

The space of minimal structures

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For a signature L with at least one constant symbol, an L -structure is called minimal if it has no proper substructures. Let \mathcal{S}_L be the set of isomorphism types of minimal L -structures. The elements of \mathcal{S}_L can be identified with ultrafilters of the Boolean algebra of quantifier-free L -sentences, and therefore one can define a Stone topology on \mathcal{S}_L . This topology on \mathcal{S}_L generalizes the topology of the space of n -marked groups. We introduce a natural ultrametric on \mathcal{S}_L , and show that the Stone topology on \mathcal{S}_L coincides with the topology of the ultrametric space \mathcal{S}_L iff the ultrametric space \mathcal{S}_L is compact iff L is locally finite (that is, L contains finitely many n -ary symbols for any $n < \omega$). As one of the applications of compactness of the Stone topology on \mathcal{S}_L , we prove compactness of certain classes of metric spaces in the Gromov-Hausdorff topology. This slightly refines the known result based on Gromov's ideas that any uniformly totally bounded class of compact metric spaces is precompact.

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1 Introduction

In the final remarks in his paper [7], Gromov explained how to deduce from the main result—virtual nilpotency of any finitely generated group of polynomial growth—the following more precise version of the result:

For any positive integers k, d, n , there exists a positive integer m such that any n -generated group, in which for all $r = 1, \dots, m$ the size of the ball of radius r centered at the identity is at most kr^d , has a subgroup of index and nilpotency class at most m .

For a proof of that version, he introduced and used a notion of limit of a sequence of groups with distinguished n generators. Implicitly, he defined a topology on the class of such groups, and used its compactness, as well as closedness of a certain subclass. Van den Dries and Wilkie gave a new proof of the result above by means of model-theoretic compactness theorem instead of Gromov's topological compactness argument [12].

Formalizing Gromov's idea, Grigorchuk suggested a precise definition of the topology used by M. Gromov, and showed that the defined topological space is metrizable, separable, compact, and has a base consisting of clopen sets [6]. That topological space, the *space of n -marked groups*, has been the subject of papers [3–5].

In the present paper we look at the space of marked groups from a model-theoretic point of view, and introduce a more general *space of minimal structures*.

For a signature L containing at least one constant symbol, an L -structure is called *minimal* if it has no proper substructures. For example, any n -marked group is a minimal L -structure, where L is the language of groups with added n constant symbols.

It is easy to show that the isomorphism type of a minimal L -structure is completely determined by its quantifier-free theory (Proposition 2.1), and a set S of quantifier-free L -sentences is the quantifier-free theory of a minimal L -structure iff S is a maximal finitely satisfiable set of quantifier-free L -sentences (Proposition 2.2). The set \mathcal{S}_L of all such S can be equipped with a topology τ , a basis of which consists of the sets $\{S \in \mathcal{S}_L : \varphi \in S\}$, where φ is a quantifier-free L -sentence. The topological space (\mathcal{S}_L, τ) is naturally homeomorphic to the Stone space of the Boolean algebra of quantifier-free L -sentences; therefore it is compact and totally disconnected. Therefore we

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call (\mathcal{S}_L, τ) the Stone space of isomorphism types of minimal L -structures. The space of isomorphism types of n -marked groups is just a clopen set in the Stone space \mathcal{S}_L for a certain L .

We show that the ‘bounded’ version of Gromov’s theorem formulated above can be deduced from its standard version using not model-theoretic compactness theorem as it was done in [12], but only compactness of the Stone space \mathcal{S}_L .

For any universally axiomatizable class \mathcal{K} of L -structures, the set \mathcal{K}_* of isomorphism types of minimal L -structures in \mathcal{K} is closed in \mathcal{S}_L (Proposition 3.5). Let \mathcal{W} be a variety of L -structures and \mathcal{V} its subvariety. We show that \mathcal{V}_* is clopen in \mathcal{W}_* iff the \mathcal{V} -free minimal L -structure is finitely presentable in \mathcal{W} (Proposition 3.7). For example, for any group variety \mathcal{V} , the set of isomorphism types of n -marked \mathcal{V} -groups is clopen in the space of isomorphism types of n -marked groups iff the \mathcal{V} -free group of rank n is finitely presentable.

For an arbitrary set X of minimal L -structures, we characterize in terms of ultraproducts the limit points of X in the Stone topology (Proposition 3.10).

As the Stone space of a Boolean algebra is metrizable iff the Boolean algebra is at most countable, the space (\mathcal{S}_L, τ) is metrizable iff L is at most countable. For an arbitrary L , we define a natural ultrametric on \mathcal{S}_L as follows. For two minimal L -structures M and N , the distance between their quantifier-free theories is defined to be equal to $1/m$, where m is maximal with the property that M and N satisfy the same atomic L -sentences of length at most m . We study the properties of that ultrametric and its relation with the Stone topology on \mathcal{S}_L . We show that the topology of the ultrametric space \mathcal{S}_L is finer or equal than the Stone topology on \mathcal{S}_L ; the two topologies coincide iff the signature L is locally finite. (We call L locally finite if L contains finitely many n -ary symbols for any n .) In particular, the ultrametric space \mathcal{S}_L is compact iff L is locally finite (Theorem 4.4).

As an application of compactness of the Stone space of minimal structures we give a proof of compactness of certain subclasses in the Gromov-Hausdorff space of metric spaces (Theorem 5.1 & Corollary 5.2). This refines the known result based on Gromov’s ideas [2], [8] that any uniformly totally bounded class of compact metric spaces is precompact in the Gromov-Hausdorff topology. For the proof, we associate with every semi-metric space certain relational structures with the same universe called semi-metric structures; the class of such structures is shown to be universally axiomatizable.

For basics of model theory, cf. [9]. The facts and notions of metric geometry we need can be found in [2].

2 Minimal structures

Let L be a signature containing at least one constant symbol; in this case the set \mathcal{T}_L of ground L -terms (that is, the terms without free variables) is not empty. We call an L -structure *minimal* if it has no proper substructures, or, equivalently, is generated by the empty set. Clearly, an L -structure is minimal iff any its element is the value of some ground L -term in the structure. For any L -structure M the substructure generated by the empty set is a unique minimal substructure; we call it the *core of M* and denote by $\text{core}(M)$. We denote the class of all minimal L -structures by \mathcal{M}_L .

Let L_0 be an arbitrary signature, C a nonempty set of constant symbols disjoint with L_0 , and $L = L_0(C)$. Clearly, an L -structure M is minimal if and only if the set $\{c^M : c \in C\}$ generates its L_0 -reduct M_0 . Thus, any structure becomes minimal after naming its generators. We call minimal L -structures *C -marked L_0 -structures*. For any L -structure M its core is a C -marked L_0 -structure—it is the minimal substructure generated by the set $\{c^M : c \in C\}$.

The notion of marked structure generalizes the notion of marked group [4], which is defined to be a group with distinguished generators (not necessarily all distinct). In this case $L_0 = \{\cdot, ^{-1}, e\}$, and C consists of names of generators of the group. Note that here we do not assume that the group is finitely generated, and C is finite. If C is finite, $|C| = n$, then C -marked groups are called n -marked groups.

Let QF_L be the set of all quantifier-free L -sentences. For an L -structure M we denote by $\text{qf}(M)$ the quantifier-free theory of M , that is, the set of sentences in QF_L that hold in M , and by $\text{at}(M)$ the set consisting of all atomic or negated atomic L -sentences from $\text{qf}(M)$. We shall need the following essentially known facts:

Proposition 2.1 *For minimal L -structures M and N the following are equivalent:*

1. $M \simeq N$;
2. $\text{qf}(M) = \text{qf}(N)$;
3. $\text{at}(M) = \text{at}(N)$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) is obvious. If (3) then the map $t^M \mapsto t^N$ is a well-defined isomorphism from M onto N , and so (1). \square

Due to this fact, we call $\text{qf}(M)$ the *isomorphism type* of a minimal L -structure M .

