

ESSAYS ON CONSISTENCY AND CONVERSE CONSISTENCY IN
MATCHING PROBLEMS

DUYGU NİZAMOĞULLARI
109806008

İSTANBUL BİLGİ ÜNİVERSİTESİ
SOSYAL BİLİMLER ENSTİTÜSÜ
EKONOMİ DOKTORA PROGRAMI

PROF. DR. İPEK ÖZKAL-SANVER
2013

Essays on Consistency and Converse Consistency in Matching Problems

Eşleşme Problemlerinde Tutarlılık ve Ters Tutarlılık üzerine Denemeler

Duygu Nizamoğulları
109806008

Tez Danışmanı : Prof. Dr. Ipek Özkal-Sanver



Jüri Üyesi : Prof. Dr. Jean Laine



Jüri Üyesi : Yrd. Doç. Dr. Uğur Özdemir



Jüri Üyesi : Yrd. Doç. Dr. Ipek Gürsel Tapkı



Jüri Üyesi : Yrd. Doç. Dr. Esra Şengelen



Tezin Onaylandığı Tarih : 08/07/2013

Toplam Sayfa Sayısı : 132

Anahtar Kelimeler (Türkçe)

- 1) Eşleşme
- 2) Ev arkadaşı problemleri
- 3) Çekirdek çözüm
- 4) Tutarlılık
- 5) Ters Tutarlılık

Anahtar Kelimeler (İngilizce)

- 1) Matching
- 2) Roommate problems
- 3) Core
- 4) Consistency
- 5) Converse Consistency

Özet

Bu doktora tezi eşleşme problemlerinde tutarlılık ve ters tutarlılık ile ilgili dört deneme içermektedir. İlk deneme eşleşme problemleri için tutarlılık ve ters tutarlılık aksiyomlarının kullanımı ile ilgili bir incelemedir. Tez danışmanım İpek Özkal-Sanver ile ortak bir çalışma olan ikinci deneme "evlilik problemleri" diye adlandırılan, çift taraflı birebir eşleşme problemlerinde çekirdeğin karakterizasyonu ile ilgilidir. Üçüncü denemede Pareto optimal kuralın ters tutarlı olan en büyük alt-kuralı hesaplanmıştır. Yine tez danışmanım İpek Özkal-Sanver ile ortak bir çalışma olan son deneme "ev arkadaşı problemleri" diye adlandırılan tek taraflı birebir eşleşme problemlerinde tutarlılık aksiyomu ilgili bir çalışmadır.

Abstract

This Ph.D. thesis consists of four essays about consistency and converse consistency in matching problems. The first essay is a survey about the use of consistency and converse consistency in the literature for matching problems. The second essay is about characterization of the core by using consistency and converse consistency in two sided one to one matching problems (marriage problems) in general domains which is a joint work with my advisor İpek Özkal-Sanver. In the third essay, I compute maximal conversely consistent subsolution of the Pareto optimal solution for marriage problems. The final essay which is again a joint work with my advisor İpek Özkal-Sanver is about consistency on one-sided one-to-one matching problems, the so called roommate problems.

Essays on Consistency and Converse
Consistency in Matching Problems

Duygu Nizamogulları

Advisor: Prof. Dr. İpek Özkal-Sanver

Istanbul Bilgi University

Ph.D. in Economics

July 8, 2013

Contents

1	Introduction	6
2	A Survey on Consistency and Converse Consistency for Matching Problems	10
2.1	Introduction	10
2.2	Formal Definitions for Matching Problems	14
2.3	Characterization Results	21
2.4	Minimal Extensions and Maximal Subsolutions	24
2.5	Concluding Remarks	27
3	Characterization of the Core in Full Domain Marriage Problems	29
3.1	Introduction	29
3.2	Notations and Definitions	31
3.3	Converse Consistency	35
3.4	Results	40
3.5	Weak Preferences	50
3.6	Concluding Remarks	62
4	A Maximal Conversely Consistent Subsolution of the Pareto Optimal Solution	64

4.1	Introduction	64
4.2	Model and Axioms	67
4.3	Maximal Conversely Consistent Subsolution of the Pareto- Optimal Solution	73
4.4	Concluding Remarks	86
5	Consistent Enlargements of the Core in Roommate Prob- lems	88
5.1	Introduction	88
5.2	Model	90
5.3	Results	93
5.4	Concluding Remarks	98
6	Conclusion	100
7	Appendices	102
8	References	123

Abstract

This Ph.D. thesis consists of four essays about consistency and converse consistency in matching problems. The first essay is a survey about the use of consistency and converse consistency in the literature for matching problems. The second essay is about characterization of the core by using consistency and converse consistency in two sided one to one matching problems (marriage problems) in general domains which is a joint work with my advisor İpek Özkal-Sanver. In the third essay, I compute maximal conversely consistent subsolution of the Pareto optimal solution for marriage problems. The final essay which is again a joint work with my advisor İpek Özkal-Sanver is about consistency on one-sided one-to-one matching problems, the so called roommate problems.

Acknowledgements

It would not have been possible to write this doctoral thesis without the help and support of the kind people around me, the only some of whom it is possible to give particular mention here.

Above all, I would like to express my deepest gratitude to my advisor, Prof. Dr. İpek Özkal-Sanver, for her excellent guidance, caring, patience and providing me with an excellent atmosphere for doing research.

Next, I would like to thank to the members of my thesis committee who guided me in preparing my thesis. I especially thank to Jean Laine who contributed a lot to the improvement of this thesis. I would like to thank Uğur Özdemir for his interest in my study. I benefited a lot from his friendly comments and critiques. I would also like to thank İpek Gürsel-Tapkı and Esra Şengelen for their sincere support.

I would like to acknowledge the academic and financial support of İstanbul Bilgi University. I spent eleven years at the İstanbul Bilgi University, five years as a undergraduate student and six years as a graduate student. I would like to thank all of my professors both in Mathematics and Economics departments for their contributions to my education.

I owe special thanks to my parents. I would like to thank my husband Alper for his personal support and great patience at all times. I also would

like to thank my daughters Duru and Derin for giving me unlimited happiness and pleasure.

Last, but by no means least, I thank my friends for their support and encouragement throughout this thesis. For any errors or inadequacies that may remain in this work, of course, the responsibility is entirely my own.

1 Introduction

This Ph.D. thesis consists of four essays about consistency and converse consistency in matching problems.

The first essay is a survey about the use of consistency and converse consistency in the literature. Consistency and converse consistency are two well-known axioms that have recently played fundamental role in axiomatic analysis. Consistency says that if an agreement is made between the both sides of the economy, if some of the agents leave the society with their endowments, payments, partners etc., then the same agreements should be made between the remaining agents. On the other hand, converse consistency says that, the agreement on the payoffs, endowments, partners etc. depends on the agreement for the two agent restricted problems. In this essay, we shortly survey applications of these axioms for many economics problems. We thoroughly present the results for matching problems.

The next three essays which are the main parts of this thesis analyse these axioms for matching problems. Matching processes are usually related to so-called one sided markets and two-sided markets. In one-sided markets coalitions are formed by same type of agents. The most well known example is roommate problem where university rooms are allocated to couples of students, each having preferences over potential mates. In two-sided markets,

coalitions are formed by two types of agents where each type has preferences over the other one. Coalitions can be pairs (one-to-one matching known as marriage markets) or subsets including either one agent of some type and several of the other (many-to-one matching), or several agent of each type (many-to-many matching). In such settings the main addressed question is how to form coalitions in a strategically stable way? In all the matchings models mentioned above stability is a central property. A matching for marriage problems or for roommate problems is stable if each agent in the society is matched with an acceptable agent and no two agent would prefer to be matched with each other rather than their current mates. Alvin E. Roth who shared the 2012 Nobel Prize in Economic Sciences with Lloyd S. Shapley said that “Many of the important things that we do in our lives are matching, from getting into a university, from getting married to getting a job.” As Alvin E. Roth emphasized, Matching Theory has many applications for real life situations such as marriage markets, school and university placement, job markets, organ donations.

The second essay is about characterization of the core by using consistency and converse consistency in two sided one to one matching problems, the so called marriage problems in general domains which is a joint work with my advisor İpek Özkal-Sanver. In this essay we characterize the core of marriage problems in general domains. First in a model where agents have

strict preferences over their potential mates, and agents are allowed to stay single, we characterize the core as the unique solution which satisfies individual rationality, Pareto optimality, gender fairness, consistency and converse consistency. Next, relaxing the constraint that agents have strict preferences over their potential mates, we show that there exists no solution satisfying Pareto optimality, anonymity and converse consistency. In this full domain, we characterize the core by individual rationality, weak Pareto optimality, monotonicity, gender fairness, consistency and converse consistency.

In the third essay, I compute maximal conversely consistent subsolution of the Pareto optimal solution for marriage problems. This essay which is very related to the second essay we study marriage problems in a restricted domain where agents have strict preferences over their potential mates, and agents are not allowed to stay single. We consider the most well-known solution concept: the Pareto-optimal solution. The Pareto-optimal solution fails to satisfy converse consistency. In this essay, we compute a maximal conversely consistent subsolution of the Pareto optimal solution. To do this, we introduce the concept of serial men-ordering. Also we show that this result is not valid for general domains where there are unequal number of men and women and being single is allowed.

The final essay which is again a joint work with my advisor İpek Özkal-Sanver is about consistency on one-sided one to one matching problems, the

so called roommate problems. We compute consistent enlargements of the core in roommate problems. By computing it, we evaluate the extent to which the core would have to be expanded in order to be well-defined and consistent. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. We also show that for any fixed order on the set of agents the solution which picks all stable matchings and the serial dictatorship matching with respect to this order is a minimal consistent enlargement of the core. Since for different orders there may be different enlargements, minimal consistent core enlargement is not unique.

2 A Survey on Consistency and Converse Consistency for Matching Problems

2.1 Introduction

The objective of this essay is to present two well-known axioms, consistency and converse consistency that have recently played fundamental role in axiomatic analysis. We briefly survey the applications of these axioms for many economic models. For matching problems, we present applications of these axioms in a more detail way.

Consistency and converse consistency require to be invariant for population changes. **Consistency** says that if an alternative is acceptable for some problem then it should be an acceptable alternative of its restrictions to all subgroups for the associated reduced problems. On the other hand, **converse consistency** allows a dual operation, if an alternative is acceptable of its restrictions to all subgroups of cardinality two for the associated reduced problems then this alternative should be acceptable for the whole group. For the definitions of these axioms two notions are crucial; a solution and a reduced problem. A solution picks for each decision problem in some domain one or several of its feasible alternatives. The notion of reduced problem is an important fact, since different formulations of a reduced problem lead to

different definitions of these axioms.

Converse consistency is a kind of two-agent decentralization axiom. Thomson (1996) give an example about the agreement of a proposal in a political convention to emphasize the importance of this axiom. In a political convention, delegates first meet each other in committees of size two and each committee argue on the proposal. If the proposal successfully passes this stage, it is examined by committees of size three. The process is repeated until the proposal is either rejected at some stage by some committee, or finally it is accepted by the whole convention in plenary session. If the decision process satisfies converse consistency, acceptance by all committees of size two will guarantee acceptance at the plenary session. Hence, the formation of the committees of size greater than two will be unnecessary.

First, we survey applications of consistency and converse consistency for many economic problems. The domains in which these axioms are used are classified by Thomson (1996) in four main classes; game theory, public economies and cost allocations, fair allocation, and some other models. The models for game theory include bargaining, games in coalitional form with and without transferable utility and games in strategic form. For public economies and cost allocations there are studies about these axioms for bankruptcy and taxation, quasi linear cost allocation, general cost allocation and pricing. As a third class, these axioms are studied for fair division

in classical private good economies, fair division in economies with single peaked preferences and fair allocation in economies with indivisible goods. The last class includes apportionment and matching problems. For each of these classes, except matching problems without stating formal definitions of the axioms we briefly present central results of the related literature. For matching problems in the proceeding sections we analyse these axioms in a detail way.

In the literature, for each of the classes there are many characterization results of the well-known solution concepts by using these axioms. In these results there are also other axioms but the central axioms are consistency and converse consistency. For bargaining problems by using consistency, Lensberg (1988) characterized the Nash solution, Lensberg (1987) characterized the lexicographic egalitarian solution and the seperable additive solution. For the games in coalitional form with transferable utility whose core is non-empty there are some characterization results of the core by using different versions of consistency; Tadenuma (1992) used complement consistency and Peleg (1986) used max consistency. For this class of problems, by using complement consistency Moulin (1985) characterized the equal allocation of non-seperable benefits solution. Sobolev (1975) characteized the prenucleous by using max consistency. There are also some studies by using some versions of converse consistency in this class of problems. By using max consistency

and max converse consistency, Peleg (1989, 1992) characterized the core for the domain of market games and Peleg (1986) characterized the prekernel. By using another version of consistency, namely self consistency, Hart and Mas-Colell (1989) characterized the Shapley value. For the games in coalitional form without transferable utility, by using complement consistency, Tadenuma (1992) characterized the core whenever it is non-empty. For this class, by using consistency and converse consistency Peleg and Tijs (1996) characterized the Nash equilibrium solution and the coalition proof Nash equilibrium. For bankruptcy and taxation, Young (1987) characterized the continuous parametric solution by using consistency. By a different version of consistency, limited consistency, Chun (1988) characterized the proportional solution. For general cost allocation problems, Moulin and Shenker (1994) characterized the average cost sharing by using consistency and the serial cost sharing by limited consistency. For pricing mechanisms, McLean, Pazgal and Sharkey (2004) characterized the Shapley value by using self consistency. For fair division in economies with single-peaked preferences, Thomson (1994a) characterized the uniform rule by using consistency. For apportionment, Balinski and Young (1982) characterized the divisor solution by consistency.

For matching problems, there are (at least) two approaches dealing with consistency and converse consistency. The first approach is characterization

of well-known solution concepts by these axioms. Since some of the well-known solutions do not satisfy these principals, as a second approach, there are papers about extending or reducing the solutions in such a way that these axioms are satisfied. Alternatively, there are some studies about the identification of conditions under which these solutions satisfy consistency and converse consistency.

