definitions 17 and 18 jointly imply that

$$F(P_{N^{-v}}, P_v) = F_n(P_{N^{-v}}, P_v) \quad \text{and}$$
$$F(P_{N^{-v}}) = F_n(P_{N^{-v}}, R_0).$$

Since each F_n satisfies PART, it follows that

$$F_n(P_{N^{-v}}, P_v) \neq F_n(P_{N^{-v}}, R_0) \Longrightarrow F_n(P_{N^{-v}}, P_v)P_vF_n(P_{N^{-v}}, R_0).$$

However, combining the previous implication with the described equivalence between the SCR F and each FSCR F_n, proves that F satisfies PART, as desired. □

By using Theorem 14, we can transfer results on the satisfaction of PART in the variable electorate setting to the fixed electorate one. More precisely, Moulin (1988) proves that with four or more alternatives and with at least 25 voters, no Condorcet rule satisfies PART. Recently, Brandt et al. (2016) proves that the minimal number of voters to obtain this incompatibility is exactly 12 using computational techniques.[12] As a consequence of these results, we can conclude that scoring rules satisfy PART in the fixed electorate setting as well or that the result of Moulin on the non-existence of Condorcet extensions that satisfy PART prevails when the electorate is fixed. To be more precise regarding the latter observation, there exists some $n \in \mathbf{N}$ where no FSCR F_n is Condorcet Consistent and satisfies PART. This also shows that MON does not imply PART in this setting as well, as a monotonic Condorcet extension such as the Copeland rule illustrates. As a matter of fact, the general logical independence between PART and MON prevails in this setting as shown by the following adaptation of the Campbell and Kelly (2002) example to the fixed electorate setting.

Example: PART does not imply MON. Fix some $x \in A$. We define a FSCR F_n such that, for any profile P_N, (1) if $P_v = R_0$ for every $v \in N$,

12 For a recent contribution on the No-Show paradox with social choice correspondences, see Brandl et al. (2015).

then $F_n(P_N) = x$, (2) if x is ranked last by every $P_v \neq R_0$, then $F_n(P_N) = x$ and (3) if there is some $P_v \neq R_0$ where some alternative y different from x is ranked last, then F_n selects the most preferred alternative of the voter with the lowest index among those with a strict preference where x is not ranked last. One can check that F_n satisfies PART but fails MON.

As the following theorem states, for FSCRs, MON and PART are equivalent when there are two alternatives only.

Theorem 15. *Let $\#A := 2$. A FSCR F_n satisfies MON if and only if it satisfies PART.*

Proof. Let $A := \{x, y\}$. Take some F_n that satisfies PART but fails MON. Since F_n fails MON, w.l.o.g., one of the following two exhaustive cases holds.

Case 1: there exist some profile $P_{N-v} \in \overline{\Pi}^{N-v}$ and some pair P'_v, P_v with $yP'_v x$ and $xP_v y$ with $F_n(P_{N-v}, P'_v) = x$ and $F_n(P_{N-v}, P_v) = y$. However, due to PART, $F_n(P_{N-v}, P'_v) = x$ implies $F_n(P_{N-v}, R_0) = x$ which in turn implies $F_n(P_{N-v}, P_v) = x$, giving a contradiction.

Case 2: there exist some profile $P_{N-v} \in \overline{\Pi}^{N-v}$ and some pair $P'_v = R_0, P_v$ with $xP_v y$ with $F_n(P_{N-v}, P'_v) = x$ and $F_n(P_{N-v}, P_v) = y$. However, as $F_n(P_{N-v}, R_0) = x$, PART implies $F_n(P_{N-v}, P_v) = x$, giving a contradiction. We leave the reader to check that MON implies PART. □

We now define WPART in the fixed electorate framework.

Definition 19. A FSCR F_n satisfies weak participation (WPART) iff $\forall v \in N, \forall P_{N-v} \in \overline{\Pi}^{n-1}, \forall P_v \in \overline{\Pi}$,

$$F_n(P_{N-v}, P_v) \neq F_n(P_{N-v}, R_0) \implies \exists x \in A \quad \text{s.t. } xP_v F_n(P_{N-v}, R_0).$$

Theorem 16. *Let $\#A \geq 3$. If a FSCR F_n satisfies MON, then it satisfies WPART.*

Proof. Take some F_n that fails WPART. Since F_n fails WPART, there must some profile $P_{N-v} \in \overline{\Pi}^{N-v}$ and some pair $P'_v, P_v = R_0$ with $xP'_v z\ \forall z \in A \setminus \{x\}$ while $F_n(P_{N-v}, P_v) = x$ and $F_n(P_{N-v}, P'_v) \neq x$. As (P_{N-v}, P'_v) is an improvement for x w.r.t. (P_{N-v}, P_v), this violates MON and concludes the proof. □

The following is the weakening of MON in the same spirit as the weakening introduced in Section 2.