Proposition 2.2 For $S \subseteq \text{QF}_L$ the following are equivalent:

1. $S = \text{qf}(M)$, for some minimal L -structure M ,
2. S is a maximal finitely satisfiable subset of QF_L ;
3. S is finitely satisfiable, and for any $\varphi \in \text{QF}_L$ either $\varphi \in S$ or $\neg\varphi \in S$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) is easy; we prove (3) \Rightarrow (1). By (3), $t = s \in S$ is an equivalence relation on \mathcal{T}_L . Denote by $[t]$ the equivalence class of $t \in \mathcal{T}_L$. Let M be the L -structure whose universe is $\{[t] : t \in \mathcal{T}_L\}$, and $f^M([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)]R^M = \{([t_1], \dots, [t_n]) : R(t_1, \dots, t_n) \in S\}$, for any function L -symbol f and relation L -symbol R of arity n . Due to (3), f^M and R^M are well-defined. By induction, $t^M = [t]$, for any $t \in \mathcal{T}_L$. Then $t = s \in S$ iff $t^M = s^M$, for any $t, s \in \mathcal{T}_L$. Using (3), it is easy to show by induction that $\varphi \in S$ iff $M \models \varphi$, for any $\varphi \in \text{QF}_L$. Thus $S = \text{qf}(M)$. Since $[t]$ is t^M for any $t \in \mathcal{T}_L$, the structure M is minimal. \square

Since, by Zorn's lemma, any finitely satisfiable subset of QF_L can be completed to a maximal such subset, we have:

Corollary 2.3 ((Herbrand's theorem)) Any finitely satisfiable subset of QF_L has a minimal model.

Herbrand's theorem is a weak version of model-theoretic compactness theorem. This version admits a simple proof given above, and in the present paper we need only this version of compactness theorem.

Denote by \mathcal{S}_L the set of all maximal finitely satisfiable subsets of QF_L . By Proposition 2.2, this is the set of isomorphism types of minimal L -structures.

3 The Stone space of minimal structures

3.1 Topology on \mathcal{S}_L

It is easy to see that, for any $S \in \mathcal{S}_L$ and $\varphi, \psi \in \text{QF}_L$,

1. $\varphi \wedge \psi \in S$ iff $\varphi \in S$ and $\psi \in S$;
2. $\varphi \vee \psi \in S$ iff $\varphi \in S$ or $\psi \in S$;
3. $\neg\varphi \in S$ iff $\varphi \notin S$.

In other words, if $U_\varphi = \{S \in \mathcal{S}_L : \varphi \in S\}$, we have

- (1) $U_{\varphi \wedge \psi} = U_\varphi \cap U_\psi$;
- (2) $U_{\varphi \vee \psi} = U_\varphi \cup U_\psi$;
- (3) $U_{\neg\varphi} = U_\varphi^c$.

Due to (1), $\{U_\varphi : \varphi \in \text{QF}_L\}$ is a basis of a topology on \mathcal{S}_L ; we denote the topology by τ . Due to (3), the sets U_φ are clopen in τ . It is easy to show that $U_\varphi = U_\psi$ iff φ and ψ are equivalent.

Let T be the set of finite conjunctions of atomic or negated atomic L -sentences. Since any $\varphi \in \text{QF}_L$ is equivalent to a finite disjunction of sentences from T then, due to (2), $\{U_\varphi : \varphi \in T\}$ is a basis of τ as well.

Proposition 3.1 The topological space (\mathcal{S}_L, τ) is totally disconnected and compact.

Proof. (totally disconnected) Let S and P be different elements of \mathcal{S}_L . Let, say, $\varphi \in S$ and $\varphi \notin P$. Then $S \in U_\varphi$, and $P \in U_\varphi^c$. Since $U_\varphi^c = U_{\neg\varphi}$, by (3), both U_φ and U_φ^c are open, and the result follows.

(compact) Suppose $\{U_\varphi : \varphi \in T\}$ covers \mathcal{S}_L , where $T \subseteq \text{QF}_L$. Then $\bigcap_{\varphi \in T} U_{\neg\varphi} = \emptyset$, that is, there is no $S \in \mathcal{S}_L$ with $\{\neg\varphi : \varphi \in T\} \subseteq S$. Then, for some finite $F \subseteq T$, there is no $S \in \mathcal{S}_L$ with $\{\neg\varphi : \varphi \in F\} \subseteq S$; otherwise

$\{\neg\varphi : \varphi \in T\}$ would be finitely satisfiable, and so could be completed to a member of \mathcal{S}_L , by Zorn's lemma. Hence $\bigcap_{\varphi \in F} U_{\neg\varphi} = \emptyset$, and so $\{U_\varphi : \varphi \in F\}$ covers \mathcal{S}_L . \square

The proof of compactness of the topology τ did not use the model-theoretic compactness theorem (nor its weaker version, Herbrand's theorem).

Proposition 3.2 *Any set clopen in τ is U_φ , for some $\varphi \in \text{QF}_L$.*

Proof. Any set U open in τ is $\bigcup_{\varphi \in T} U_\varphi$, for some $T \subseteq \text{QF}_L$. If U is closed, it is compact, by Proposition 3.1(ii), and hence $U = \bigcup_{\varphi \in F} U_\varphi$, for some finite $F \subseteq T$. By (2), $U = U_\psi$, where $\psi = \bigvee_{\varphi \in F} \varphi$. \square

For $\varphi \in \text{QF}_L$, denote by $[\varphi]$ the set of all $\psi \in \text{QF}_L$ equivalent to φ . The sets $[\varphi]$ form a Boolean algebra with the operations induced by the logical operators \wedge , \vee , and \neg . We denote that Boolean algebra by \mathcal{QF}_L , and its Stone space by $\text{St}(\mathcal{QF}_L)$.

Recall that for a Boolean algebra \mathcal{B} its Stone space $\text{St}(\mathcal{B})$ is defined to be the topological space whose points are ultrafilters of \mathcal{B} , and a basis of topology is $\{U_b : b \in \mathcal{B}\}$, where

$$U_b = \{p : p \text{ is an ultrafilter of } \mathcal{B} \text{ with } b \in p\}.$$

It is known that $\text{St}(\mathcal{B})$ is compact and totally disconnected [10, Section 8]; it is metrizable iff it has a countable basis iff $|\mathcal{B}| \leq \aleph_0$; its clopen sets are exactly the sets U_b . Any closed subspace X of $\text{St}(\mathcal{B})$ is a compact, totally disconnected space; its clopen sets are exactly the sets $U_b \cap X$, and they form a basis of X .

For $T \subseteq \text{QF}_L$ denote $\{[\varphi] : \varphi \in T\}$ by $[T]$. It is not difficult to show that $S \mapsto [S]$ is a bijection between \mathcal{S}_L and the set of ultrafilters of \mathcal{QF}_L . Moreover, $[U_\varphi] = U_{[\varphi]}$, for any $\varphi \in \text{QF}_L$. Therefore $S \mapsto [S]$ is a natural homeomorphism between the topological space (\mathcal{S}_L, τ) and the Stone space $\text{St}(\mathcal{QF}_L)$. Because of that, we call τ the *Stone topology on \mathcal{S}_L* . Since \mathcal{S}_L is the set of isomorphism types of minimal L -structures, we call the topological space (\mathcal{S}_L, τ) the *Stone space of isomorphism types of minimal L -structures*, or, for short, the Stone space \mathcal{S}_L .

As $|\mathcal{QF}_L| \leq \aleph_0$ iff $|L| \leq \aleph_0$, the Stone space $\text{St}(\mathcal{QF}_L)$ is metrizable iff $|L| \leq \aleph_0$. Since any compact metric space is separable, $\text{St}(\mathcal{QF}_L)$ is separable if $|L| \leq \aleph_0$. Thus, the Stone space of minimal L -structures is metrizable and separable if $|L| \leq \aleph_0$.