This essay proceeds as follows: Section 2.2 presents the basic notations and definitions of the axioms for matching problems. Section 2.3 gives characterization results of the core. Section 2.4 gives the results about the minimal extensions or maximal subsolutions of the solutions. Section 2.5 concludes.

2.2 Formal Definitons for Matching Problems

First we define formal definitions of consistency and converse consistency for marriage problems. Let M and W be two disjoint universal sets. Let M be a nonempty and finite subset of M . Similarly, let W be a nonempty and finite subset of W . A society is a union of some $M \subset M$ and some $W \subset W$. In the context of marriage, the set M stands for a set of men and the set W for a set of women.

Let $A=M \cup W$ denote the universal set of agents. Let $\mathcal{A} = \{M \cup W\}_{M \subset M, W \subset W}$ be the set of all possible societies. For each society

$A = M \cup W \in \mathcal{A}$ and for each agent $i \in A$ the set of potential mates of i , denoted by $A(i)$, is defined as

$$A(i) \equiv \{i\} \cup \begin{cases} W & \text{if } i \in M \\ M & \text{if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by P_i . Let \mathcal{P} denote the set of all possible preference profiles $P \equiv (P_i)_{i \in A}$.

A **matching** is a function $\mu : A \rightarrow A$ such that for all $i \in A$, $\mu(i) \in A(i)$ and for all $i \in A$, $\mu^2(i) = i$. Here, $\mu(i)$ is the mate of agent i under matching μ . If $\mu(i) = i$, we say that agent i is selfmatched or single. Let $\mathcal{M}(A)$ denote the set of all matchings for A .

A **(matching) problem** p is a pair $p = (A, P)$, where A is a society, P is the profile of their preferences over potential mates. Let \mathbf{P} denote the set of all problems.

Let $p = (A, P) \in \mathbf{P}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if for all $i \in A$, $\mu(i) P_i i$ or $\mu(i) = i$. Let $\mathcal{IR}(p)$ denote the set of all individually rational matchings. A pair of agents (i, j) **blocks** a matching $\mu \in \mathcal{M}(A)$ if $j P_i \mu(i)$ and $i P_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if it is individually rational for p and there is no pair (i, j) blocking μ at p . Let $\mathcal{S}(p)$ denote the set of all stable matchings.

Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$, μ **dominates** μ' if there exists a coalition $K \subseteq A$ such that for all $i \in K$,

$\mu(i) \in K$ and $\mu(i) P_i \mu'(i)$. A matching μ is **undominated** if there exists no matching $\mu' \in \mathcal{M}(A)$ which dominates μ . The **core** of p is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor(1990)). Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ **Pareto dominates** μ' if for all $i \in A$, $\mu(i) P_i \mu'(i)$ whenever $\mu(i) \neq \mu'(i)$. A matching $\mu \in \mathcal{M}(A)$ is **Pareto optimal** for p if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates μ . Let $\mathcal{PO}(p)$ denote the set of all Pareto optimal matchings.

Given a problem $p = (A, P)$ and a subset of set of agents N , a **reduced problem** of p with respect to N is a problem where the preference profile P is restricted to agents in N . Formally; for all $p = (A, P) \in \mathbf{P}$ and all $N \subseteq A$, $p' = (N, P_N) \in \mathbf{P}$ is the reduced problem of p with respect to N . Given a matching $\mu \in \mathcal{M}(A)$, a **reduced matching** of μ with respect to N is a matching $\mu|_N : N \cup \mu(N) \longrightarrow N \cup \mu(N)$ such that for all $i \in N$, $\mu|_N(i) = \mu(i)$.

We define **extension** of a problem $p' = (M' \cup W', P')$ of $p = (M \cup W, P) \in \mathbf{P}$ if $M \subseteq M'$, $W \subseteq W'$ and $P'|_{M \cup W} = P$. In particular, if $M \neq M'$ and $W = W'$, p' is M -extension of p and if $M = M'$ and $W \neq W'$ p' is W -extension of p .

A **solution** φ is a correspondence which associates with each p a non-empty subset $\varphi(p) \subseteq \mathcal{M}(A)$.

Now, we define axioms on solutions that are used for axiomatic analysis in the literature. The first axiom requires that at each problem the solution recommends individually rational matchings:

Individual rationality (IR): For each $p \in \mathbf{P}$, $\varphi(p) \subseteq \mathcal{IR}(p)$.

The second axiom requires that at each problem the solution recommends Pareto optimal matchings:

Pareto optimality (PO): For each $p \in \mathbf{P}$, $\varphi(p) \subseteq \mathcal{PO}(p)$.

The third axiom requires that renaming men among men and renaming women among women do not change the result:

Anonymity (AN): For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |M'|$ and $|W| = |W'|$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = M'$ and $\pi(W) = W'$. For all $p = (A, P)$, let P' be such that for all $i \in A$ and all $j, k \in A(i)$, $jP_i k$ if and only if $\pi(j)P'_{\pi(i)} \pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

The fourth axiom imposes same treatment of men and women, in the sense that renaming men as women and women as men do not change the result.

Gender fairness (GF): For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |W'|$ and $|W| = |M'|$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = W'$ and $\pi(W) = M'$. For all $p = (A, P)$, let P' be such that for all

$i \in A$ and all $j, k \in A(i)$, $j P_i k$ if and only if $\pi(j) P'_{\pi(i)} \pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

The fifth axiom says that if there exists a matching which is preferred by every agent it should be the unique recommendation.

Weak unanimity (WU): For each $p = (M \cup W, P) \in \mathbf{P}$, if there exists a matching $\mu \in \mathcal{M}(M \cup W)$ such that $\mu(a)$ is most preferred by each agent $a \in M \cup W$, then $\varphi(p) = \{\mu\}$.

The sixth axiom which is stronger than weak unanimity requires that a pair of mutually best agents to be matched at every solution outcome.

Mutually best (MB): For each $p = (M \cup W, P) \in \mathbf{P}$, for any $m \in M$ and $w \in W$ if m and w most prefer each other then $\mu(m) = w$ for each $\mu \in \varphi(p)$.

The next axiom requires that if the number of agents in one side of the market increases while the opposite sides remains fixed no incumbent on the same side as the entrants is strictly better off.

Population Monotonicity (PMON): For each $p = (M \cup W, P) \in \mathbf{P}$ and each M -extension p' of p if $\mu \in \varphi(p)$, then there exists $\mu' \in \varphi(p')$ such that $\mu(m) R_m \mu'(m)$ for each $m \in M$. If p' is a W -extension of p , there exists μ' satisfying a symmetric requirement for each $w \in W$.

The next axiom says that if the preference profile changes in such a way

that ranking does not decrease in each agents preferences then the outcome is still recommended.

Let $p = (M \cup W, P) \in \mathbf{P}$ be a problem and $\mu \in \mathcal{M}(M \cup W)$. For each $m \in M$, define $L(\mu, R_m) = \{a \in W \mid \mu(m)R_m a\}$.

Similarly, for each $w \in W$, $L(\mu, R_w)$ is defined. It is said that $p' = (M \cup W, P')$ is obtained from p by a monotonic transformation at μ if $L(\mu, R_a) \subseteq L(\mu, R'_a)$ for each $a \in M \cup W$.

Maskin Monotonicity (MMON.): For each $p = (M \cup W, P) \in \mathbf{P}$, and $\mu \in \mathcal{M}(M \cup W)$ if $p' = (M \cup W, P')$ is obtained from p by a monotonic transformation at μ then $\mu \in \varphi(p')$.

Now we define our main axioms; consistency and converse consistency.

Consider some problem p and some solution φ . Take any matching μ recommended by φ at p . If the reduced matching of μ with respect to each subgroup of matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup of matched pairs, then we say that the solution φ is consistent. More formally:

Consistency (CON): For each $p = (A, P) \in \mathbf{P}$, each $\mu \in \varphi(p)$ we have $\mu|_N \in \varphi(N, P|_N)$ for any society $N \subseteq A$ such that $\mu(N) = N$.

We define converse consistency only imposing the requirement on the subgroups formed exactly by two matched pairs: Consider some problem p and some solution φ . Take any matching μ . The requirement is that if the

reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of these two matched pairs, then μ must be recommended by φ at the original problem p .

Converse Consistency (CCON): For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$, then $\mu \in \varphi(p)$.

In the following chapter, we consider three versions of converse consistency.¹ In a restricted domain where the number of men and women coincide and agents are not allowed being single, all these three versions are equivalent.

We also analyse the roommate problems. Now, we give formal definitions of the axioms that are used for roommate problems.

To define the following axioms, let $p = (A, P)$ be a roommate problem and let $p' = (A', P')$ be an extension of p where $A' = A \cup \tilde{A}$.

The next axiom, competition sensitivity which is first defined by Klaus (2008) requires that if two incumbents are newly matched after a set of newcomers arrived, then one of them suffers.

Competition Sensitivity (CS): Each $\mu \in \varphi(p)$ there is $\mu' \in \varphi(p')$ such

¹Thomson (2004, 2009) introduced and studied the converse consistency axiom for various economic models.

that for any $i, j \in A$ with $\mu(i) \neq j$ and $\mu'(i) = j$ we have either $\mu(i)P'_i\mu'(i)$ or $\mu(i)P'_i\mu'(i)$.

A solution is **weakly competition sensitive** if competition sensitivity is satisfied when we only add one newcomer at a time.

Resource sensitivity requires that if two incumbents are unmatched after a set of newcomers arrived, then one of them benefits. Formally,

Resource Sensitivity (RS): For each $\mu' \in \varphi(p')$ there is $\mu \in \varphi(p)$ such that for any $i, j \in A$ with $\mu(i) = j$ and $\mu'(i) \neq j$ we have either $\mu'(i)P'_i\mu(i)$ or $\mu'(i)P'_i\mu(i)$.

A solution is **weakly resource sensitive** if resource sensitivity is satisfied when we only add only one newcomer at a time.

2.3 Characterization Results

In the literature, there are many studies about the characterization of the core for matching problems by using consistency and converse consistency.

For the sake of completeness we state these results as theorems:

Consistency and converse Consistency are introduced to the matching theory literature by Sasaki and Toda (1992) who characterize the core of marriage (two sided, one-to-one matching) problems as the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse

consistency.

Theorem 2.1 (*Sasaki and Toda, 1992*) *The core is the unique solution satisfying Pareto optimality, anonymity, consistency and converse consistency.*

For marriage problems, Toda (2006) give another two characterization of the core by using weak unanimity, population monotonicity, Maskin monotonicity and consistency.

Theorem 2.2 (*Toda, 2006*) *The core is the unique solution satisfying weak unanimity, population monotonicity and Maskin monotonicity..*

Theorem 2.3 (*Toda, 2006*) *The core is the unique solution satisfying weak unanimity, population monotonicity and consistency.*

For marriage problems when agents have weak preferences over their potential mates Toda (2006) give a characterizaiton of the core without using converse consistency.

Theorem 2.4 (*Toda, 2006*) *If agents have weak preferences then the core is the unique solution satisfying weak unanimity, population monotonicity, Maskin monotonicity and consistency.*

In a similar way, Toda (2006) uses consistency in characterizing the core of two sided many-to-one matching problems (College admissions problems).

Theorem 2.5 (*Toda, 2006*) *The core of college admissions problems is the unique solution satisfying weak unanimity, population monotonicity and consistency.*

For matching problems with money which is introduced first by Shapley and Shubik (1972), Sasaki (1995) characterized the core for these problems by using a weaker version of consistency, namely *seperation independence*.

Theorem 2.6 (*Sasaki, 1995*) *On the domain of matching problems with money the core is the unique solution satisfying Pareto optimality, continuity, individual rationality, couple rationality, seperation independence and worth monotonicity.*

On the domain of one sided one-to-one matching problems (roommate problems) Özkal-Sanver (2010) showed that no solution satisfies Pareto optimality, anonymity and converse consistency. Klaus (2013) characterizes the core by using consistency and converse consistency on the domain of no odd rings roommate problems. She also proves that extending her result to the domain of solvable roommate problems is not possible. Can and Klaus (2012) give two characterizations of the core of roommate problems by using consistency for no odd rings domains and for solvable domains.

Theorem 2.7 (*Can and Klaus, 2012*)

a) *On the domain of no odd rings roommate problems the core is the unique solution satisfying weak unanimity, weak competition sensitivity and consistency.*

b) *On the domain of solvable roommate problems the core is the unique solution satisfying weak unanimity, competition sensitivity and consistency.*

Theorem 2.8 (*Can and Klaus 2012*)

a) *On the domain of no odd rings roommate problems the core is the unique solution satisfying weak unanimity, weak resource sensitivity and consistency.*

b) *On the domain of solvable roommate problems the core is the unique solution satisfying weak unanimity, resource sensitivity and consistency.*

By using these axioms, Can and Klaus (2012) get an impossibility result for all roommate problems. They show that, on the domain of all roommate problems there exists no solution satisfying weak unanimity, competition sensitivity and consistency.

2.4 Minimal Extensions and Maximal Subsolutions

If a solution does not satisfy consistency or converse consistency, we would like to know how serious violations of these axioms. There are two procedures which are developed by Thomson (1994b, 1996) to recover these properties.

The first attempt is minimally enlarge the solution so that consistency or converse consistency are satisfied. Since consistency and converse consistency are preserved under intersections Minimal consistent extension of a solution (or minimal conversely consistent extension) is defined by intersection of consistent solutions which include that solution. More formally, given a solution φ the minimal consistent extension of φ , denoted by MCE_φ , is defined by Thomson (1996) as follows:

$$MCE_\varphi = \bigcap_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is consistent}\}$$

Similarly, the minimal conversely consistent extension of a solution φ can be defined as follows:

$$MCCE_\varphi = \bigcap_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is conversely consistent}\}$$

As a second attempt, Thomson (1996) introduced the concept of maximal consistent subsolution. By computing maximal consistent subsolution we reduce the solution maximally to get consistency. Formally, given a solution φ that contains at least one consistent subsolution, the maximal consistent subsolution of φ , denoted by $MXCS_\varphi$, is defined as follows:

$$MXCS_\varphi = \bigcup_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \subseteq \varphi, \psi \text{ is consistent}\}$$

As Thomson (1996) noticed, union of two conversely consistent solutions may not be conversely consistent. So, we can not carry the concept of maxi-

mal consistent subsolution directly to the conversely consistent subsolution. Therefore, for a given solution φ we define maximal conversely consistent subsolution $MXCCS_\varphi$ as a maximal conversely consistent solution that includes φ .