Definition 20. Given any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$,
If $P'_v \neq R_0$, then P_N is a strong improvement for x w.r.t. P'_N if

(1) $yP'_v x$ for every $y \in A \setminus \{x\}$,
(2) $yP'_v z \iff yP_v z$ for every $y, z \in A \setminus \{x\}$.

If $P'_v = R_0$, then P_N is a strong improvement for x w.r.t. P'_N if

(1) $xP_v y$ for every $y \in A \setminus \{x\}$.

Definition 21. A FSCR F_n is weakly monotonic (WMON) iff given $x \in A, P_N, P'_N \in \overline{\Pi}^N$ such that P_N is a strong improvement for x w.r.t. P'_N

$$x = F_n(P'_N) \quad \text{and} \quad F_n(P_N) \neq F_n(P'_N) \implies F_n(P_N) P_v x.$$

Theorem 17. *Let $\#A \geq 3$. If a FSCR F_n satisfies PART, then it satisfies WMON.*

Proof. Take some F_n that satisfies PART but fails WMON. Since F_n fails WMON, one of the following two exhaustive cases holds.

Case 1: there exist some profile $P_{N-v} \in \overline{\Pi}^{N-v}$ and some pair P'_v, P_v with $zP'_v x \forall z \in A \setminus \{x\}$ and $xP_v y$ with $F_n(P_{N-v}, P'_v) = x$ and $F_n(P_{N-v}, P_v) = y$. However, due to PART, $F_n(P_{N-v}, P'_v) = x$ implies $F_n(P_{N-v}, R_0) = x$ which in turn implies $F_n(P_{N-v}, P_v) \neq y$, giving a contradiction.

Case 2: there exist some profile $P_{N-v} \in \overline{\Pi}^{N-v}$ and some pair $P'_v = R_0, P_v$ with $xP_v z \forall z \neq x$ while $F_n(P_{N-v}, P'_v) = x$ and $F_n(P_{N-v}, P_v) = y$. However, due to PART, $F_n(P_{N-v}, R_0) = x$ implies $F_n(P_{N-v}, P_v) = x$, giving a contradiction.

Thus, there is no F_n satisfying PART but failing WMON, which concludes the proof. □

4. Concluding remarks

Although the logical independence between the no-show paradox and the failure of monotonicity has already been observed, our findings suggest that this observation does not mean a major conceptual gap between the two conditions. In fact, under two different interpretations of "a new-comer to the society", we are able to present instances where participation and monotonicity get very close to each other—in some particular cases to the extent that the established general logical independence between them vanishes.

This closeness is rather expected to us because, as we discuss in the introduction, both conditions are related to the manipulability of SCRs. In fact, until their logical independence was established by Nurmi (1999) and Campbell and Kelly (2002), there was a prevailing intuition that the two conditions were somehow related, in particular that PART could imply MON (see, for example, Nurmi, 1999, p. 62). Our findings point to a wisdom in this intuition: Although Campbell and Kelly (2002) show that PART does not imply MON, as the unorthodoxy of their examples suggests, PART almost implies MON, more precisely implies its weaker version WMON. On the other hand, as our Proposition 3 suggests, the fact that MON does not imply PART is not a mere consequence of the fact that PART is a condition for SCRs defined over variable size societies while MON is not.

Our discussions on the relationship between PART and MON paved the way to general results on certain interesting classes of SCRs. In particular, we show that all threshold scoring rules but one fail PART; all point runoff procedures but one (namely Borda) fail WMON; and all point runoff procedures fail PART.

It is worth noting that although the fixed-electorate interpretation has no considerable effect on the class of social choice rules that satisfy PART, it results in MON and PART getting closer. This is also rather expected because, again as discussed in the introduction, under this interpretation the link between the no-show paradox and manipulability of SCRs is more direct. A point we wish to emphasize is the equivalence between PART and MON under the fixed-electorate interpretation, when there are two alternatives. In this framework, majority rules with quorums are known to fail MON and they are supposed to give room to manipulation by abstention (see Houy, 2009). Our Theorem 15 is a formal expression of this supposition.

Finally, we wish to remark that PART has been mostly considered in the literature for single-valued SCRs which led our analysis to be held in this framework. However, there are a relatively few considerations of PART for multi-valued SCRs, such as Jimeno et al. (2009), and how our analysis would carry to that framework remains as an open question.

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