For an L -sentence φ denote by $\text{Mod}_{\mathcal{M}_L}(\varphi)$ the class of minimal models of φ , and by $\text{Mod}_{\mathcal{S}_L}(\varphi)$ the set of isomorphism types of minimal models of φ . In other words,

$$\begin{aligned} \text{Mod}_{\mathcal{M}_L}(\varphi) &= \{M \in \mathcal{M}_L : M \models \varphi\}, \\ \text{Mod}_{\mathcal{S}_L}(\varphi) &= \{\text{qf}(M) : M \in \mathcal{M}_L, M \models \varphi\}. \end{aligned}$$

Clearly, for $\varphi \in \text{QF}_L$,

$$\text{Mod}_{\mathcal{S}_L}(\varphi) = U_\varphi.$$

Thus for any $\varphi \in \text{QF}_L$ the set $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is a clopen subspace of the Stone space \mathcal{S}_L .

Proposition 3.3 *If φ is an existential L -sentence then $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is open.*

Proof. Let φ be $\exists v_1 \dots v_n \psi(v_1, \dots, v_n)$, where ψ is quantifier-free. Clearly, φ holds in a minimal L -structure M iff $M \models \psi(t_1, \dots, t_n)$ for some $t_1, \dots, t_n \in \mathcal{T}_L$. Therefore $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is the union of all sets $\text{Mod}_{\mathcal{S}_L}(\psi(t_1, \dots, t_n))$, where $t_1, \dots, t_n \in \mathcal{T}_L$. Since all $\text{Mod}_{\mathcal{S}_L}(\psi(t_1, \dots, t_n))$ are clopen, $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is open. \square

Proposition 3.4 *If φ is a universal L -sentence then $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is closed.*

Proof. The sentence φ is equivalent to $\neg\vartheta$ for some existential L -sentence ϑ . Then the complement of $\text{Mod}_{\mathcal{S}_L}(\varphi)$ in \mathcal{S}_L is the set $\text{Mod}_{\mathcal{S}_L}(\vartheta)$, which is open by Proposition 3.3. \square

For an L -theory T , denote by $\text{Mod}_{\mathcal{S}_L}(T)$ the set of isomorphism types of minimal models of T .

Proposition 3.5 *If T is a universal L -theory then $\text{Mod}_{\mathcal{S}_L}(T)$ is closed.*

Proof. Since $\text{Mod}_{\mathcal{S}_L}(T) = \bigcap_{\varphi \in T} \text{Mod}_{\mathcal{S}_L}(\varphi)$, this follows from Proposition 3.4. \square

Similarly to the Stone topology on \mathcal{S}_L , one can define a topology on the class \mathcal{M}_L a basis of which consists of the classes $\text{Mod}_{\mathcal{M}_L}(\varphi)$, where $\varphi \in \text{QF}_L$. We call that topology the *Stone topology on \mathcal{M}_L* . The class \mathcal{M}_L equipped with that topology is called the *Stone space of minimal L -structures*, or, for short, the Stone space \mathcal{M}_L . Obviously, analogues of Propositions 3.1–3.5 hold for it, with one exception: the Stone space \mathcal{M}_L is not Hausdorff (and so not totally disconnected), because any isomorphic but different members of \mathcal{M}_L cannot be separated by open sets. Note that compactness of the Stone space \mathcal{M}_L is based on Herbrand's theorem.

If $L = L_0(C)$, we call the Stone space \mathcal{S}_L the *Stone space of isomorphism types of C -marked L_0 -structures*. Let $L_0 = \{\cdot, {}^{-1}, e\}$, and γ be the universal L_0 -sentence that axiomatizes the class of groups. Then $\text{Mod}_{\mathcal{S}_L}(\gamma)$ is a closed subspace of the Stone space \mathcal{S}_L , by Proposition 3.5. Its points are isomorphism types of groups with generators marked by elements of C . We call this topological space the *space of isomorphism types of C -marked groups* and denote it by \mathcal{G}_C . The space \mathcal{G}_C is compact and totally disconnected.

For $\psi \in \text{QF}_L$, the set $U_\psi \cap \mathcal{G}_C$ is the set of isomorphism types of C -marked groups satisfying ψ ; it is clopen in \mathcal{G}_C . Any clopen set in \mathcal{G}_C is of that form, and the sets $U_\psi \cap \mathcal{G}_C$ form a basis of \mathcal{G}_C . Moreover, for the set Ψ of finite conjunctions of L -sentences of the form $w = e$ or $w \neq e$, where w is a group word over C , the set $\{U_\psi \cap \mathcal{G}_C : \psi \in \Psi\}$ is a basis of the space \mathcal{G}_C .

For a finite set of constant symbols C with $|C| = n$, the space \mathcal{G}_C is exactly the space of isomorphism types of n -marked groups introduced in [6]; we denote it by \mathcal{G}_n .

Proposition 3.6 *The set \mathcal{G}_C is clopen in \mathcal{S}_L iff C is finite.*

Proof. Suppose C is finite. Let Θ be the set of quantifier-free L -sentences

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad c \cdot e = e \cdot c = c, \quad c \cdot c^{-1} = c^{-1} \cdot c = e$$

for all constant symbols a, b, c in C . Clearly, Θ is finite. It is easy to show that $\mathcal{G}_C = \text{Mod}_{\mathcal{S}_L}(\Theta) = U_\vartheta$, where $\vartheta = \bigwedge \Theta$. Therefore \mathcal{G}_C is clopen.

Now we show that if C is infinite then \mathcal{G}_C is not clopen. Suppose not, and $\mathcal{G}_C = U_\vartheta$, where $\vartheta \in \text{QF}_L$. Let C^* be the finite set of all $c \in C$ that occurs in ϑ . Consider any C^* -marked group M^* . It is easy to construct a minimal L -structure N such that M^* is a substructure of its $L_0(C^*)$ -reduct, and the L_0 -reduct of N is not a group. Since any L -expansion of M^* belongs to \mathcal{G}_C , we have $M^* \models \vartheta$. Therefore $N \models \vartheta$, and hence $\text{qf}(N) \in \mathcal{G}_C$. Contradiction. \square

A special case of Proposition 3.4 was proven in [4, Section 5.2]: *for any universal sentence ϑ in the group language, the set of isomorphism types of n -marked groups satisfying ϑ is closed in \mathcal{G}_n* . This fact is slightly weaker than Proposition 3.4: for example, it does not imply closedness of the set \mathcal{K} of isomorphism types of n -marked centerless groups, because the class of centerless groups is not closed under subgroups and therefore is not universally axiomatizable. However, Proposition 3.4 implies that \mathcal{K} is closed in \mathcal{G}_n , because for any finite C the class of C -marked centerless groups is axiomatizable by the universal sentence

$$\forall v \left(\bigwedge_{c \in C} [v, c] = 1 \rightarrow v = 1 \right).$$

Note that \mathcal{K} is not open in \mathcal{G}_n if $n = |C| > 1$. Indeed, let G be a free group of rank n , and N_k a free k -nilpotent group of rank n ; then $N_k \simeq G/G_k$, where G_k is the k -th member of the lower central series of G . Consider G and N_k as groups with marked free generators. Then G is a limit of the sequence N_1, N_2, \dots ; this follows from the well-known fact that $\bigcap_{k=1}^{\infty} G_k = 1$. But G is centerless, and all N_k are not.

For a variety \mathcal{V} of L -structures, we call a \mathcal{V} -free structure generated by the empty set a \mathcal{V} -free minimal structure. Denote by \mathcal{V}_* the set of isomorphism types of minimal L -structures from \mathcal{V} .

Proposition 3.7 *Let \mathcal{V} and \mathcal{W} be varieties of L -structures, and $\mathcal{V} \subseteq \mathcal{W}$. The following are equivalent:*

1. \mathcal{V}_* is clopen in \mathcal{W}_* ;
2. the \mathcal{V} -free minimal structure N is finitely presentable in \mathcal{W} .

Proof. (2) \Rightarrow (1). Suppose N is finitely presented in \mathcal{W} by atomic L -sentences $\varphi_1, \dots, \varphi_n$. Then

$$\mathcal{V}_\star = \text{Mod}_{S_L}(\varphi) \cap \mathcal{W}_\star,$$

where $\varphi = \bigwedge_i \varphi_i$. So \mathcal{V}_\star is clopen in \mathcal{W}_\star .