In the literature, there are few papers computing the minimal consistent extension of solution, as well as the minimal conversely consistent extension of solutions for different economic problems. For instance, for the domain of fair allocation problems Bevia (1996), Kolm (1973), for bargaining problems Thomson (1995) and Korthues (2000). For matching problems, Özkal-Sanver (2012) compute the minimal conversely consistent extension of the men-optimal solution. For this aim, she introduced the concept of men-barterproofness. Men barter-proof solution MB is defined in the following way: A matching is men-barterproof whenever there is no such a pair of men who benefit from switching their mates among themselves. More formally, a matching $\mu \in \mathcal{M}(A)$ is **men-barterproof** for $p = (A, P) \in \mathbf{P}$, if there exists no pair of men $\{m, m'\} \subset M$ bartering at μ . Let $MB(p)$ denote the set of the men-barterproof matchings for p . The **men-barterproof solution** is the correspondence MB that associates with each problem p the set of men-barterproof matchings $MB(p)$.

Theorem 2.9 (Özkal-Sanver, 2013) *The minimal conversely consistent*

extension of the men-optimal solution is $MCC E_{MO}(p) = MO(p) \cup [S(p) \cap MB(p)]$ for all $p \in \mathbf{P}$.

Also, there are some studies for roommate problems about consistency. As Gale and Shapley (1962) showed, the core may be empty for roommate problems. Özkal-Sanver (2010) define the concept of core extension as a solution which picks the core whenever it is non-empty. Formally, a core extension is a solution φ such that for any $p \in \mathbf{P}$ with $\mathcal{S}(p) \neq \emptyset$, $\varphi(p) = \mathcal{S}(p)$. She showed that no core extension is consistent.

Theorem 2.10 (*Özkal-Sanver, 2010*) *No core extension is consistent.*

If a solution does not satisfy consistency and converse consistency, as an alternative approach there are some studies which provide necessary and sufficient conditions for the solution to satisfy these axioms. For instance, for College Admissions problems, Klaus and Klijn (2011) established necessary and sufficient conditions for student optimal solution to be conversely consistent.

2.5 Concluding Remarks

In this essay, we survey applications of consistency and converse consistency in the literature. First, we state both informal and formal definitions of these

axioms. Then we present results of related literature. There are several related open questions to be investigated further: In the proceedings chapters we deal with some of them. First, we characterize the core of marriage problems in general domains. Then, we compute a maximal conversely consistent subsolution of the Pareto optimal solution in marriage problems. Finally, we study on consistent enlargements of the core in roommate problems. For many to one matching problems (college admission problems) characterization of the core by using converse consistency is still open questions worth to be investigated.

3 Characterization of the Core in Full Domain Marriage Problems

Duygu Nizamoğulları and İpek Özkal-Sanver

3.1 Introduction

Aim of this essay is to characterize the core of two-sided one-to-one matching problems in general domains. Sasaki and Toda (1992)'s characterization result is valid for a domain where agents have strict preferences over their potential mates and agents are not allowed to be single.²

First we consider a model where agents are allowed to stay single, but still agents have strict preferences over their potential mates. We show that the core is the unique solution satisfying individual rationality, Pareto optimality, gender fairness, consistency and converse consistency.

Consistency and converse consistency are two fundamental properties of solutions for allocation problems, in variable population models. The consistency axiom is some kind of independence of irrelevant alternatives axiom. Consistency of a solution imposes that if some matched pairs are leaving

²Sasaki and Toda (1992) characterize the core by Pareto optimality, anonymity, consistency and converse consistency. Toda (1993) showed that there exists some other solution than the core satisfying individual rationality, Pareto optimality, anonymity, consistency, and a weaker version of converse consistency; when we allow that agents stay single.

the society; the reduced matching is among the matchings recommended by the solution for the reduced problem. Consistency is trivially adapted to the model where agents may stay single. However; there are several possible ways to adapt the converse consistency to this model. Converse consistency is a kind of decentralization axiom. Thomson (2004) illustrates the notion of converse consistency by a jigsaw puzzle as “correct positioning of pieces two-by-two guarantees correct positioning altogether.” This axiom allows us to deduce from the simpler calculations on all of the meaningfully smallest subproblems whether an alternative should be chosen for the original big problem. We discuss alternative definitions of converse consistency in this more general framework. Gender fairness is a stronger version of anonymity.³ Roughly speaking, if a solution is gender fair, by renaming men as women, and women as men, and applying the solution, we end up with the outcome permuted accordingly.

Next we consider what would happen if we further relax the condition that agents have strict preferences over their potential mates. If we allow indifference on preferences, the core fails to satisfy Pareto optimality. We show that there exists no solution satisfying Pareto optimality, anonymity and converse consistency. We characterize the core as the unique solution satisfying individual rationality, weak Pareto optimality, gender fairness, consistency,

³See Özkal-Sanver (2004).

converse consistency and monotonicity. Another characterization of the core can be found in Toda (2006) where the core of many-to-one matching problems is characterized by weak unanimity, consistency, Maskin monotonicity and population monotonicity.⁴

This essay proceeds as follows: Section 3.2 presents the basic notations and definitions. Section 3.3 discusses converse consistency of a solution. Section 3.4 gives the results of the first model where agents are allowed to stay single, and agents have strict preferences over their potential mates. Section 3.5 gives the results of the second model where agents may have weak preferences. Section 3.6 concludes.

3.2 Notations and Definitions

Let M and W be two disjoint universal sets. Let M be a nonempty and finite subset of M . Similarly, let W be a nonempty and finite subset of W . A society is a union of some $M \subset M$ and some $W \subset W$. In the context of marriage, the set M stands for a set of men and the set W for a set of women.

Let $A=M \cup W$ denote the universal set of agents. Let $\mathcal{A} = \{M \cup W\}_{M \subset M, W \subset W}$ be the set of all possible societies. For each society $A = M \cup W \in \mathcal{A}$ and for each agent $i \in A$ the set of potential mates of

⁴The characterization result of Toda (2006) can be carried to one-to-one matching problems.

i , denoted by $A(i)$, is defined as

$$A(i) \equiv \{i\} \cup \begin{cases} W & \text{if } i \in M \\ M & \text{if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by P_i . Let \mathcal{P} denote the set of all possible preference profiles $P \equiv (P_i)_{i \in A}$.

A **matching** is a function $\mu : A \rightarrow A$ such that for all $i \in A$, $\mu(i) \in A(i)$ and for all $i \in A$, $\mu^2(i) = i$. Here, $\mu(i)$ is the mate of agent i under matching μ . If $\mu(i) = i$, we say that agent i is selfmatched or single. Let $\mathcal{M}(A)$ denote the set of all matchings for A .

A **(matching) problem** p is a pair $p = (A, P)$, where A is a society, P is the profile of their preferences over potential mates. Let \mathbf{P} denote the set of all problems.

Let $p = (A, P) \in \mathbf{P}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if for all $i \in A$, $\mu(i) P_i i$ or $\mu(i) = i$. Let $\mathcal{IR}(p)$ denote the set of all individually rational matchings. A pair of agents (i, j) **blocks** a matching $\mu \in \mathcal{M}(A)$ if $j P_i \mu(i)$ and $i P_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if it is individually rational for p and there is no pair (i, j) blocking μ at p . Let $\mathcal{S}(p)$ denote the set of all stable matchings.

Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$, μ **dominates** μ' if there exists a coalition $K \subseteq A$ such that for all $i \in K$, $\mu(i) \in K$ and $\mu(i) P_i \mu'(i)$. A matching μ is **undominated** if there exists

no matching $\mu' \in \mathcal{M}(A)$ which dominates μ . The **core** of p is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor(1990)). Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ **Pareto dominates** μ' if for all $i \in A$, $\mu(i) P_i \mu'(i)$ whenever $\mu(i) \neq \mu'(i)$. A matching $\mu \in \mathcal{M}(A)$ is **Pareto optimal** for p if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates μ . Let $\mathcal{PO}(p)$ denote the set of all Pareto optimal matchings.

Given a problem $p = (A, P)$ and a subset of set of agents N , a **reduced problem** of p with respect to N is a problem where the preference profile P is restricted to agents in N . Formally; for all $p = (A, P) \in \mathbf{P}$ and all $N \subseteq A$, $p' = (N, P|_N) \in \mathbf{P}$ is the reduced problem of p with respect to N . Given a matching $\mu \in \mathcal{M}(A)$, a **reduced matching** of μ with respect to N is a matching $\mu|_N : N \cup \mu(N) \longrightarrow N \cup \mu(N)$ such that for all $i \in N$, $\mu|_N(i) = \mu(i)$.

A **solution** φ is a correspondence which associates with each p a non-empty subset $\varphi(p) \subseteq \mathcal{M}(A)$. The **core solution** is the correspondence \mathcal{S} which associates with each p its set of stable matchings $\mathcal{S}(p)$.

Now, we are ready to define our axioms on solutions:

The first axiom requires that at each problem the solution recommends individually rational matchings:

Individual rationality (IR): For each $p \in \mathbf{P}$, $\varphi(p) \subseteq \mathcal{IR}(p)$.

The second axiom requires that at each problem the solution recommends Pareto optimal matchings:

Pareto optimality (PO): For each $p \in \mathbf{P}$, $\varphi(p) \subseteq \mathcal{PO}(p)$.

The third axiom requires that renaming men among men and renaming women among women do not change the result:

Anonymity (AN): For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |M'|$ and $|W| = |W'|$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = M'$ and $\pi(W) = W'$. For all $p = (A, P)$, let P' be such that for all $i \in A$ and all $j, k \in A(i)$, $j P_i k$ if and only if $\pi(j) P'_{\pi(i)} \pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

The fourth axiom imposes same treatment of men and women, in the sense that renaming men as women and women as men do not change the result.

Gender Fairness (GF): For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |W'|$ and $|W| = |M'|$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = W'$ and $\pi(W) = M'$. For all $p = (A, P)$, let P' be such that for all $i \in A$ and all $j, k \in A(i)$, $j P_i k$ if and only if $\pi(j) P'_{\pi(i)} \pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.⁵

⁵Note that gender fairness is stronger than anonymity.

The next axiom is central to our analysis. Consider some problem p and some solution φ . Take any matching μ recommended by φ at p . If the reduced matching of μ with respect to each subgroup of matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup of matched pairs, then we say that the solution φ is consistent.

Consistency (CON): For each $p = (A, P) \in \mathbf{P}$, each $\mu \in \varphi(p)$ we have $\mu|_N \in \varphi(N, P|_N)$ for any society $N \subseteq A$ such that $\mu(N) = N$.

The next section is devoted to converse consistency axiom.

3.3 Converse Consistency

In this section we consider three versions of “converse consistency”.⁶ In a restricted domain where the number of men and women coincide and agents are not allowed being single, all these three versions are equivalent to its definition used in Sasaki and Toda (1992).

First we define "weakest converse consistency". Consider some problem p and some solution φ . Take any matching μ . The requirement is that if the reduced matching of μ with respect to each subgroup consisting of at most two men and two women matched at μ is among the recommendations made

⁶Thomson (2004, 2009) introduced and studied the converse consistency axiom for various economic models.

by the solution φ for the reduced problem of p with respect to this subgroup, then μ must be recommended by φ at the original problem p . More formally,

Weakest Converse Consistency (WWCCON): For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $M' \cup W' \subseteq A$ with $|M'| \leq 2$ and $|W'| \leq 2$, $\mu|_{M' \cup W'} \in \varphi(M' \cup W', P_{M' \cup W'})$, then $\mu \in \varphi(p)$.⁷

WWCCON will consider a subgroup of agents consisting of two self-matched women and two selfmatched men; as well as a subgroup of agents consisting of a pair matched to each other, a selfmatched woman and a self-matched man. From our point of view, the former subgroup consists of four matched pairs, and the latter subgroup consists of three matched pairs.

Next we define "weak converse consistency". It is stronger than WWCCON, in the sense that the subgroups consisting of three and four matched pairs are not considered here, but only the subgroups consisting of one and two matched pairs are taken into account: Consider some problem p and some solution φ . Take any matching μ . The requirement is that if the reduced matching of μ with respect to each subgroup consisting of at most two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup, then μ must be recommended by φ at the original problem p .

Weak Converse Consistency (WCCON): For each $p = (A, P) \in \mathbf{P}$

⁷Toda (1993) uses this version of converse consistency.

and each $\mu \in \mathcal{M}(A)$, if for each subset $N \subseteq A$ with $|N| = 2$, $\mu|_N \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$, then $\mu \in \varphi(p)$.

Finally, we define converse consistency, only imposing the requirement on the subgroups formed exactly by two matched pairs: Consider some problem p and some solution φ . Take any matching μ . The requirement is that if the reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of these two matched pairs, then μ must be recommended by φ at the original problem p .

Converse Consistency (CCON): For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu|_N \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$, then $\mu \in \varphi(p)$.

Throughout the essay, we will use the strongest version of converse consistency, which we call converse consistency. Indeed, since we deal with individually rational and Pareto optimal solutions, weak converse consistency and converse consistency axioms are equivalent.