(1) \Rightarrow (2). Suppose \mathcal{V}_\star is clopen in \mathcal{W}_\star . Since \mathcal{W}_\star is closed by Proposition 3.5, $\mathcal{V}_\star = \text{Mod}_{S_L}(\varphi) \cap \mathcal{W}_\star$, for some $\varphi \in \text{QF}_L$. We may assume that φ is a finite disjunction of sentences of the form

$$\varphi_1 \wedge \dots \wedge \varphi_n \wedge \neg\psi_1 \wedge \dots \wedge \neg\psi_k,$$

where all φ_i, ψ_j are atomic L -sentences. Then one of these disjuncts—say, the disjunct written above—holds in N . Let M be the minimal L -structure presented in \mathcal{W} by the relations $\varphi_1, \dots, \varphi_n$. Then there is a homomorphism from M onto N . Hence all $\neg\psi_i$ hold in M . Therefore φ holds in M , and so $M \in \mathcal{V}$. Since N is \mathcal{V} -free there is a homomorphism from N onto M . Hence this homomorphism is an isomorphism. Thus N is finitely presented in \mathcal{W} . \square

Corollary 3.8 *Let \mathcal{V} be a group variety, and $n \geq 1$. Then the set of isomorphism types of n -marked \mathcal{V} -groups is clopen in \mathcal{G}_n iff the \mathcal{V} -free group of rank n is finitely presented.*

For example, if \mathcal{V} is any nilpotent group variety then the class of n -marked \mathcal{V} -groups is clopen in \mathcal{G}_n . Since, by Šmel'kin's theorem [11], for $n, m \geq 2$ the n -generated free m -solvable group is not finitely presented (for a more general result, cf. [1]), the class of n -marked m -solvable groups is not open in \mathcal{G}_n . The latter fact was explained in [4, Section 2.6] in a completely different way based on some D. V. Osin's result. Note that there is an open question posed by A. Yu. Olshanski whether any finitely presented relatively free group is virtually nilpotent.

3.2 Compactness of \mathcal{G}_n and Gromov's theorem

Now we explain how one can use compactness of \mathcal{G}_n to deduce from Gromov's theorem its 'bounded' version formulated at the beginning of the present paper.

Fix n, k , and d . Let $L_0 = \{\cdot, ^{-1}, e\}$, and $L = L_0(C)$, where $C = \{c_1, \dots, c_n\}$.

It is easy to construct $\sigma_m \in \text{QF}_L$ which says about a C -marked group that for all $r = 1, \dots, m$ the size of the ball of radius r centered at the identity is $\leq kr^d$. Also, there is $\tau_m \in \text{QF}_L$ which says about a C -marked group that it has a nilpotent subgroup of class $\leq m$ and index $\leq m$. We explain how to construct such τ_m using the idea suggested in [12, Section 7].

Let F be the free group with basis C . It is known that F has only finitely many subgroups of index $\leq m$, say, H_1, \dots, H_s ; each of them is finitely generated. Let X_i be a finite set of group words over C generating H_i , and ψ_i be the conjunction of all equalities $[v_1, \dots, v_m] = 1$, where $v_1, \dots, v_m \in X_i$. We show that $\bigvee_{i=1}^s \psi_i$ can be taken as τ_m .

Let M be a C -marked group; then there is a unique epimorphism $\rho : F \rightarrow M$ of C -marked groups. Suppose M has a nilpotent subgroup N of class $\leq m$ and index $\leq m$. Then $|F : \rho^{-1}(N)| \leq m$, and hence $\rho^{-1}(N) = H_i$ for some i . Since N is nilpotent of class $\leq m$, we have $M \models \psi_i$, and so $M \models \tau_m$. Now suppose $M \models \tau_m$; then $M \models \psi_i$ for some i . Therefore the subgroup $\rho(H_i)$ of M is nilpotent of class $\leq m$ and index $\leq m$. Indeed, since $\rho(X_i)$ generates $\rho(H_i)$, and $[u_1, \dots, u_m] = 1$ for all $u_1, \dots, u_m \in \rho(X_i)$, the subgroup $\rho(H_i)$ is nilpotent of class $\leq m$. This subgroup is of index $\leq m$ in M because $|F : H_i| \leq m$.

Let φ_m denote $\sigma_m \rightarrow \tau_m$. It is easy to see that if $m < l$ then φ_m implies φ_l . Every C -marked group M satisfies φ_m for some m (possibly, depending on M). Indeed, if M is virtually nilpotent then $M \models \tau_m$ for some m ; if M is not virtually nilpotent then, by Gromov's theorem, M is not of polynomial growth, and therefore $M \not\models \sigma_m$, for some m .

Let \mathcal{K}_m denote $U_{\varphi_m} \cap \mathcal{G}_C$. Then \mathcal{K}_m is clopen in \mathcal{G}_C . Thus $\{\mathcal{K}_m : m \geq 1\}$ is an open cover of \mathcal{G}_C . It has a finite subcover because \mathcal{G}_C is compact. Since $\mathcal{K}_m \subseteq \mathcal{K}_{m+1}$ for all m , we have $\mathcal{G}_C = \mathcal{K}_m$ for some m . Thus there is m such that every C -marked group satisfies $\sigma_m \rightarrow \tau_m$, and the result follows.

Note that the proof above is based on compactness of the Stone space of C -marked groups, which follows from a general fact on compactness of Stone spaces of Boolean algebras. For a proof of the latter fact one needs only Zorn's lemma but not model-theoretic compactness theorem. The proof of the result given in [12, Section 7] is based on model-theoretic compactness theorem; so our proof is different, even though uses the same idea.

Another way to realize that idea is to use ultraproducts. Towards a contradiction, suppose for every i there is a C -marked group M_i with $M_i \models \neg\varphi_i$. If $j > i$ then $M_j \models \neg\varphi_i$ because φ_i implies φ_j . Then, by Łoś's theorem, for any non-principal ultraproduct M of the C -marked groups M_i we have $M \models \neg\varphi_j$, for all j . Then all φ_j fail in the C -marked group $\text{core}(M)$, contrary to Gromov's theorem.

3.3 Topology on \mathcal{M}_L and ultraproducts

In general, there is a link between ultraproducts and the Stone topology on the class of minimal structures (cf. [4, Proposition 6.4], where a link between ultraproducts and convergence of groups in the space of marked groups had been demonstrated).

Proposition 3.9 *Let X be a subset of \mathcal{M}_L , and $M \in \mathcal{M}_L$. Then the following are equivalent:*

1. M belongs to the closure of X in the Stone space \mathcal{M}_L ;
2. M is isomorphic to the core of an ultraproduct of structures from X ;
3. M is embeddable into an ultraproduct of structures from X .

Proof. Obviously, (2) \Rightarrow (3).

(3) \Rightarrow (1). Suppose M is embeddable into an ultraproduct $\prod_{i \in I} M_i/D$ of structures from X . We show that any basic neighbourhood $\text{Mod}_{\mathcal{M}_L}(\varphi)$ of M , where $\varphi \in \text{QF}_L$, contains an element of X . Since φ is quantifier-free and holds in M , it holds in the ultraproduct. Therefore, by Łoś's theorem,

$$I_\varphi = \{i \in I : M_i \models \varphi\} \in D.$$

Hence $I_\varphi \neq \emptyset$, and so $\text{Mod}_{\mathcal{M}_L}(\varphi)$ contains an element of X .

(1) \Rightarrow (2). Let $X = \{M_i : i \in I\}$. For $\varphi \in \text{qf}(M)$ denote

$$I_\varphi = \{i \in I : M_i \models \varphi\};$$

then $I_\varphi \neq \emptyset$, because $X \cap U_\varphi \neq \emptyset$, by (1). The set

$$P = \{I_\varphi : \varphi \in \text{qf}(M)\}$$

is closed under finite intersections, because if $\varphi_1, \dots, \varphi_n \in \text{qf}(M)$ then

$$I_{\varphi_1} \cap \dots \cap I_{\varphi_n} = I_{\varphi_1 \wedge \dots \wedge \varphi_n}, \quad \text{and} \quad \varphi_1 \wedge \dots \wedge \varphi_n \in \text{qf}(M).$$

Therefore P has the finite intersection property, and hence can be completed to an ultrafilter D on I . For any $\varphi \in \text{qf}(M)$ we have $I_\varphi \in D$, and hence $\prod_{i \in I} M_i/D \models \varphi$, by Łoś's theorem. It follows that any $\varphi \in \text{qf}(M)$ holds in the core of the ultraproduct. Therefore M is isomorphic to the core, by Proposition 2.1. \square

A point M of the Stone space \mathcal{M}_L is called a *limit point* of a subset X of \mathcal{M}_L if every open neighbourhood of M in \mathcal{M}_L contains a member of X which is non-isomorphic to M .

Proposition 3.10 *Let X be a subset of \mathcal{M}_L , and M a structure in \mathcal{M}_L , which is non-isomorphic to any member of X . Then the following are equivalent:*

1. M is a limit point of X in the Stone space \mathcal{M}_L ;
2. M is isomorphic to the core of a non-principal ultraproduct of pairwise non-isomorphic structures from X ;
3. M is embeddable into a non-principal ultraproduct of pairwise non-isomorphic structures from X .

Proof. Obviously, (2) \Rightarrow (3).

(3) \Rightarrow (1). Suppose M is embeddable into $\prod_{i \in I} M_i/D$, where $\{M_i : i \in I\}$ is a family of pairwise non-isomorphic structures from X , and D is a non-principal ultrafilter on I . We need to show that any basic neighbourhood $\text{Mod}_{\mathcal{M}_L}(\varphi)$ of M contains an element of X non-isomorphic to M . Since φ is quantifier-free and holds in M , it holds in the ultraproduct. Therefore, by Łoś's theorem,

$$I_\varphi = \{i \in I : M_i \models \varphi\} \in D.$$

Since the ultrafilter D is non-principal, $|I_\varphi| > 1$. Since all M_i are pairwise non-isomorphic, there is $j \in I_\varphi$ such that M_j is not isomorphic to M . Then $M_j \in X$, and $M_j \in \text{Mod}_{\mathcal{M}_L}(\varphi)$.

(1) \Rightarrow (2). Let $\{M_i : i \in I\}$ be a family of representatives of all isomorphism types of structures in X , which are not isomorphic to M . For any $\varphi \in \text{qf}(M)$, the set

$$I_\varphi = \{i \in I : M_i \models \varphi\}$$

is infinite. Indeed, suppose not. By Proposition 2.1, for each i there is $\vartheta_i \in \text{qf}(M)$ such that $M_i \not\models \vartheta_i$. Since M is a limit point of X , there is $N \in X$ which is non-isomorphic to M and such that $N \models \bigwedge_i \vartheta_i$. Then none of M_i is isomorphic to N . Contradiction. The set

$$P = \{I_\varphi : \varphi \in \text{qf}(M)\}$$

is closed under finite intersections, as in the proof of (1) \Rightarrow (2) at Proposition 3.9. Let F be the Fréchet filter on I . The set $P \cup F$ has the finite intersection property: otherwise, for some $\varphi \in \text{qf}(M)$ the set I_φ is disjoint with a set from F , and hence is finite. Hence $P \cup F$ is contained in an ultrafilter D on I . The ultrafilter D is non-principal because it contains F . For any $\varphi \in \text{qf}(M)$ we have $I_\varphi \in D$, and therefore $\prod_{i \in I} M_i / D \models \varphi$, by Łoś's theorem. It follows that any $\varphi \in \text{qf}(M)$ holds in the core of the ultraproduct. Therefore M is isomorphic to the core, by Proposition 2.1. \square

4 The ultrametric space of minimal structures

For $m \geq 1$, we say that L -structures M and N are m -close if

$$M \models \vartheta \Leftrightarrow N \models \vartheta,$$

for any atomic L -sentence ϑ of length $\leq m$.

Note that minimal L -structures M and N are m -close for arbitrary large m iff $\text{at}(M) = \text{at}(N)$ iff $M \simeq N$, by Proposition 2.1.

For minimal L -structures M and N we define $d(M, N)$, the distance between M and N , as follows. If $M \simeq N$, put $d(M, N) = 0$. Otherwise $d(M, N)$ is defined to be $1/m$, where m be the maximal positive integer such that M and N are m -close.

It is easy to see that d is *semi-ultrametric* on \mathcal{M}_L , that is, for any $M, N, Q \in \mathcal{M}_L$

1. $d(M, N) \geq 0$, and $d(M, M) = 0$;
2. $d(M, N) = d(N, M)$;
3. $d(M, P) \leq \max\{d(M, N), d(N, Q)\}$.

Since $d(M, N) = 0$ iff $M \simeq N$, the semi-ultrametric d induces an ultrametric on the set of isomorphism types of minimal L -structures, that is, on \mathcal{S}_L . We denote the induced ultrametric on \mathcal{S}_L by the same letter d ; so for any $S, P \in \mathcal{S}_L$, we have $d(S, P) = d(M, N)$, where $S = \text{qf}(M)$ and $P = \text{qf}(N)$.

Clearly, for any $S, P \in \mathcal{S}_L$, we have

- (i) $d(S, P) \in \{1/m : m \geq 1\} \cup \{0\}$, and
- (ii) $d(S, P) \leq 1/m$ means that $\vartheta \in S$ iff $\vartheta \in P$, for any atomic L -sentence ϑ of length $\leq m$.

We call (\mathcal{M}_L, d) and (\mathcal{S}_L, d) the *semi-ultrametric* and *ultrametric space of minimal L -structures*, respectively.

Clearly, in \mathcal{M}_L and \mathcal{S}_L for any point x the open ball $B(x, \varepsilon)$ is the whole space if $\varepsilon > 1$. If for a positive integer m

$$1/(m+1) < \varepsilon \leq 1/m,$$

then the open ball $B(x, \varepsilon)$ is equal to the closed ball $\bar{B}(x, 1/m)$. Thus in the spaces \mathcal{M}_L and \mathcal{S}_L any open ball is a closed set. It follows that *the metric space \mathcal{S}_L is totally disconnected*.

Proposition 4.1 *For any $\varphi \in \text{QF}_L$, the set $\text{Mod}_{\mathcal{S}_L}(\varphi)$ is clopen in the ultrametric space \mathcal{S}_L .*

Proof. Since a boolean combination of clopen sets is clopen, we may assume that φ is atomic. Let m be the length of φ . Denote $\text{Mod}_{S_L}(\varphi)$ by U . For any $S \in \mathcal{M}_L$, if $\varphi \in S$ then $B(S, 1/m) \subseteq U$, and if $\varphi \notin S$ then $B(S, 1/m) \subseteq U^c$. So U is clopen. \square

Since $\{\text{Mod}_{S_L}(\varphi) : \varphi \in \text{QF}_L\}$ is a basis of the Stone topology on S_L , we have

Corollary 4.2 *The ultrametric topology is equal to or finer than the Stone topology on S_L .*

In general, the two topologies do not coincide: in the Stone space S_L the clopen sets are exactly $\text{Mod}_{S_L}(\varphi)$, where $\varphi \in \text{QF}_L$, but in the ultrametric space S_L it is not always so. For example, in Proposition 3.6 we proved that if C is infinite then $\mathcal{G}_C \neq \text{Mod}_{S_L}(\varphi)$, for any $\varphi \in \text{QF}_L$. However,

Proposition 4.3 *The set \mathcal{G}_C is clopen in the ultrametric space S_L , for any C .*

Proof. Let Θ be defined as in the proof of Proposition 3.6; then

$$\mathcal{G}_C = \text{Mod}_{S_L}(\Theta) = \bigcap \{\text{Mod}_{S_L}(\vartheta) : \vartheta \in \Theta\}.$$

Since all $\text{Mod}_{S_L}(\vartheta)$ are clopen, \mathcal{G}_C is closed. Also, \mathcal{G}_C is open because if $S \in \mathcal{G}_C$, and m is the maximal length of sentences in Θ , then $B(S, 1/m) \subseteq \mathcal{G}_C$. \square

We call a signature L *locally finite* if for every n the set of n -ary symbols in L is finite. Clearly, any locally finite signature is finite or countable.