Proposition 3.1 *If a solution satisfies IR and PO, then CCON is equivalent to WCCON.*

Proof. Clearly, converse consistency implies weak converse consistency. For the other direction, let φ be a solution which satisfies individual ratio-

nality, Pareto optimality and weak converse consistency but fails to satisfy converse consistency. Then there are a problem $p = (A, P)$ and a matching $\mu \in \mathcal{M}(A)$ such that $\mu \notin \varphi(p)$ and $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for any subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$. Since φ satisfies weak converse consistency we have $\mu|_N \notin \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for some $\{i, j\} = N \subseteq A$ with $\mu(i) = j$. Say $i = m$ and $j = w$. If $w P_m m$ and $m P_w w$, then by Pareto optimality of φ , $\{\mu|_N\} = \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$. Hence, there are three possibilities, we have either (i) $m P_m w$ and $m P_w w$ or (ii) $w P_m m$ and $w P_w m$ or (iii) $m P_m w$ and $w P_w m$. For all these three cases, take any agent $\hat{m} \in M \setminus \{m\}$ and let $N' = \{m, \hat{m}\}$. By individual rationality of φ , we have $\mu|_{N'} \notin \varphi(N' \cup \mu(N'), P|_{N' \cup \mu(N')})$, a contradiction. ■

Remark 3.1 *There exists some IR, PO and WWCCON solution, which fails to satisfy WCCON. To see that take the solution $\tilde{\varphi}$ defined for each $p = (A, P)$ as $\tilde{\varphi}(p) = \{\mu \in IR(p) \cap PO(p)$ such that there is no blocking pair (m, w) with $\mu(m) \neq m$ and $\mu(w) \neq w\}$ and consider the example below:*

Example 3.1 *Let $\tilde{\varphi}$ be defined for each $p = (A, P)$ as $\tilde{\varphi}(p) = \{\mu \in IR(p) \cap PO(p)$ such that there is no blocking pair (m, w) with $\mu(m) \neq m$ and $\mu(w) \neq w\}$. By its definition, $\tilde{\varphi}$ is an individually rational and Pareto optimal solution. Toda (2003) shows that $\tilde{\varphi}$ is weakest conversely consistent. Since for each $p \in \mathbf{P}$, $\tilde{\varphi}(p) \subseteq IR(p) \cap PO(p)$, by Proposition 3.1, it is suffi-*

cient to show that $\tilde{\varphi}$ fails to satisfy converse consistency. Let $M = \{m_1, m_2\}$,

$W = \{w_1, w_2\}$. Let P be defined as follows:

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_1	m_2	m_1
w_1	m_2	m_1	w_2
m_1		w_1	

Let $p = (M \cup W, P)$. Consider the matching $\mu \in \mathcal{M}(M \cup W)$ where $\mu(m_1) = w_1, \mu(m_2) = m_2$ and $\mu(w_2) = w_2$. Since μ is Pareto dominated by μ^* where $\mu^*(m_1) = w_2$ and $\mu^*(m_2) = w_1$, $\mu \notin \tilde{\varphi}(p)$. One can easily check that for any proper subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$.

To see that, consider the following three subproblems: First, letting $N = \{m_1, m_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$. Indeed, (m_2, w_1) forms a blocking pair, but $\mu(m_2) = m_2$. Next, letting $N = \{m_1, w_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$. Indeed, (m_1, w_2) forms a blocking pair, but $\mu(w_2) = w_2$. Finally, letting $N = \{m_2, w_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$, by individual rationality of $\tilde{\varphi}$. Hence, $\tilde{\varphi}$ is not conversely consistent. By Proposition 3.1, $\tilde{\varphi}$ is not weak conversely consistent.

3.4 Results

In this section, we consider a model where agents have strict preferences and agents may stay single. In this more general model, the characterization result of Sasaki and Toda (1992) is not valid. The core is no more characterized by IR, PO, AN, CON and CCON. To see that take the solution φ^* defined for each $p = (A, P)$ as $\varphi^*(p) = \{\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m\}$ and consider the example below.

Example 3.2 *Let $M = \{m_1, m_2\}$ and $W = \{w_1\}$. Let $w_1 P_{m_1} m_1$, $w_1 P_{m_2} m_2$ and $m_1 P_{w_1} m_2 P_{w_1} w_1$. Let $p = (M \cup W, P)$. Consider the matchings $\mu_1 \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$ where $\mu_1(m_1) = w_1, \mu_1(m_2) = m_2$ and $\mu_2 \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$ where $\mu_2(m_2) = w_1$ and $\mu_2(m_1) = m_1$. Note that $\mathcal{S}(p) = \{\mu_1\}$ and $\varphi^*(p) = \{\mu_1, \mu_2\}$.⁸*

For our characterization result, we use a strong anonymity condition which is called gender fairness. This axiom does not allow that a matching rule favors a gender at the expense of the other. It imposes same treatment of men and women, in the sense that renaming men as women and women as men does not change the result. Since gender fairness implies anonymity we also utilize from anonymity for the proposition below.

⁸Refer to Lemma 7.1 in the Appendix I to see that φ^* satisfies CCON.

Proposition 3.2 *Let φ be a solution satisfying IR, PO, GF and CCON. If $\mu_{|N} \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$ for any $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$, then for any $\mu \in \varphi(A, P)$ we have $\mu_{|N} \in \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$.*

Proof. Let φ be a solution satisfying IR, PO, GF and CCON. Let $p = (A, P)$ be a problem. Take $\mu \in \varphi(p)$ such that $\mu_{|N} \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$ for all $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$, but $\mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$ for some $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$.

Let $N = \{i, j\}$. First we assume that $\mu(i) = i$ and $\mu(j) = j$. Wlog, say $i = m_1$ and $j = w_1$. Then $\mu(m_1) = m_1$ and $\mu(w_1) = w_1$. Since $\mu_{|N} \in \mathcal{IR}(N \cup \mu(N), P_{|N \cup \mu(N)})$ and $\mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$, (m_1, w_1) forms a blocking pair. We have $w_1 P_{m_1} m_1$ and $m_1 P_{w_1} w_1$. Then by Pareto optimality of φ , we have $\mu_{|N} \notin \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Next, let $\mu(i) = i$ and $\mu(j) \neq j$. Again, say $i = m_1$ and $j = w_1$. Then $\mu(m_1) = m_1$ and $\mu(w_1) = m_2$. Since $\mu_{|N} \in \mathcal{IR}(N \cup \mu(N), P_{|N \cup \mu(N)})$ and $\mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$, there (m_1, w_1) forms a blocking pair. Hence, there is only one possible preference profile:

P_{m_1}	P_{m_2}	P_{w_1}
w_1	w_1	m_1
m_1	m_2	m_2
		w_1

Introduce an agent w_2 and extend the preferences of the agents of N to the larger set $N \cup \{w_2\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{w_1}^*$	$P_{w_2}^*$
w_1	w_2	m_1	m_2
w_2	w_1	m_2	w_2
m_1	m_2	w_1	m_1

Consider the matching μ^* where $\mu^*(m_2) = w_1, \mu^*(m_1) = m_1$ and $\mu^*(w_2) = w_2$. Since μ^* is Pareto dominated by the matching μ' where $\mu'(m_1) = w_1$ and $\mu'(m_2) = w_2$ we have $\mu^* \notin \varphi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset A$ with $\mu^*(i) \neq j$.

To see that, consider the following three subproblems: Letting $N' = \{m_1, m_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$. (Indeed, $N \cup \mu(N) = N' \cup \mu^*(N')$). Next, letting $N' = \{m_1, w_2\}$, by individual rationality of φ , we have $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$. Finally, let $N' = \{m_2, w_2\}$. Let $\pi(m_1) = w_2, \pi(m_2) = w_1$ and $\pi(w_1) = m_2$. Let P' be the preference profile in the permuted problem. Note that $P' = P_{|N \cup \mu(N)}$, $\pi_\mu(m_2) = w_1$ and $\pi_\mu(w_2) = w_2$. Since $\pi_\mu = \mu_{|N'}^*$, then by gender fairness we have $\mu_{|S}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$. Finally, a similar argument works for the symmetric cases where $\mu(i) \neq i$ and $\mu(j) = j$.

Now, assume that $\mu(i) \neq i$ and $\mu(j) \neq j$. Say $i = m_1$ and $j = m_2$. Then $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Since $\mu|_N \in \mathcal{IR}(N \cup \mu(N), P_{|_{N \cup \mu(N)}})$ and $\mu|_N \notin \mathcal{S}(N \cup \mu(N), P_{|_{N \cup \mu(N)}})$, then there is a blocking pair, wlog assume (m_1, w_2) forms a blocking pair. Then we have nine possible preferences profiles, four of them are the same as Sasaki and Toda (1992) where each agent puts being single at the bottom. The other five possible preference profiles are:

i)

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_2	m_1	m_1
w_1	m_2	m_2	m_2
m_1	w_1	w_1	w_2

ii)

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_2	m_2	m_1
w_1	m_2	m_1	m_2
m_1	w_1	w_1	w_2

iii)

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_1	m_1	m_1
w_1	w_2	w_1	m_2
m_1	m_2	m_2	w_2

iv)

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_2	m_1	m_1
w_1	w_1	w_1	m_2
m_1	m_2	m_2	w_2

v)

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_2	w_2	m_1	m_1
w_1	m_2	w_1	m_2
m_1	w_1	m_2	w_2

For the case (i), two new agents m_3 and w_3 enter the society. We extend the preferences of the agents of N to the larger set $N \cup \{m_3, w_3\}$ of agents in the following way

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_2	w_3	w_1	m_3	m_1	m_2
w_1	w_2	w_3	m_1	m_2	m_3
m_1	m_2	m_3	m_2	m_3	m_1
w_3	w_1	w_2	w_1	w_2	w_3

Consider the matching μ^* where $\mu^*(m_1) = w_1, \mu^*(m_2) = w_2$ and $\mu^*(m_3) = w_3$. Since μ^* is Pareto dominated by the matching μ' where $\mu'(m_1) = w_2, \mu'(m_2) = w_3$ and $\mu'(m_3) = w_1$, we have $\mu^* \notin \varphi(M^*, W^*, P^*)$.

However, $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset A$ with $\mu^*(i) \neq j$.

To see that, consider the following three subproblems: Letting $N' = \{m_1, m_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' = \{m_1, m_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_1	w_1	m_3	m_3
m_1	w_3	m_1	m_1
w_3	m_3	w_1	w_3

Let $\pi(m_1) = m_3, \pi(m_2) = m_1, \pi(w_1) = w_3$ and $\pi(w_2) = w_1$. Let P' be the preference profile in the permuted problem. Note that $P' = P_{|N \cup \mu(N)}$. Since $\pi_{|N'} = \mu_{|N'}^*$, by anonymity of φ , we have $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Finally, letting $N' = \{m_2, m_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_3	w_3	m_2	m_2
w_2	m_3	m_3	m_3
m_2	w_2	w_2	w_3

Let $\pi(m_1) = m_2, \pi(m_2) = m_3, \pi(w_1) = w_2$ and $\pi(w_2) = w_3$. Let P' be the preference profile in the permuted problem. Note that $P' = P_{|N \cup \mu(N)}$. Since

$\pi|_{\mu} = \mu^*_{|N'}$, by anonymity of φ , we have $\mu^*_{|N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')})$

Similar arguments work for the case (ii), (iii) and (iv).

For the case (v), two new agents m_3 and w_3 enter the society. We extend the preferences of the agents of N to the larger set $N \cup \{m_3, w_3\}$ of agents in the following way

$P^*_{m_1}$	$P^*_{m_2}$	$P^*_{m_3}$	$P^*_{w_1}$	$P^*_{w_2}$	$P^*_{w_3}$
w_2	w_3	w_1	m_3	m_1	m_2
w_1	w_2	w_3	m_1	m_2	m_3
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_2	m_3	m_1

Consider the matching μ^* where $\mu^*(m_1) = w_1, \mu^*(m_2) = w_2$ and $\mu^*(m_3) = w_3$. Since μ^* is Pareto dominated by the matching μ' where $\mu'(m_1) = w_2, \mu'(m_2) = w_3$ and $\mu'(m_3) = w_1$, we have $\mu^* \notin \varphi(M^*, W^*, P^*)$. However, $\mu^*_{|N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')})$ for each $N' = \{i, j\} \subset A$ with $\mu^*(i) \neq j$.

To see that, consider the following three subproblems: Letting $N' = \{m_1, m_2\}$, we have $P^*_{|N' \cup \mu^*(N')} = P_{|N \cup \mu(N)}$ and $\mu^*_{|N'} = \mu_{|N} \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' = \{m_1, m_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_1	w_1	m_3	m_3
m_1	w_3	m_1	w_3
w_3	m_3	w_1	m_1

Let $\pi(m_1) = m_3$, $\pi(m_2) = m_1$, $\pi(w_1) = w_3$ and $\pi(w_2) = w_1$. Let P' be the preference profile in the permuted problem. Note that $P' = P_{|N \cup \mu(N)}$. Since $\pi_{|\mu} = \mu_{|N'}^*$, by anonymity of φ , we have $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Finally, letting $N' = \{m_2, m_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_3	w_3	m_2	m_2
w_2	m_3	w_2	m_3
m_2	w_2	m_3	w_3

Let $\pi(m_1) = m_2$, $\pi(m_2) = m_3$, $\pi(w_1) = w_2$ and $\pi(w_2) = w_3$. Let P' be the preference profile in the permuted problem. Note that $P' = P_{|N \cup \mu(N)}$. Since $\pi_{|\mu} = \mu_{|N'}^*$, by anonymity of φ , we have $\mu_{|N'}^* \in \varphi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Hence, by CCON we have $\mu^* \in \varphi(M^*, W^*, P^*)$, a contradiction. ■

Proposition 3.3 *If a solution φ satisfies IR, PO, GF, CON and CCON, then $\varphi(p) \subseteq \mathcal{S}(p)$ for each problem $p \in \mathbf{P}$.*

Proof. Let φ be a solution satisfying IR, PO, GF, CON and CCON.

Let $p = (A, P)$ be a problem. Let $\mu \in \varphi(p)$. By consistency of φ , we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$. By Proposition 4.2, $\mu|_N \in \mathcal{S}(N \cup \mu(N), P|_{N \cup \mu(N)})$. Since \mathcal{S} satisfies CCON, we have $\mu \in \mathcal{S}(p)$. ■

To show that no proper subsolution of the core satisfies consistency in many-to-one matching problems, Toda (2006) uses a bracing lemma. The lemma below is its adaptation for one-to-one matching problems.

Lemma 3.1 (*Bracing Lemma*) For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{S}(p)$, there exists \tilde{p} such that $\mathcal{S}(\tilde{p}) = \{\tilde{\mu}\}$, p is the reduced problem of \tilde{p} at $\tilde{\mu}$ and $\tilde{\mu}|_{M \cup W} = \mu$.

Proposition 3.4 *No proper subsolution of the core satisfies CON.*

Proof. Let φ be a subsolution of the core satisfying CONS. Let $p = (A, P)$ be a problem. Let $\mu \in \mathcal{S}(p)$. Then by Lemma 3.1, there exists \tilde{p} such that $\mathcal{S}(\tilde{p}) = \{\tilde{\mu}\}$, p is reduced problem of \tilde{p} at $\tilde{\mu}$ and $\tilde{\mu}|_{M \cup W} = \mu$. Since φ is a subsolution of the core we have $\varphi(\tilde{p}) = \mathcal{S}(\tilde{p}) = \{\tilde{\mu}\}$. Since φ satisfies CON, $\mu \in \varphi(p)$. Hence, $\varphi \equiv \mathcal{S}$. ■

Theorem 3.1 *The core \mathcal{S} is the unique solution satisfying IR, PO, GF, CON and CCON.*

Proof. One can easily check that the core \mathcal{S} satisfies the above listed axioms. For the uniqueness, let φ be a solution satisfying these five axioms. Then by Proposition 3.3, $\varphi(p) \subseteq \mathcal{S}(p)$ for each $p \in \mathbf{P}$. By Proposition 3.4, no proper subsolution of the core satisfies CON. Hence, $\varphi \equiv \mathcal{S}$. ■

We also check the independence of the axioms. By doing that, we also show that dropping any one of the five axioms leads to the failure of Theorem 3.1.

- A solution which satisfies IR, PO, GF, CON, but not CCON.

For each $p \in \mathbf{P}$, $\varphi(p) = \mathcal{IR}(p) \cap \mathcal{PO}(p)$.

- A solution which satisfies IR, PO, GF, CCON, but not CON.

Let φ be defined as follows:

$$\varphi(p) = \begin{cases} \mathcal{S}(p) & \text{for any } p = (N \cup \mu(N), P_{|N \cup \mu(N)}) \text{ with } N = \{i, j\} \text{ and } \mu(i) \neq j \\ \mathcal{IR}(p) \cap \mathcal{PO}(p) & \text{otherwise.} \end{cases}$$

- A solution which satisfies IR, PO, CON, CCON, but not GF.

Let φ be defined for each $p = (A, P)$ as $\varphi(p) = \{\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m\}$.⁹

- A solution which satisfies IR, GF, CON, CCON, but not PO.

⁹Refer to Lemma 7.1 in the Appendix I to see that φ satisfies CCON.

For each $p \in \mathbf{P}$, $\varphi(p) = \mathcal{IR}(p)$.

- A solution which satisfies PO, GF, CON, CCON, but not IR.

Let φ be defined for each $p = (A, P)$ as $\varphi(p) = \{\mu \in \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w)\}$.¹⁰

3.5 Weak Preferences

In this section, we relax the assumption that each agent has a strict preference over his/her potential mate. Each agent $i \in A$ may be indifferent between matching with any two possible mates $j, k \in A(i)$; if it is the case, then we write $j I_i k$. For each agent $i \in A$, each $j, k \in A(i)$ we write $j R_i k$ if and only if either $j P_i k$ holds or $j I_i k$ holds. From this section on, each agent $i \in A$ has a complete and transitive preference relation over $A(i)$, denoted by R_i . Let \mathcal{R} denote the set of all possible preference profiles $R \equiv (R_i)_A$. A **problem** p is redefined as a pair $p = (A, R)$, where A is a society, R is the profile of their preferences over potential mates. Let \mathbf{R} denote the set of all problems.

Let $p = (A, R) \in \mathbf{R}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if for all $i \in A$, $\mu(i) R_i i$ or $\mu(i) = i$. Let $\mathcal{IR}(p)$ denote the set of all individually rational matchings. A pair of agents

¹⁰Refer to Lemma 7.2 in the Appendix I to see that φ satisfies CCON.

(i, j) **blocks** a matching $\mu \in \mathcal{M}(A)$ if $j P_i \mu(i)$ and $i P_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if it is individually rational for p and there is no pair (i, j) blocking μ at p . Let $S(p)$ denote the set of all stable matchings. Note that $S(p)$ is nonempty.¹¹

Given a problem $p = (A, R) \in \mathbf{R}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ **Pareto dominates** μ' if for all $i \in A$, $\mu(i) R_i \mu'(i)$ and for some $j \in A$, $\mu(j) P_j \mu'(j)$. A matching $\mu \in \mathcal{M}(A)$ is **Pareto optimal** for p if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates μ . Let $PO(p)$ denote the set of all Pareto optimal matchings. The other axioms defined in Section 2 are simply carried to the world of weak preferences just by replacing P by R .

First note that the Pareto Stable rule, denoted by PS and defined for each $p \in \mathbf{R}$ as $\mathcal{PS}(p) = \mathcal{PO}(p) \cap S(p)$, fails to satisfy converse consistency.¹²

Example 3.3 Let $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$. Let R be

¹¹The stronger version of stability is defined as follows: A pair of agents (i, j) **weakly blocks** a matching $\mu \in \mathcal{M}(A)$ if $j R_i \mu(i)$, $i R_j \mu(j)$ and for some $k \in \{i, j\}$, $R_k = P_k$. A matching $\mu \in \mathcal{M}(A)$ is **strongly stable** for p if it is individually rational for p and there is no pair (i, j) weakly blocking μ at p . Note that the set of all strongly stable matchings may be empty for some problems.

¹²When we have strict preferences $\mathcal{PS}(p) = \mathcal{PO}(p) \cap S(p)$. But when we allow indifferences, a stable matching may not be Pareto optimal. To see this, let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let \mathbf{P} be defined as follows:

$$\begin{array}{cc|cc} P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\ \hline w_1 w_2 & w_1 & m_1 m_2 & m_1 \\ \dots & w_2 & \dots & m_2 \end{array}$$

Let $p = (M \cup W, \mathbf{P})$. Consider the matching μ where $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Note that $\mu \in S(p)$, but $\mu \notin \mathcal{PO}(p)$.

defined as follows:

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
w_1	w_2	w_3	w_4	$m_1m_2m_3m_4$	$m_1m_3m_4$	$m_1m_2m_4$	m_2m_3
w_2	w_3	w_4	w_1	w_1	m_2	m_3	m_4
w_3	w_4	w_1	w_2		w_2	w_3	m_1
w_4	w_1	w_2	w_3				w_4
m_1	m_2	m_3	m_4				

Let $p = (M \cup W, R)$. Consider the matching μ where $\mu(m_1) = w_1$, $\mu(m_2) = w_4$, $\mu(m_3) = w_2$, $\mu(m_4) = w_3$. Since μ is Pareto dominated by μ' where $\mu'(m_1) = w_1$, $\mu'(m_2) = w_3$, $\mu'(m_3) = w_4$, $\mu'(m_4) = w_2$, we have $\mu \notin \mathcal{PO}(p)$. Hence, $\mu \notin \mathcal{PS}(p)$. But $\mu|_N \in \mathcal{PS}(N \cup \mu(N), R|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subset A$ with $\mu(i) \neq j$.

Indeed, we have an impossibility result: there exists no solution satisfying Pareto optimality, anonymity and converse consistency.¹³

Theorem 3.2 *No solution satisfies PO, AN and CCON.*

Proof. Let φ be a solution satisfying PO, AN and CCON. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let R be defined as follows:

¹³This result is a similar spirit of Proposition 4.2 in Özkal-Sanver (2010).

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
w_1	w_1	m_1m_2	m_1m_2
w_2	w_2	w_1	w_2
m_1	m_2		

Let $p = (M \cup W, R)$. Note that $\mathcal{PO}(p) = \{\mu_1, \mu_2\}$ where $\mu_1(m_1) = w_1$, $\mu_1(m_2) = w_2$ and $\mu_2(m_1) = w_2$, $\mu_2(m_2) = w_1$. By Pareto optimality of φ , we have $\varphi(p) \subseteq \{\mu_1, \mu_2\}$.

Suppose $\mu_1 \in \varphi(p)$. Introduce two agents m_3 and w_3 and extend the preferences of the agents of $M \cup W$ to the larger set $M \cup W \cup \{m_3, w_3\}$ of agents in the following way:

$R_{m_1}^*$	$R_{m_2}^*$	$R_{m_3}^*$	$R_{w_1}^*$	$R_{w_2}^*$	$R_{w_3}^*$
w_1w_3	w_1	w_2	m_1m_2	$m_1m_2m_3$	$m_1m_2m_3$
w_2	w_2	w_3	m_3	w_2	w_3
m_1	w_3	w_1	w_1		
	m_2	m_3			

Let $p^* = (M \cup W \cup \{m_3, w_3\}, R^*)$. Consider the matching μ^* where $\mu^*(i) = \mu_1(i)$ for all $i \in M \cup W$ and $\mu^*(m_3) = w_3$. Since μ^* is Pareto dominated by μ' where $\mu'(m_1) = w_3$, $\mu'(m_2) = w_1$ and $\mu'(m_3) = w_2$, we have $\mu^* \notin \varphi(p^*)$. However, $\mu_{|N}^* \in \varphi(N \cup \mu^*(N), R_{|N \cup \mu^*(N)}^*)$ for each $N = \{i, j\} \subset A$ with $\mu^*(i) \neq j$.

To see that, consider the following three subproblems: Letting $N = \{m_1, m_2\}$, we have $(N \cup \mu^*(N), R_{|N \cup \mu^*(N)}^*) = p$ and $\mu_{|N}^* = \mu_1 \in \varphi(p)$.

Next, letting $N = \{m_1, m_3\}$, we have $\mathcal{PO}(N \cup \mu^*(N), R_{|N \cup \mu^*(N)}^*) = \{\tilde{\mu}\}$ where $\tilde{\mu}(m_1) = w_1$ and $\tilde{\mu}(m_3) = w_3$. Since $\mu_{|N}^* = \tilde{\mu}$, we have $\mu_{|N}^* \in \varphi(N \cup \mu^*(N), R_{|N \cup \mu^*(N)}^*)$.

Finally, letting $N = \{m_2, m_3\}$, note that $R_{|N \cup \mu^*(N)}^*$ is

$R_{m_2}^*$	$R_{m_3}^*$	$R_{w_2}^*$	$R_{w_3}^*$
w_2	w_2	$m_2 m_3$	$m_2 m_3$
w_3	w_3	w_2	w_3
m_2	m_3		

Let $\pi(m_1) = m_2$, $\pi(m_2) = m_3$, $\pi(w_1) = w_2$ and $\pi(w_2) = w_3$. Let R' be the preference profile in the permuted problem. Note that $R' = R$. Since $\pi_{|\mu} = \mu_{|N}^*$, by anonymity of φ , we have $\mu_{|N}^* \in \varphi(N \cup \mu^*(N), R_{|N \cup \mu^*(N)}^*)$.

Then by converse consistency $\mu^* \in \varphi(p^*)$. Since φ satisfies Pareto optimality, we have $\varphi(p) = \{\mu_2\}$. Next, introduce two agents m_4 and w_4 and extend the preferences of the agents of $M \cup W$ to the larger set $M \cup W \cup \{m_4, w_4\}$ of agents in the following way:

\widehat{R}_{m_1}	\widehat{R}_{m_2}	\widehat{R}_{m_4}	\widehat{R}_{w_1}	\widehat{R}_{w_2}	\widehat{R}_{w_4}
w_1	w_1w_4	w_2	m_1m_2	$m_1m_2m_4$	$m_1m_2m_4$
w_2	w_2	w_4	m_4	w_2	w_4
w_4	m_2	w_1	w_1		
m_1		m_4			

Let $\widehat{p} = (M \cup W \cup \{m_4, w_4\}, \widehat{R})$. Consider the matching $\widehat{\mu}$ where $\widehat{\mu}(i) = \mu_2(i)$ for all $i \in M \cup W$ and $\widehat{\mu}(m_4) = w_4$. Since $\widehat{\mu}$ is Pareto dominated by the matching μ' where $\mu'(m_1) = w_1$, $\mu'(m_2) = w_4$ and $\mu'(m_4) = w_2$, we have $\widehat{\mu} \notin \varphi(\widehat{p})$. However, $\widehat{\mu}_{|N \cup \mu^*(N)} \in \varphi(N \cup \widehat{\mu}(N), \widehat{R}_{|N \cup \widehat{\mu}(N)})$ for each $N = \{i, j\} \subset A$ with $\widehat{\mu}(i) \neq j$.

To see that, consider the following three subproblems: Letting $N = \{m_1, m_2\}$, we have $(N \cup \widehat{\mu}(N), \widehat{R}_{|N \cup \widehat{\mu}(N)}) = p$ and $\widehat{\mu}_{|N \cup \widehat{\mu}(N)} = \mu_2 \in \varphi(p)$.

Next, letting $N = \{m_2, m_4\}$, we have $PO(N \cup \widehat{\mu}(N), \widehat{R}_{|N \cup \widehat{\mu}(N)}) = \{\bar{\mu}\}$ where $\bar{\mu}(m_2) = w_1$ and $\bar{\mu}(m_4) = w_4$. Since $\widehat{\mu}_{|N} = \bar{\mu}$, we have $\widehat{\mu}_{|N} \in \varphi(N \cup \widehat{\mu}(N), \widehat{R}_{|N \cup \widehat{\mu}(N)})$.