Theorem 4.4 *The following are equivalent:*

1. any clopen set in the ultrametric space S_L is $\text{Mod}_{S_L}(\varphi)$ for some $\varphi \in \text{QF}_L$;
2. any open ball in the ultrametric space S_L is $\text{Mod}_{S_L}(\varphi)$ for some $\varphi \in \text{QF}_L$;
3. the Stone and ultrametric topologies on S_L coincide;
4. the ultrametric space S_L is compact;
5. the ultrametric space S_L is separable;
6. the signature L is locally finite.

Proof. We prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (2) and (2) \wedge (4) \Rightarrow (1).

(1) \Rightarrow (2) because every open ball in S_L is closed.

(2) \Rightarrow (3). Due to (2), every open set in the ultrametric space S_L is open in the Stone topology on S_L . Together with Corollary 4.2, this gives (3).

(3) \Rightarrow (4) because the Stone space S_L is compact, which is a consequence of Herbrand's theorem.

(4) \Rightarrow (5) because any compact metric space is separable.

(5) \Rightarrow (6). Suppose there are infinitely many n -ary symbols in L . We show that S_L is not separable.

First we show that there is a family $\{\vartheta_i : i < \omega\}$ of atomic L -sentences of the same length m such that for any $I \subseteq \omega$ the set of sentences

$$\Theta_I = \{\vartheta_i : i \in I\} \cup \{\neg\vartheta_i : i \notin I\}$$

holds in some minimal L -structure N_I .

Let c be a constant symbol in L . If L contains infinitely many distinct and different from c constant symbols c_0, c_1, \dots , one can take the sentence $c_i = c$ as ϑ_i . If L contains infinitely many distinct n -ary function symbols f_0, f_1, \dots , where $n \geq 1$, one can take the sentence $f_i(c, \dots, c) = c$ as ϑ_i . If L contains infinitely many distinct n -ary relation symbols P_0, P_1, \dots one can take the sentence $P_i(c, \dots, c)$ as ϑ_i . Clearly, for such choice of ϑ_i the set Θ_I holds in some L -structure, and hence in its core N_I .

We prove that no countable subset is dense in S_L . To show that, we construct for any sequence $(M_i : i < \omega)$ in \mathcal{M}_L a member of \mathcal{M}_L which is not m -close to M_i for every $i < \omega$. Let

$$I = \{i < \omega : M_i \not\models \vartheta_i\}.$$

Then for any i the structure N_I is not m -close to M_i . Indeed, if $i \in I$ then $N_I \models \vartheta_i$ and $M_i \not\models \vartheta_i$; if $i \notin I$ then $N_I \not\models \vartheta_i$ and $M_i \models \vartheta_i$.

(6) \Rightarrow (2). Let $S \in \mathcal{S}_L$, and $m \geq 1$. We show that $B(S, 1/m) = \text{Mod}_{\mathcal{S}_L}(\varphi)$ for some $\varphi \in \text{QF}_L$. Since L is locally finite, the set of atomic L -sentences of length $\leq m + 1$ is finite. Let φ be the conjunction of all sentences from $\text{at}(M)$ of length $\leq m + 1$. Then $P \in \text{Mod}_{\mathcal{S}_L}(\varphi)$ means exactly that P and S are $(m + 1)$ -close, that is, $P \in B(S, 1/m)$.

(2) \wedge (4) \Rightarrow (1). Let U be a clopen set in the ultrametric space \mathcal{S}_L . Since U is closed, it is compact, by (4). Since U is open, it is a union of open balls, and so a union of finitely many open balls B_i , by compactness of U . By (2), each B_i is $\text{Mod}_{\mathcal{S}_L}(\varphi_i)$, for some $\varphi_i \in \text{QF}_L$. Then $U = \text{Mod}_{\mathcal{S}_L}(\varphi)$, where $\varphi = \bigvee_i \varphi_i$. \square

Corollary 4.5 *If L is locally finite then all subspaces of the ultrametric space \mathcal{S}_L are separable.*

Proof. For metric spaces separability is equivalent to existence of a countable base, which is a hereditary property. \square

5 Gromov-Hausdorff spaces and compactness

5.1 Gromov-Hausdorff distance

First we recall some notions and facts of metric geometry [2, Chapter 7]. We already used above the notion of semi-metric; we shall need a bit more general definition of semi-metric, in which distances between points can be infinite.

A map $d : X \times X \rightarrow \mathbb{R} \cup \{\infty\}$ is called a *semi-metric* on X if d is nonnegative, symmetric, satisfies the triangle inequality, and $d(x, x) = 0$ for every $x \in X$. A semi-metric is called a *metric* if $d(x, y) > 0$ for different $x, y \in X$.

A set (or, more generally, a class) equipped with a (semi-)metric is said to be a *(semi-)metric space*. Usually, the set and the space are denoted with the same letter, and the (semi-)metric of the space X is denoted by d_X .

Like a metric, any semi-metric d on X defines a topology on X in a usual way; this topology is Hausdorff iff d is a metric.

We shall use the following easy observations. Let X and Y be semi-metric spaces, and $f : X \rightarrow Y$ be surjective and distance-preserving. Then if A is a compact subset of X then $f(A)$ is a compact subset of Y , and if B is a compact subset of Y then $f^{-1}(B)$ is a compact subset of X .

For a semi-metric d on X , the relation $d(x, y) = 0$ is an equivalence relation on X . Denote by $[x]$ the equivalence class of $x \in X$, and by X/d the set of all equivalence classes. Clearly, $([x], [y]) \mapsto d(x, y)$ is a well-defined metric on X/d ; thus $x \mapsto [x]$ is a surjective and distance-preserving map from the semi-metric space X to the metric space X/d .

The *Hausdorff distance* $d_H(X, Y)$ between subspaces X and Y of a metric space Z is defined to be the infimum of all $r > 0$ such that for any $x \in X$ there is $y \in Y$ with $d(x, y) < r$, and for any $y \in Y$ there is $x \in X$ with $d(x, y) < r$. If there is no such r then $d_H(X, Y) := \infty$. Clearly, $d_H(X, Y) < \infty$ for bounded X and Y .

The map d_H is a semi-metric on the set of all subspaces of Z . In general, it is not a metric: for example, $d_H(X, \bar{X}) = 0$, for any subspace X of Z and its closure \bar{X} in Z . However, d_H is a metric on the set of closed subsets of Z .

Any two metric spaces X and Y are isometrically embeddable into a third metric space Z ; for each such embeddings the Hausdorff distance between the isometric images of X and Y is defined. The infimum of Hausdorff distances between X and Y for all such embeddings is called the *Gromov-Hausdorff distance* between X and Y (cf. [2, 8]); it is denoted by $d_{\text{GH}}(X, Y)$. An equivalent, often more convenient, definition [2, Theorem 7.3.25]:

$$d_{\text{GH}}(X, Y) = \frac{1}{2} \inf \sup \{|d_X(x_i, x_j) - d_Y(y_i, y_j)| : i, j \in I\}, \quad (\star)$$

where the infimum is taken over all maps $i \mapsto x_i, i \mapsto y_i$ from sets I onto X, Y .

The map d_{GH} is a semi-metric on the class of all metric spaces; we denote the corresponding semi-metric space by \mathcal{GH} .

The semi-metric d_{GH} can be extended to a semi-metric on the class of all semi-metric spaces: for semi-metric spaces X and Y put

$$d_{\text{GH}}(X, Y) = d_{\text{GH}}(X/d_X, Y/d_Y).$$

It is easy to show that (\star) holds for semi-metric spaces X and Y as well.