Finally, letting $N = \{m_1, m_4\}$, note that $\widehat{R}_{|N \cup \widehat{\mu}(N)}$ is

\widehat{R}_{m_1}	\widehat{R}_{m_4}	\widehat{R}_{w_2}	\widehat{R}_{w_4}
w_2	w_2	$m_1 m_4$	$m_1 m_4$
w_4	w_4	w_2	w_4
m_1	m_4		

Let $\pi(m_1) = m_4$, $\pi(m_2) = m_1$, $\pi(w_1) = w_2$ and $\pi(w_2) = w_4$. Let R'' be the preference profile in the permuted problem. Note that $R'' = R$. Since $\pi_\mu = \widehat{\mu}|_N$, by anonymity of φ , we have $\widehat{\mu}|_N \in \varphi(N \cup \widehat{\mu}(N), \widehat{R}|_{N \cup \widehat{\mu}(N)})$. Then by converse consistency of φ , we have $\widehat{\mu} \in \varphi(\widehat{p})$, contradicting that φ satisfies Pareto optimality, and completing the proof. ■

By weakening the Pareto optimality axiom and adding a monotonicity condition, we end up with a characterization result.

Given a problem $p = (A, R) \in \mathbf{R}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ **strongly Pareto dominates** μ' if for all $i \in A$ we have $\mu(i) R_i \mu'(i)$ and for all $j \in A$ with $\mu(j) \neq \mu'(j)$, we have $\mu(j) P_j \mu'(j)$. A matching $\mu \in \mathcal{M}(A)$ is **weakly Pareto optimal** for p if there exists no matching $\mu' \in \mathcal{M}(A)$ which strongly Pareto dominates μ . Let $\mathcal{WPO}(p)$ denote the set of all weakly Pareto optimal matchings.

A weak Pareto optimal solution recommends at each problem among weakly Pareto optimal matchings:

Weak Pareto optimality(WPO): For each $p \in \mathbf{R}$, $\varphi(p) \subseteq \mathcal{WPO}(p)$.

Remark 3.2 Note that for each problem $p = (A, P)$, $\mathcal{PO}(p) = \mathcal{WPO}(p)$. If a solution φ satisfies *IR*, *WPO*, *GF*, *CON* and *CCON* then for each $p \in \mathbf{P}$, $\varphi(p) = \mathcal{S}(p)$.

Consider some problem $p = (A, R^*)$ and any matching $\mu \in \mathcal{M}(A)$. We define complete strict preference P^μ over A as a **monotonic stretching** of R^* for μ in the following way: For each $i \in A$, for any $j, k \in A(i)$; let $j P_i^\mu k$ if $j P_i^* k$ holds and let $\mu(i) P_i^\mu j$ for any $j \in A(i)$ such that $\mu(i) I_i^* j$. Using monotonic stretching we switch from weak preferences to some adequate form of strict preferences. Roughly speaking, monotonicity axiom allows us to check whether a matching μ is recommended or not for a problem (A, R^*) by considering whether μ is recommended for (A, P^μ) where P^μ is monotonic stretching of R^* for μ . In other words, monotonicity of a solution φ requires that for any matching μ and for any monotonic stretching P^μ of R^* , either μ must be recommended by φ both for (A, P^μ) and (A, R^*) , or μ is not recommended by φ for (A, P^μ) and (A, R^*) .

Monotonicity(MON): For each $p = (A, R^*) \in \mathbf{R}$ and $\mu \in \mathcal{M}(A)$, and for any monotonic stretching P^μ of R^* , we have $\mu \in \varphi(A, P^\mu)$ if and only if $\mu \in \varphi(A, R^*)$

Proposition 3.5 If a solution φ satisfies *IR*, *WPO*, *GF*, *CCON* and *MON* then for any $\mu \in \varphi(A, R)$, if $\mu|_N \in \varphi(N \cup \mu(N), R|_{N \cup \mu(N)})$ for any $N =$

$\{i, j\} \subseteq A$ with $\mu(i) \neq j$, then we have $\mu|_N \in \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$.

Proof. Let φ be a solution satisfying IR, WPO, GF, CCON and MON.

Let $p = (A, R)$ be a problem. Take $\mu \in \varphi(p)$ such that $\mu|_N \in \varphi(N \cup \mu(N), R|_{N \cup \mu(N)})$ for all $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$, we want to show that $\mu|_N \in \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$. Suppose, by contrary, $\mu|_N \notin \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$ for some $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$. Let $N = \{i, j\}$.

Case 1: First we consider the case that $\mu(i) = i$ and $\mu(j) = j$. Wlog, say $i = m_1$ and $j = w_1$. Then $\mu(m_1) = m_1$ and $\mu(w_1) = w_1$. Since $\mu|_N \in \mathcal{IR}(N \cup \mu(N), R|_{N \cup \mu(N)})$ and $\mu|_N \notin \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$, (m_1, w_1) forms a blocking pair. We have $w_1 P_{m_1} m_1$ and $m_1 P_{w_1} w_1$. Then by weak Pareto optimality of φ , m_1 is matched with w_1 we have $\mu|_N \notin \varphi(N \cup \mu(N), R|_{N \cup \mu(N)})$, a contradiction.

Case 2: Next we consider the case where only one agent is self matched. Let $\mu(i) = i$ and $\mu(j) \neq j$. Again, say $i = m_1$ and $j = w_1$. Then $\mu(m_1) = m_1$ and $\mu(w_1) = m_2$. Since $\mu|_N \in \mathcal{IR}(N \cup \mu(N), R|_{N \cup \mu(N)})$ and $\mu|_N \notin \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$, there (m_1, w_1) forms a blocking pair. Then, there are three possible preference profiles:

i)

R_{m_1}	R_{m_2}	R_{w_1}
w_1	$w_1 m_2$	m_1
m_1		$m_2 w_1$

ii)

R_{m_1}	R_{m_2}	R_{w_1}
w_1	$w_1 m_2$	m_1
m_1		m_2
		w_1

iii)

R_{m_1}	P_{m_2}	R_{w_1}
w_1	w_1	m_1
m_1	m_2	m_2
		w_1

For the subcase 2 (i) and 2(ii), let P^μ be a monotonic stretching of R for μ . Note that $\mu|_N \notin S(N \cup \mu(N), P^\mu_{|N \cup \mu(N)})$. By Theorem 3.1. $\mu|_N \notin \varphi(N \cup \mu(N), P^\mu_{|N \cup \mu(N)})$. By MON, $\mu|_N \notin \varphi(N \cup \mu(N), R_{|N \cup \mu(N)})$, a contradiction.

For the Subcase 2(iii), Note that $\mu|_N \notin S(N \cup \mu(N), R_{|N \cup \mu(N)})$. Since $R_{|N \cup \mu(N)}$ is strict, by Theorem 3.1.; $\mu|_N \notin \varphi(N \cup \mu(N), R_{|N \cup \mu(N)})$, a contradiction.

Case 3: Finally, let $\mu(i) \neq i$ and $\mu(j) \neq j$. Again, say $i = m_1$ and $j = m_2$.

Then $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Since $\mu|_N \in IR(N \cup \mu(N), R|_{N \cup \mu(N)})$ and $\mu|_N \notin \mathcal{S}(N \cup \mu(N), R|_{N \cup \mu(N)})$, assume (m_1, w_2) forms a blocking pair. Then we have

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
w_1		m_1	
w_2		m_2	

For any agent who is indifferent between the other agents apply the monotonic stretching to the preferences profile say $p' = (A, P^\mu)$ and we are in the one of the nine cases of Proposition 3.2. For all these cases we know that $\mu|_N \notin \mathcal{S}(N \cup \mu(N), P^\mu|_{N \cup \mu(N)})$. By Theorem 3.1, we have $\mu|_N \notin \varphi(N \cup \mu(N), P^\mu|_{N \cup \mu(N)})$. By MON, $\mu|_N \notin \varphi(N \cup \mu(N), R|_{N \cup \mu(N)})$, a contradiction. ■

Proposition 3.6 *If a solution φ satisfies IR, WPO, GF, CON, CCON and MON then $\varphi(p) \subseteq \mathcal{S}(p)$ for each problem $p \in \mathbf{R}$.*

Proof. Let φ be a solution satisfying IR, WPO, GF, CON, CCON and MON. Let $p = (A, R)$ be a problem. Let $\mu \in \varphi(p)$. By consistency of φ , we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$. By Proposition 3.5, $\mu|_N \in \mathcal{S}(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subseteq A$ with

$\mu(i) \neq j$. Since \mathcal{S} satisfies CCON, we have $\mu \in \mathcal{S}(p)$. ■

Proposition 3.7 *No proper subsolution of the core satisfies CON and MON.*

Proof. Let φ be a subsolution of \mathcal{S} satisfying both CON and MON, let $p = (A, R)$ be a problem and let $\mu \in \mathcal{S}(p)$. Let P^μ be a monotonic stretching of R for μ . Since \mathcal{S} satisfies MON, by MON, $\mu \in \mathcal{S}(A, P^\mu)$. By the same arguments as in the proof of Lemma 3.1, there exists a problem $p' = (A', P')$ such that $\mathcal{S}(p') = \{\mu'\}$, $\mu'_{|A} = \mu$, and (A, P^μ) is a reduced problem of p' at μ . Since $\varphi(p') = \{\mu'\}$ and φ satisfies CON, $\mu \in \varphi(A, P^\mu)$. Since P^μ is a monotonic stretching of R for μ , by MON we have $\mu \in \varphi(A, R)$. This proves that $\varphi \equiv \mathcal{S}$.¹⁴ ■

Theorem 3.3 *The core \mathcal{S} is the unique solution satisfying IR, WPO, GF, CON, CCON and MON.*

Proof. One can easily check that the core \mathcal{S} satisfies the above listed axioms. For the uniqueness, let φ be a solution satisfying these six axioms. Then by Proposition 3.6, $\varphi(p) \subseteq \mathcal{S}(p)$ for each $p \in \mathbf{R}$. Also by Proposition 3.7, no proper subsolution of the core satisfies CON and MON. Hence, $\varphi \equiv \mathcal{S}$.

■

¹⁴This proof is an adaptation of the proof of the Proposition 4.4 of Toda (2006).

Remark 3.3 *Whether the core could be characterized by IR, WPO, GF, CON and CCON (relaxing the monotonicity axiom) is an open question worth to be investigated.*

3.6 Concluding Remarks

We consider a marriage model where agents are allowed to stay selfmatched. First, we discuss how the definition of converse consistency can be adapted to this more general framework. Next, we show that the core is no more the unique solution satisfying individual rationality, Pareto optimality, anonymity, consistency, and converse consistency in this setting. We characterize the core by individual rationality, Pareto optimality, gender fairness, consistency and converse consistency. We also check the independence of these axioms.

Whether our characterization result can be extended to many-to one matching problems under suitable preference restrictions is on our agenda for further research. Finally, we consider what would happen if we further relax the condition that agents have strict preferences over their potential mates. We show that, if we allow indifferences on preferences, there exists no solution satisfying Pareto optimality, anonymity and converse consistency. Toda (2006) characterized the core (of many to one matching problems) by weak

unanimity, consistency, Maskin monotonicity and population monotonicity when indifferences are allowed. In this study, we characterized the core as the unique solution satisfying individual rationality, weak Pareto optimality, gender fairness, consistency, converse consistency and monotonicity.

4 A Maximal Conversely Consistent Subsolution of the Pareto Optimal Solution

4.1 Introduction

For many economic problems, in particular for matching problems consistency and converse consistency are two well-known axioms that are used for axiomatic analysis. In this study, we specially consider converse consistency for matching problems. We know that several solutions, including the Pareto-optimal solution in matching problems, fail to satisfy converse consistency. Thomson (1996) introduces the concept of maximal consistent subsolution of a solution in the general framework. By evaluating it, we maximally reduce the solution so that consistency is satisfied. Since consistency is preserved under unions, the union of consistent subsolutions can be described as the maximal consistent subsolution of a solution. However, union of two conversely consistent solutions may not be conversely consistent. So we can not carry the concept of maximal consistent subsolution when we replace consistency by converse consistency. Therefore, for a solution we define the maximal converse consistent subsolution by using inclusion operator as the maximal converse consistent solution which is included in the solution. We aimed to compute maximal conversely consistent subsolution of the Pareto

optimal solution.

We study two-sided, one-to-one matching problems which is known as marriage problems in a restricted domain where there are equal numbers of agents from each side and being single is not allowed. We consider a well-known solution concept: the Pareto-optimal solution. Pareto-optimal solution in matching problems does not satisfy converse consistency. The main aim of this essay is to compute its maximal conversely consistent subsolution. Specifically, in matching problems, converse consistency is defined as follows: Consider some problem p and some solution φ . Take any matching μ . The requirement is the following: If the reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of two matched pairs, then μ must be a matching recommended by φ at the original problem p . That is the original definition of Sasaki and Toda (1992) where they show that the core is the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse consistency. Some versions of converse consistency are studied in matching problems.¹⁵ As far as we know, in the literature this is the first study on computing the maximal conversely consistent subsolution of a solution. However, there are a few papers

¹⁵For roommate problems Özkal-Sanver (2009), for marriage problems Sasaki and Toda (1992), Özkal-Sanver (2004), Özkal-Sanver (2012), Nizamogullari and Özkal-Sanver (2013).

in the literature computing the minimal consistent extension of solutions¹⁶, as well as the minimal conversely consistent extension of solutions.¹⁷

To compute the maximal conversely consistent subsolution of the Pareto-optimal solution, we introduce the concept of serial men-ordering. For a fixed total order on the set of men it turns out that maximal converse consistent subsolution of the Pareto optimal solution is a correspondence which associates with each problem the set of all Pareto optimal matchings when the number of agents in the society larger than four, and the set of all stable matchings and the set of matchings which are chosen by serial men-ordering solution when there are four agents or less than four agents. So, maximal converse consistent subsolution of the Pareto optimal solution is not unique. For different orders on the set of men we will get different subsolutions. Also we show that this result is not true for general domains where there are unequal number of men and women and being single is allowed.

This essay proceeds as follows: Section 4.2 presents the basic notions and axioms. Section 4.3 introduces the serial men-ordering solution and gives our results. Finally, section 4.4 makes some closing remarks.

¹⁶See Thomson (1994*a*), Bevia (1996) and Korthues (2000).