5.2 Uniform boundedness and compactness

It is known that any uniformly totally bounded class of compact metric spaces is precompact in the Gromov-Hausdorff topology (that is, its closure is compact) [2, Theorem 7.4.15]. Here a class of metric spaces \mathfrak{X} is called *uniformly totally bounded* if for every $\varepsilon \geq 0$ there is a positive integer n_ε such that

1. the diameter of every space in \mathfrak{X} is $\leq n_0$;
2. for any $\varepsilon > 0$ any space in \mathfrak{X} can be covered by $\leq n_\varepsilon$ closed balls of radius ε .

Our goal is to prove compactness of certain subclasses of \mathcal{GH} using compactness of the Stone space \mathcal{S}_L for a certain L .

We call semi-metric spaces satisfying (1) and (2) ν -bounded, where

$$\nu : [0, \infty) \rightarrow \mathbb{Z}^{>0}, \quad \nu(\varepsilon) = n_\varepsilon.$$

We denote the class of ν -bounded metric spaces by \mathfrak{X}_ν . So a class \mathfrak{X} of metric spaces is uniformly totally bounded if $\mathfrak{X} \subseteq \mathfrak{X}_\nu$, for some ν .

Theorem 5.1 *For any ν , the class \mathfrak{X}_ν is compact in \mathcal{GH} .*

We postpone the proof until Subsection 5.4, because for that we need a certain correspondence between semi-metric spaces and structures, which requires some preparatory work.

Theorem 5.1 has a corollary which is a refinement of the result on precompactness of any uniformly totally bounded class of compact metric spaces in the Gromov-Hausdorff topology.

Corollary 5.2 *For any ν , the class \mathfrak{C}_ν of ν -bounded compact metric spaces is compact in \mathcal{GH} .*

Proof. It suffices to show that the map $X \mapsto \hat{X}$, where \hat{X} is a completion of X , is a surjective distance-preserving map from \mathfrak{X}_ν to \mathfrak{C}_ν .

First we note that if $X \in \mathfrak{X}_\nu$ then $\hat{X} \in \mathfrak{C}_\nu$. If $X \in \mathfrak{X}_\nu$ is a dense subspace of a metric space Y , then $Y \in \mathfrak{X}_\nu$. (Indeed, first, X and Y have the same diameter, and, second, if for some $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ the closed balls $\bar{B}_X(x_i, \varepsilon)$ cover X then the closed balls $\bar{B}_Y(x_i, \varepsilon)$ cover Y because otherwise the complement of $\bigcup_i \bar{B}_Y(x_i, \varepsilon)$ in Y is open and nonempty but does not meet X , contrary to density of X in Y .) So $\hat{X} \in \mathfrak{X}_\nu$. Since a metric space is compact iff it is complete and totally bounded, \hat{X} is compact. So $\hat{X} \in \mathfrak{C}_\nu$.

Now we show that $X \mapsto \hat{X}$ maps \mathfrak{X}_ν onto \mathfrak{C}_ν . For $Y \in \mathfrak{C}_\nu$ and $i = 1, 2, \dots$ choose n_i closed balls $\bar{B}_Y(y_{ij}, 1/i)$ which cover Y . Let X be the subspace of all y_{ij} . Then $X \in \mathfrak{X}_\nu$ and $Y = \hat{X}$.

Clearly, $d_{\mathcal{GH}}(X, \hat{X}) = 0$; therefore the map $X \mapsto \hat{X}$ preserves $d_{\mathcal{GH}}$. \square

5.3 Semi-metric structures

Now we introduce some relational signature L_0 , and associate with any semi-metric space X a set of certain L_0 -structures with universe X ; we call them X -structures. An L_0 -structure, which is an X -structure for some semi-metric space X , will be called a *semi-metric structure*. We call L_0 the *signature of semi-metric structures*.

The signature L_0 consists of binary relational symbols R_ε , where $\varepsilon \in \mathbb{R}^{>0}$. An L_0 -structure M with a universe X is called an X -structure if for any $\varepsilon > 0$ and any $x, y \in X$

$$[d_X(x, y) < \varepsilon] \Rightarrow [M \models R_\varepsilon(x, y)] \Rightarrow [d_X(x, y) \leq \varepsilon].$$

An example of X -structure is the L_0 -structure M_X on X in which for any ε

$$R_\varepsilon^{M_X} = \{(x, y) \in X \times X : d_X(x, y) \leq \varepsilon\}.$$

We shall use the structure M_X in the proof of Theorem 5.1. This example is not unique: replacing \leq with $<$ in the definition of M_X , we obtain another example of X -structure.

Now we show that any X -structure completely determines the space X .

For an L_0 -structure M and $x, y \in M$ denote

$$d_M(x, y) = \begin{cases} \inf\{\varepsilon : M \models R_\varepsilon(x, y)\}, & \text{if } M \models R_\varepsilon(x, y) \text{ for some } \varepsilon; \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition 5.3 *If M is an X -structure then $d_M = d_X$. In particular, if M is an X -structure and Y -structure then the semi-metric spaces X and Y coincide.*

Proof. Let $x, y \in X$. For any $\varepsilon > 0$ with $M \models R_\varepsilon(x, y)$ we have $d_X(x, y) \leq \varepsilon$; so

$$d_X(x, y) \leq \inf\{\varepsilon : M \models R_\varepsilon(x, y)\} = d_M(x, y).$$

Suppose $d_X(x, y) < d_M(x, y)$. Choose ε with

$$d_X(x, y) < \varepsilon < d_M(x, y).$$

Since $d_X(x, y) < \varepsilon$, we have $M \models R_\varepsilon(x, y)$, and hence $d_M(x, y) \leq \varepsilon$, contrary to the choice of ε . \square

Proposition 5.4 *The class of semi-metric structures is universally axiomatizable.*

Proof. Let Γ be the set of universal L_0 -sentences

- (a) $\forall uv (R_\delta(u, v) \rightarrow R_\varepsilon(v, u))$, for $\delta < \varepsilon$;
- (b) $\forall uvw ((R_\varepsilon(u, v) \wedge R_\delta(v, w)) \rightarrow R_\eta(u, w))$, for $\varepsilon + \delta < \eta$;
- (c) $\forall u R_\varepsilon(u, u)$,
- (d) $\forall uv (R_\delta(u, v) \rightarrow R_\varepsilon(u, v))$, for $\delta < \varepsilon$;

where $\delta, \varepsilon, \eta$ run over $\mathbb{R}^{>0}$.

It is easy to check that any X -structure is a model of Γ . We show that any model M of Γ is an X -structure for some semi-metric space X .

Let M be a model of Γ . We show that d_M is a semi-metric on the universe of M . Obviously, d_M is non-negative. The axiom (a) implies that d_M is symmetric. Indeed, suppose, say, $d_M(x, y) < d_M(y, x)$. Choose ε with

$$d_M(x, y) < \varepsilon < d_M(y, x).$$

By definition of d_M , there is $\delta < \varepsilon$ with $M \models R_\delta(x, y)$. By (a), $M \models R_\varepsilon(y, x)$. Hence $d_M(y, x) \leq \varepsilon$. Contradiction.

The axiom (b) implies that d_M satisfies the triangle inequality. Towards a contradiction, suppose

$$d(x, z) > d_M(x, y) + d_M(y, z).$$

Choose reals α and β such that

$$d_M(x, y) < \alpha, \quad d_M(y, z) < \beta, \quad \alpha + \beta < d_M(x, z).$$

By definition of d_M , there are ε and δ such that $M \models R_\varepsilon(x, y)$ and $M \models R_\delta(y, z)$. By (b), $M \models R_{\alpha+\beta}(x, y)$. Then $d_M(x, z) \leq \alpha + \beta$. Contradiction.

By (c), $d_M(x, x) = 0$ for any $x \in M$. Let X be the semi-metric space which is the universe of M equipped with d_M . We show that M is an X -structure. By definition of d_M , if $M \models R_\varepsilon(x, y)$ then $d_M(x, y) \leq \varepsilon$; if $d_M(x, y) < \varepsilon$ then $M \models R_\delta(x, y)$ for some $\delta < \varepsilon$, and therefore $M \models R_\varepsilon(x, y)$, by (d). \square

5.4 Compactness of \mathfrak{X}_ν

In this subsection we give a proof of Theorem 5.1.

Let \mathfrak{Y}_ν be the class of ν -bounded semi-metric spaces of cardinality at most 2^{\aleph_0} . Then $X \mapsto X/d_X$ is a distance-preserving map from \mathfrak{Y}_ν to \mathfrak{X}_ν . Since any ν -bounded metric space is of cardinality at most 2^{\aleph_0} , the map is surjective. Therefore \mathfrak{X}_ν is compact iff \mathfrak{Y}_ν is compact. We shall prove compactness of \mathfrak{Y}_ν .

For that we define: an extension L of the signature L_0 of semi-metric structures by some constants, a universally axiomatizable subclass of \mathcal{M}_L , and a continuous surjective map from that subclass onto \mathfrak{Y}_ν . Since the subclass is compact in the Stone topology on \mathcal{M}_L , due to results of Section 3, this implies compactness of \mathfrak{Y}_ν .

Let $L = L_0(C)$, where C is the union of a family of pairwise disjoint sets of constant symbols $\{C_\varepsilon : \varepsilon \geq 0\}$, with $|C_0| = 2^{\aleph_0}$, and $|C_\varepsilon| = n_\varepsilon$ for $\varepsilon > 0$.

Let Γ_ν be the union of Γ and the set of universal L -sentences

- (e) $\forall uv R_{n_0}(u, v)$;
- (f) $\forall u \bigvee \{R_\varepsilon(u, c) : c \in C_\varepsilon\}$, $\varepsilon > 0$.

Denote by \mathcal{M}_ν the class $\text{Mod}_{\mathcal{M}_L}(\Gamma_\nu)$. By results of Section 3, the class \mathcal{M}_ν is compact in Stone topology on \mathcal{M}_L .

- Lemma 5.5** (1) For any $X \in \mathfrak{Y}_\nu$ there exists $M \in \mathcal{M}_\nu$ such that the L_0 -reduct of M is an X -structure;
 (2) for any $M \in \mathcal{M}_\nu$ there is a unique $X \in \mathfrak{Y}_\nu$ such that the L_0 -reduct of M is an X -structure.

Proof. (1) Let $X \in \mathfrak{Y}_\nu$. Since $|X| \leq 2^{\aleph_0}$ and X is ν -bounded, there is a $f : C \rightarrow X$ such that $f(C_0) = X$, and for every $\varepsilon > 0$ the closed balls of radius ε centered at $f(c)$, where $c \in C_\varepsilon$, cover X .

The X -structure M_X defined in Subsection 5.3 has the following property:

$$M_X \models R_\varepsilon(x, y) \quad \text{iff} \quad d_X(x, y) \leq \varepsilon,$$

for all $x, y \in X$ and all $\varepsilon < 0$. By Proposition 5.4, M_X is a model of Γ .

Consider the L -expansion M of M_X such that $c^M = f(c)$ for all $c \in C$. Then M satisfies (e) and (f), by the choice of f . Since $f(C_0) = X$, the L -structure M is minimal. Thus $M \in \mathcal{M}_\nu$, and its L_0 -reduct is the X -structure M_X .

(2) As M satisfies Γ , the L_0 -reduct of M is an X -structure for some semi-metric space X , which is unique, by Proposition 5.3. Since M is a minimal L -structure, $|M| \leq 2^{\aleph_0}$, and so $|X| \leq 2^{\aleph_0}$. As M satisfies (e), the diameter of X is $\leq n_0$. Since M satisfies (f), X is covered by the close balls of radius ε centered at c^M with $c \in C_\varepsilon$. Thus $X \in \mathfrak{Y}_\nu$. \square

For $M \in \mathcal{M}_\nu$ let $\chi(M)$ be the unique $X \in \mathfrak{Y}_\nu$ such that the L_0 -reduct of M is an X -structure, which exists by Lemma 5.5(2). The map

$$\chi : \mathcal{M}_\nu \rightarrow \mathfrak{Y}_\nu$$

is surjective, by Lemma 5.5(1). Now, to complete the proof of compactness of \mathfrak{Y}_ν , it suffices to prove

Lemma 5.6 The map χ is continuous.

Proof. To prove that χ is continuous at $M_0 \in \mathcal{M}_\nu$, we need to show that for any $\alpha > 0$ there is $\psi \in \text{QF}_L$ with $M_0 \models \psi$ such that for any $N \in \mathcal{M}_\nu$ with $N \models \psi$

$$d_{\text{GH}}(\chi(M_0), \chi(N)) < \alpha.$$

For any $\alpha > 0$ we construct a finite $\Phi \subseteq \text{QF}_L$ such that, for any $M, N \in \mathcal{M}_\nu$,

$$(M \models \varphi \text{ iff } N \models \varphi \text{ for all } \varphi \in \Phi) \Rightarrow d_{\text{GH}}(\chi(M), \chi(N)) < \alpha;$$

then we can take as ψ the conjunction of all sentences from $\Phi \cup \{\neg\varphi : \varphi \in \Phi\}$ that hold in M_0 .

Choose ε with $0 < \varepsilon < n_0$ and $5\varepsilon/2 < \alpha$. Let m be the integer with

$$0 < m\varepsilon < n_0 \leq (m+1)\varepsilon.$$

Let Φ be the set of all sentences $R_{i\varepsilon}(a, b)$, where $i \in \{1, \dots, m\}$ and $a, b \in C_\varepsilon$. We show that the finite set Φ satisfies the required conditions.

Let $M, N \in \mathcal{M}_\nu$. Denote $\chi(M) = X$ and $\chi(N) = Y$; so the L_0 -reduct of M is an X -structure, and the L_0 -reduct of N is a Y -structure. Let

$$X_\varepsilon = \{c^M : c \in C_\varepsilon\}, \quad Y_\varepsilon = \{c^N : c \in C_\varepsilon\}.$$

Since X is covered by the closed balls centered at points of the set X_ε , we have $d_{\text{GH}}(X, X_\varepsilon) \leq \varepsilon$. Similarly, $d_{\text{GH}}(Y, Y_\varepsilon) \leq \varepsilon$. Therefore by the triangle inequality,

$$d_{\text{GH}}(X, Y) \leq \varepsilon + d_{\text{GH}}(X_\varepsilon, Y_\varepsilon) + \varepsilon.$$

Hence it suffices to show that if $M \models \varphi$ iff $N \models \varphi$ for all $\varphi \in \Phi$, then

$$d_{\text{GH}}(X_\varepsilon, Y_\varepsilon) \leq \varepsilon/2,$$

because this implies

$$d_{\text{GH}}(X, Y) \leq \varepsilon + \varepsilon/2 + \varepsilon = 5\varepsilon/2 < \alpha.$$

To prove $d_{\text{GH}}(X_\varepsilon, Y_\varepsilon) \leq \varepsilon/2$, it suffices to show that for any $a, b \in C_\varepsilon$

$$|d_X(a^N, b^N) - d_Y(a^M, b^M)| \leq \varepsilon,$$

by (\star) from Subsection 5.1. The latter inequality holds because, first, the diameters of M and N are $\leq n_0$, and so

$$0 \leq d_X(a^N, b^N), d_Y(a^M, b^M) \leq n_0,$$

and, second, none of the numbers $\varepsilon, 2\varepsilon, \dots, m\varepsilon$ can be strictly between $d_X(a^N, b^N)$ and $d_Y(a^M, b^M)$: if, say,

$$d_X(a^N, b^N) < i\varepsilon < d_Y(a^M, b^M),$$

then $N \models R_{i\varepsilon}(a, b)$ and $M \not\models R_{i\varepsilon}(a, b)$. The lemma is proven. □

The proof of Theorem 5.1 is completed. □

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