¹⁷See Thomson (1996, 1999), and Özkal-Sanver (2012)

4.2 Model and Axioms

We consider matching problems with equal numbers of men and women. Let M and W be two disjoint universal sets. Let M be a nonempty and finite subset of M . Similarly, let W be a nonempty and finite subset of W . Let $A = M \cup W$ be the set of agents. We assume that $\#M = \#W$ where $\#$ denotes the cardinality of a set.

For each agent $i \in A$, the set of potential mates of i , denoted by $A(i)$, is defined as

$$A(i) = \begin{cases} W & \text{if } i \in M \\ M & \text{if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by P_i . Let \mathcal{P} denote the set of all possible preference profiles $P = (P_i)_{i \in A}$.

Let \succ be a total order on the set of men M . For any woman $w \in W$ we say that P_w is **coherent** to \succ if the following condition is satisfied: For all $m, m' \in M$, $mP_w m'$ if and only if $m \succ m'$.

Similarly, for a given total order \succ on the set of women W , for any man $m \in M$, we say that P_m is **coherent** to \succ if the following condition is satisfied: For all $w, w' \in W$, $wP_m w'$ if and only if $w \succ w'$.

A **matching** is a function $\mu : A \rightarrow A$ such that for each $i \in A$, we have $\mu(i) \in A(i)$ and for each pair $\{j, k\} \subset A$, $\mu(j) = k$ implies $\mu(k) = j$. Agent $\mu(i)$ is the mate of agent i under matching μ . Throughout the essay,

we sometimes write $(m, w) \in \mu$ to denote $\mu(m) = w$. Let $\mathcal{M}(A)$ denote the set of all matchings for A . A **(matching) problem** is a pair $p = (A, P)$, where A is the set of agents and P is the preference profile. Let \mathbf{P} denote the set of all problems.

A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if for all $i \in A$, $\mu(i) P_i i$ or $\mu(i) = i$. A pair of agents $\{i, j\}$ **blocks** a matching $\mu \in \mathcal{M}(A)$ if $j P_i \mu(i)$ and $i P_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if it is individually rational for p and there is no pair $\{i, j\} \subset A$ blocking μ . Let $S(p)$ denote the set of all stable matchings. Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$, μ **dominates** μ' if there exists a coalition $K \subseteq A$ such that for all $i \in K$, $\mu(i) \in K$ and $\mu(i) P_i \mu'(i)$. A matching μ is **undominated** if there exists no matching $\mu' \in \mathcal{M}(A)$ which dominates μ . The **core** of p is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor (1990)). Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ **Pareto dominates** μ' if for all $i \in A$, $\mu(i) P_i \mu'(i)$ whenever $\mu(i) \neq \mu'(i)$. A matching $\mu \in \mathcal{M}(A)$ is **Pareto optimal** for p if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates μ . Let $PO(p)$ denote the set of all Pareto optimal matchings.

A **solution** is a correspondence φ that associates with each problem $p = (A, P)$ a nonempty subset of $\mathcal{M}(A)$. Let Φ be the set of solutions. The

core solution is the correspondence S that associates with each problem p its set of stable matchings $S(p)$.

Given a problem $p = (A, P)$ and a subset of set of agents A' , a **reduced problem** of p with respect to A' is a problem where the preference profile P is restricted to agents in A' . Formally; for all $p = (A, P) \in \mathbf{P}$ and all $A' \subseteq A$, $p' = (A', P|_{A'}) \in \mathbf{P}$ is the reduced problem of p with respect to A' . Write $\mu(N) = \{\mu(i) : i \in N\}$. Given a matching $\mu \in \mathcal{M}(A)$ and a subset of agents $N \subseteq M \cup W$, a **reduced matching** of μ with respect to N is a matching $\mu|_N : N \cup \mu(N) \rightarrow N \cup \mu(N)$ such that for all $i \in N \cup \mu(N)$, $\mu|_N(i) = \mu(i)$.

Now, we are ready to define our axioms on solutions:

The first axiom requires that at each problem the solution recommends among Pareto optimal matchings:

Pareto optimality (PO): For each $p \in \mathbf{P}$, $\varphi(p) \subseteq PO(p)$.

The second axiom requires that renaming men among men and renaming women among women do not change the result.

Anonymity (AN): For all $A = M \cup W$ and all $A' = M' \cup W'$ with $\#M = \#M'$ and $\#W = \#W'$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = M'$ and $\pi(W) = W'$. For all $p = (A, P)$, let P' be such that for all $i \in A$ and all $j, k \in A(i)$, $jP_i k$ if and only if $\pi(j)P'_{\pi(i)}\pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$.

If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

Now we define consistency axiom. Consider some problem p and some solution φ . Take any matching μ recommended by φ at p . If the reduced matching of μ with respect to each subgroup of matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup of matched pairs, then we say that the solution φ is consistent.

Consistency: For each $p = (A, P) \in \mathbf{P}$, each $\mu \in \varphi(p)$, and each $N \subseteq A$, we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$.

The next axiom is central to our analysis. Consider some problem p and some solution φ . Take any matching μ . The requirement is that if the reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of these two matched pairs, then μ must be a matching recommended by φ at the original problem p .

Converse Consistency: For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$, then $\mu \in \varphi(p)$.

Given a solution φ , the minimal converse consistent extension of φ , denoted by $MCCE_\varphi$, is defined by Thomson (1996) as follows¹⁸:

¹⁸Converse consistency is preserved under intersections.

$$MCCE_\varphi = \bigcap_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is converse consistent}\}$$

Given a solution φ that contains at least one consistent subsolution, the maximal consistent subsolution of φ , denoted by $MXCS_\varphi$, is defined by Thomson (1996) as follows:

$$MXCS_\varphi = \bigcup_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \subseteq \varphi, \psi \text{ is consistent}\}$$

By definition of consistency it is direct consequence that consistency is preserved under unions. So the above definition is well defined. However, union of two conversely consistent solutions may not be conversely consistent. In the below example we show that converse consistency is not preserved under unions.

Example 4.1 *It is a well-known result that for any problem p , $\mathcal{S}(p)$ is conversely consistent. It is shown by Özkal-Sanver (2013) for any problem p , $MB(p)$ ¹⁹ is conversely consistent. But $\mathcal{S} \cup MB$ is not conversely consistent. To see that consider the problem $p = (M \cup W, P)$ where $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and P as follows:*

¹⁹Men barter-proof solution MB is defined by Özkal-Sanver (2012) in the following way: A matching is men-barterproof whenever there is no such a pair of men who benefit from switching their mates among themselves. More formally, a matching $\mu \in \mathcal{M}(A)$ is **men-barterproof** for $p = (A, P) \in \mathbf{P}$, if there exists no pair of men $\{m, m'\} \subset M$ bartering at μ . Let $MB(p)$ denote the set of the men-barterproof matchings for p . The **men-barterproof solution** is the correspondence MB that associates with each problem p the set of men-barterproof matchings $MB(p)$

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_2	w_1	w_4	w_4	m_1	m_2	m_3	m_3
w_1	w_2	w_3	w_1	m_3	m_3	m_1	m_4
w_3	w_3	w_1	w_2	m_4	m_4	m_2	m_1
w_4	w_4	w_2	w_3	m_2	m_1	m_4	m_2

Let $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Since (m_3, w_4) is a blocking pair and (m_1, m_2) barter we have $\mu \notin (\mathcal{S} \cup MB)(p)$. But any two pair restriction of μ is in $\mathcal{S} \cup MB$. Hence $\mathcal{S} \cup MB$ is not conversely consistent.

Therefore we can not carry the concept of maximal consistent subsolution when the point of departure is converse consistency. For a solution φ , we define the maximal converse consistent subsolution of φ , denoted by $MXCCS_\varphi$, as follows: For each $p = (A, P) \in \mathbf{P}$, we have $MXCCS_\varphi(p) \subsetneq \varphi(p)$ and there is no conversely consistent solution ψ with $MXCCS_\varphi(p) \subsetneq \psi(p) \subsetneq \varphi(p)$.

As we stated before the main aim of this essay is to compute its maximal conversely consistent subsolution. It is a well-known result that the Pareto-optimal solution fails to satisfy converse consistency. To see that consider the following example:

Example 4.2 *The Pareto-optimal solution does not satisfy converse consis-*

tency.

Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let P be defined as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_2	w_3	w_4	m_4	m_1	m_2	m_3
w_2	w_3	w_4	w_1	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_3	m_4	m_1
w_4	w_1	w_2	w_3	m_1	m_2	m_3	m_4

Let $p = (M \cup W, P)$. Consider the matching where $\mu(m_1) = w_1, \mu(m_2) = w_4, \mu(m_3) = w_2, \mu(m_4) = w_3$. There are six different subproblems with $\#N = 2$. Let $N_1 = \{m_1, m_2\}, N_2 = \{m_1, m_3\}, N_3 = \{m_1, m_4\}, N_4 = \{m_2, m_3\}, N_5 = \{m_2, m_4\}, N_6 = \{m_3, m_4\}$. One can easily check that $\mu|_{N_i} \in PO(N_i \cup \mu(N_i), P|_{N_i \cup \mu^*(N_i)})$ for all $i = 1, 2, \dots, 6$. However, $\mu \notin PO(p)$.²⁰

4.3 Maximal Conversely Consistent Subsolution of the Pareto-Optimal Solution

The maximal conversely consistent subsolution of the Pareto optimal solution whenever it is anonymous is a rule which associates with each problem the set of all Pareto optimal matchings when there are more than four agents and

²⁰ $\mu^* = \{(m_1, w_1), (m_2, w_3), (m_3, w_4), (m_4, w_2)\}$ Pareto dominates μ .

all stable matchings otherwise. We state this in the following proposition

Proposition 4.1 *The maximal conversely consistent and anonymous sub-solution of the Pareto optimal solution is*

$$MXCCS_{\mathcal{PO}}(M \cup W, P) = \begin{cases} \mathcal{S}(M \cup W, P) & \text{if } \#M = \#W = 2 \\ \mathcal{PO}(M \cup W, P) & \text{otherwise} \end{cases}$$

Proof. Sasaki and Toda (1992), Lemma2 states that if a solution φ satisfies PO, CCON and AN then $\varphi(M, W, P) \subseteq \mathcal{S}(M \cup W, P)$ for all (M, W, P) with $\#M = \#W = 2$ ■

Before to proceed we define the *serial men ordering solution* based on the names (or indexes) of men in the society. Let $p \equiv (M \cup W, P)$ be a problem and \succ be a fixed order on the set of men M . *Serial men ordering solution* SMO_{\succ} with respect to \succ is defined in the following way: Let $\mu \in \mathcal{M}(M \cup W)$ be any matching for this society and \succ_{μ} be an ordering on the set of women W defined as $w_i \succ_{\mu} w_j$ if and only if $\mu(w_j) \succ \mu(w_i)$. Let $m \in M$ be any agent in M we write $t(P_m; W)$ for the top-ranked agent in W with respect to the preference profile P_m of agent m , hence $t(P_m; W)P_m w$ for all $w \in W$. Similarly, $t(P_m; W)$ is defined for any $w \in W$. Then we look at the top-ranked agents and define s_w as the cardinality of of the agents who ranked them as the best. That is $s_w = \#\{m : t(P_m; W) = w\}$ for any $w \in W$. Similarly, s_m is defined for any $m \in M$. Let $\arg \max_{i \in M \cup W} s_i = \{i \in M \cup W \mid \text{for all } i \in M \cup W, s_j \leq s_i\}$. We consider any agent $i^* \in \arg \max_{i \in M \cup W} s_i$ if P_{i^*} is coherent to order \succ

or if P_{i^*} is coherent to order \succ_{μ} then $\mu \in SMO_{\succ}$.²¹

Remark 4.1 SMO_{\succ} does not satisfy anonymity. To see that consider the problem $p = (A, P)$ where $A = \{m_1, m_2, w_1, w_2\}$ and P as follows:

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_1	w_1	m_1	m_1
w_2	w_2	m_2	m_2
m_1	m_2	w_1	w_2

Assume that $m_1 \succ m_2$. Then $SMO_{\succ}(p) = \{\mu_1, \mu_2\}$ where $\mu_1(m_1) = w_1$, $\mu_1(m_2) = w_2$, $\mu_2(m_1) = w_2$ and $\mu_2(m_2) = w_1$. Let $\pi(m_1) = m_2$, $\pi(m_2) = m_1$, $\pi(w_1) = w_1$ and $\pi(w_2) = w_2$. Let P' be the preference profile in the permuted problem. Then P' is the following preference profile:

P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}
w_1	w_1	m_2	m_2
w_2	w_2	m_1	m_1
m_1	m_2	w_1	w_2

Note that $\pi_{\mu_2} = \{(m_1, w_1), (m_2, w_2)\}$. Since $m_1 \succ m_2$ we have $w_2 \succ_{\pi_{\mu_2}} w_1$. Note that $t(P_{m_1}; W) = w_1$, $t(P_{m_2}; W) = w_1$, $t(P_{w_1}; W) = m_2$ and

²¹Note that SMO_{\succ} is not well defined solution. This solution is useful and meaningful for the societies where there are two men and two women.

$t(P_{w_2}; W) = m_2$ then $\arg \max_{i \in M \cup W} s_i = \{w_1, m_2\}$ but P_{w_1} is not coherent to \succ and P_{m_2} is not coherent to $\succ_{\pi_{\mu_2}}$. Hence, $\pi_{\mu_2} \notin SMO_{\succ}(A, P')$.

Theorem 4.1 *A maximal conversely consistent subsolution of the Pareto optimal solution for a fixed order \succ on M is*

$$MXCCS_{\mathcal{PO}}(M \cup W, P) = \begin{cases} \mathcal{S}(M \cup W, P) \cup SMO_{\succ}(M \cup W, P) & \text{if } \#M = \#W = 2 \\ \mathcal{PO}(M \cup W, P) & \text{otherwise} \end{cases}$$

Proof. First we will show that this solution is converse consistent.

Assume $MXCCS_{\mathcal{PO}}$ is not converse consistent. Then there is a problem $p = (M \cup W, P)$ and a matching $\mu \in \mathcal{M}(M \cup W)$ such that $\mu|_N \in MXCCS_{\mathcal{PO}}(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subset M \cup W$ with $\mu(i) \neq j$. but $\mu \notin MXCCS_{\mathcal{PO}}(p)$, which means $\mu \notin \mathcal{PO}(M \cup W, P)$. Then there exists $\mu' \in \mathcal{M}(A)$ Pareto dominating μ . Let \succ be a fixed order on the set of men M . There exists at least an agent $i \in A$ such that $\mu'(i) P_i \mu(i)$. Let m be the first agent with respect to \succ who is strictly better off. Let $\mu(m) = w$ and $\mu'(w) = m'$, $\mu(m') = w'$. Let $N = \{m, m'\}$. Then $P|_{N \cup \mu(N)}$ is the following preference profile

P_m	$P_{m'}$	P_w	$P_{w'}$
	w	m'	
	w'	m	

Then there are four possibilities for the other agents:

i)		P_m	$P_{m'}$	P_w	$P_{w'}$
		w	w	m'	m'
		w'	w'	m	m

First note that $\mu|_N \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Also, $\mu|_N \notin SMO_{\succ}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Since $m \succ m'$ then we have $w' \succ_{\mu} w$. Note that $t(P_m; W) = w$ and $t(P_{m'}; W) = w$, $t(P_w; W) = m'$ and $t(P_{w'}; W) = m'$, then $\arg \max_{i \in N \cup \mu(N)} s_i = \{w, m'\}$ but P_w is not coherent to \succ and $P_{m'}$ is not coherent to \succ_{μ} . Hence $\mu|_N \notin MXCCS_{\mathcal{PO}}(N \cup \mu(N), P_{|N \cup \mu(N)})$.

ii)		P_m	$P_{m'}$	P_w	$P_{w'}$
		w	w	m'	m
		w'	w'	m	m'

First note that $\mu|_N \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Also, $\mu|_N \notin SMO_{\succ}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Since $m \succ m'$ then we have $w' \succ_{\mu} w$. Note that $t(P_m; W) = w$ and $t(P_{m'}; W) = w$, then $\arg \max_{i \in N \cup \mu(N)} s_i = \{w\}$ but P_w is not coherent to \succ . Hence $\mu|_N \notin MXCCS_{\mathcal{PO}}(N \cup \mu(N), P_{|N \cup \mu(N)})$.

iii)

P_m	$P_{m'}$	P_w	$P_{w'}$
w' w		m' m'	
w	w'	m	m

First note that $\mu|_N \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Also, $\mu|_N \notin SMO_{\succ}(N \cup \mu(N), P_{|N \cup \mu(N)})$. Since $m \succ m'$ then we have $w' \succ_{\mu} w$. Note that $t(P_w; W) = m'$ and $t(P_{w'}; W) = m'$, then $\arg \max_{i \in N \cup \mu(N)} s_i = \{m'\}$ but $P_{m'}$ is not coherent to \succ_{μ} . Hence $\mu|_N \notin MXCCS_{\mathcal{PO}}(N \cup \mu(N), P_{|N \cup \mu(N)})$.

iv)

P_m	$P_{m'}$	P_w	$P_{w'}$
w' w		m' m	
w	w'	m	m'

In this case $\mu|_N \notin \mathcal{PO}(N \cup \mu(N), P_{|N \cup \mu(N)})$.

Now, we have to show $MXCCS_{\mathcal{PO}}$ is maximal. Assume there is a conversely consistent solution ψ such that $MXCCS_{\mathcal{PO}} \subsetneq \psi \subsetneq \mathcal{PO}$. Then there is a problem $p = (M \cup W, P)$ with $\#M = \#W = 2$ and a Pareto optimal matching $\mu \in \psi(p)$ but $\mu \notin MXCCS_{\mathcal{PO}}(p)$. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. And assume that $m_1 \succ m_2$. Then we have one of the six

cases of the following preference profiles;²²

1)	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_1</td> <td>w_1</td> <td>m_2</td> <td>m_1</td> </tr> <tr> <td>w_2</td> <td>w_2</td> <td>m_1</td> <td>m_2</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_1	w_1	m_2	m_1	w_2	w_2	m_1	m_2	m_1	m_2	w_1	w_2		<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_1</td> <td>w_1</td> <td>m_2</td> <td>m_2</td> </tr> <tr> <td>w_2</td> <td>w_2</td> <td>m_1</td> <td>m_1</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_1	w_1	m_2	m_2	w_2	w_2	m_1	m_1	m_1	m_2	w_1	w_2	2)
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_1	w_1	m_2	m_1																																	
w_2	w_2	m_1	m_2																																	
m_1	m_2	w_1	w_2																																	
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_1	w_1	m_2	m_2																																	
w_2	w_2	m_1	m_1																																	
m_1	m_2	w_1	w_2																																	
3)	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_2</td> <td>w_1</td> <td>m_2</td> <td>m_2</td> </tr> <tr> <td>w_1</td> <td>w_2</td> <td>m_1</td> <td>m_1</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_2	w_1	m_2	m_2	w_1	w_2	m_1	m_1	m_1	m_2	w_1	w_2	4)	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_2</td> <td>w_2</td> <td>m_1</td> <td>m_2</td> </tr> <tr> <td>w_1</td> <td>w_1</td> <td>m_2</td> <td>m_1</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_2	w_2	m_1	m_2	w_1	w_1	m_2	m_1	m_1	m_2	w_1	w_2	
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_2	w_1	m_2	m_2																																	
w_1	w_2	m_1	m_1																																	
m_1	m_2	w_1	w_2																																	
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_2	w_2	m_1	m_2																																	
w_1	w_1	m_2	m_1																																	
m_1	m_2	w_1	w_2																																	
5)	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_1</td> <td>w_2</td> <td>m_2</td> <td>m_2</td> </tr> <tr> <td>w_2</td> <td>w_1</td> <td>m_1</td> <td>m_1</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_1	w_2	m_2	m_2	w_2	w_1	m_1	m_1	m_1	m_2	w_1	w_2	6)	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="border-bottom: 1px solid black;">P_{m_1}</th> <th style="border-bottom: 1px solid black;">P_{m_2}</th> <th style="border-bottom: 1px solid black;">P_{w_1}</th> <th style="border-bottom: 1px solid black;">P_{w_2}</th> </tr> </thead> <tbody> <tr> <td>w_2</td> <td>w_2</td> <td>m_2</td> <td>m_2</td> </tr> <tr> <td>w_1</td> <td>w_1</td> <td>m_1</td> <td>m_1</td> </tr> <tr> <td>m_1</td> <td>m_2</td> <td>w_1</td> <td>w_2</td> </tr> </tbody> </table>	P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}	w_2	w_2	m_2	m_2	w_1	w_1	m_1	m_1	m_1	m_2	w_1	w_2	
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_1	w_2	m_2	m_2																																	
w_2	w_1	m_1	m_1																																	
m_1	m_2	w_1	w_2																																	
P_{m_1}	P_{m_2}	P_{w_1}	P_{w_2}																																	
w_2	w_2	m_2	m_2																																	
w_1	w_1	m_1	m_1																																	
m_1	m_2	w_1	w_2																																	

Assume we are in the first case. For this preference profile $MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_2), (m_2, w_1)\}$ then we have $\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in

²²For the other ten cases $MXCCS_{\mathcal{PO}}$ chooses all the Pareto optimal matchings.

the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_1	w_4	w_2	m_2	m_4	m_1	m_3
w_1	w_4	w_3	w_4	m_3	m_1	m_3	m_4
w_2	w_2	w_1	w_1	m_1	m_2	m_4	m_1
w_4	w_3	w_2	w_3	m_4	m_3	m_2	m_2

Let $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_1	w_1	m_1	m_3
m_1	m_3	w_1	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_1$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCSP_{\mathcal{O}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_4	m_1	m_4
w_4	w_1	m_4	m_1
m_1	m_4	w_1	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_2	w_3	m_2	m_3
w_3	w_2	m_3	m_2
m_2	m_3	w_2	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_4	w_2	m_4	m_4
w_2	w_4	m_2	m_2
m_2	m_4	w_2	w_4

Since $m_4 \succ m_2$ then we have $w_2 \succ_{\mu^*} w_4$. Note that $t(P_{w_2}; M) = m_4$ and $t(P_{w_4}; M) = m_4$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_4\}$ and P_{m_4} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_3, w_4\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Similar arguments work for the other five cases. ²³ ■

Remark 4.2 *This solution defined in the above theorem does not satisfy converse consistency when some agents are allowed to be single. Assume to get a contradiction, in this more general domain $MXCCS_{\mathcal{PO}}$ satisfies converse consistency. Then consider the following problem $p = (M \cup W, P)$ where $M \cup W = \{m_1, m_2, m_3, w_1, w_2, w_3\}$ and P as follows:*

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_1	w_2	w_3	m_1	m_2	m_3
w_2	w_3	w_2	m_3	m_1	m_2
m_1	w_1	w_1	m_2	m_3	w_3
w_3	m_2	m_3	w_1	w_2	m_1

with an order $m_1 \succ m_2 \succ m_3$.

Let $\mu^* = \{(m_1), (m_2, w_1), (m_3, w_2), (w_3)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$, we have $\mu^* \notin MXCCS_{\mathcal{PO}}(M, W, P)$. However, $\mu^*_{|N'} \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ for each $N' = \{i, j\} \subset M \cup W$.

²³For details, see the appendix II.

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_1}	P_{m_2}	P_{w_1}
w_1	w_1	m_1
m_1	m_2	m_2
		w_1

Note that $t(P_{m_1}; W) = w_1$ and $t(P_{m_2}; W) = w_1$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\}$ and P_{w_1} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_1}	P_{m_3}	P_{w_2}
w_2	w_2	m_1
m_1	m_3	m_3
		w_2

Note that $t(P_{m_1}; W) = w_2$ and $t(P_{m_3}; W) = w_2$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\}$ and P_{w_2} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next letting $N' \cup \mu^*(N') = \{m_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_1}	P_{w_3}
m_1	w_3
w_3	m_1

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_2\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}
w_2	w_2	m_3	m_2
w_1	w_1	m_2	m_3
m_2	m_3	m_2	w_2

Note that $t(P_{m_2}; W) = w_2$ and $t(P_{m_3}; W) = w_2$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\}$ and P_{w_2} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next letting $N' \cup \mu^*(N') = \{m_2, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_2}	P_{w_1}	P_{w_3}
w_3	m_2	m_2
w_1	w_1	w_3
m_2		

Note that $t(P_{w_1}; M) = m_2$ and $t(P_{w_3}; M) = m_2$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_2\}$

and P_{m_2} is coherent to \succ_{μ^*} since $w_3 \succ_{\mu^*} w_1$. Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_2, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}$ is

P_{m_3}	P_{w_2}	P_{w_3}
w_3	m_3	m_3
w_2	w_1	w_3
m_2		

Note that $t(P_{w_2}; M) = m_3$ and $t(P_{w_3}; M) = m_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_3\}$

and P_{m_3} is coherent to \succ_{μ^*} since $w_3 \succ_{\mu^*} w_2$. Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in MXCCS_{\mathcal{PO}}(M, W, P)$, which gives a contradiction.

4.4 Concluding Remarks

In this essay, we study two sided one-to-one matching problems where there are equal numbers of agents from each sides. We compute a maximal conversely consistent subsolution of the Pareto optimal solution as a correspondence which associates with each problem the set consisting of all the Pareto optimal matchings if there are at least three men and three women, all stable

matchings and pareto optimal matchings which are chosen by serial men-ordering solution for a fixed order on the set of men. if there are two men and two women. Therefore, it turns out that maximal conversely consistent subsolution of Pareto optimal rule is not unique. We also show that failure of this result on general domains.

5 Consistent Enlargements of the Core in Roommate Problems

Duygu Nizamogulları and İpek Özkal-Sanver

5.1 Introduction

The consistency axiom is introduced to the matching theory literature by Sasaki and Toda (1992) who characterize the core of marriage (two sided, one-to-one matching) problems as the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse consistency. Their analysis is followed by Sasaki (1995) and Toda (2005) who give two different characterizations of the core of assignment problems with indivisible goods and money, each of which uses consistency as one of the axioms. In a similar way, Toda (2006) uses consistency in characterizing the core of many-to-one matching problems. All of these characterizations are in environments which ensure the nonemptiness of the core. We consider roommate (one-sided matching) problems where, as Gale and Shapley (1962) show, the core may be empty. There is a huge literature considering core extensions which are solutions coinciding with the core whenever it is non-empty. Given the critical role of consistency in characterizing the core of various types of match-

ing problems, Özkal-Sanver (2010) aimed at characterizing the class of consistent core extensions. Nevertheless, she ended up with an impossibility: There exists no refinement of a core extension which is consistent, a fortiori no consistent core extension. This result covers several solution concepts of the literature. For instance, absorbing matchings (Inarra et al, 2008a), maximum stable matchings (Tan 1990), almost stable matchings (Abraham et al. 2006), P-Stable matchings (Inarra et al, 2008b), stochastically stable matchings (Klaus et al, 2008) are all core extensions. On the other hand, a solution which picks the median stable matchings (Serhuraman and Teo 2001, Klaus and Klijn 2009) whenever the core is nonempty, is not a core extension, but a pseudo refinement of the core.

Thomson (1990) and Thomson (1996) introduce the concept of minimal consistent extension of well-defined solution in the general framework. Our aim is to compute consistent enlargements of the core. By computing it, we evaluate the extent to which the core would have to be expanded in order to be well-defined and consistent. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. We also show that for any fixed order on the set of agents the solution which picks all stable matchings and the serial dictatorship matching with respect to this order is a minimal consistent enlargement of the core. Since for different orders there may be different

enlargements, minimal consistent core enlargement is not unique.

Section 5.2 presents the model. Section 5.3 introduces the axioms we consider. Section 5.4 states our results.

5.2 Model

We consider roommate problems. We write \mathbf{A} for the universal set of agents. We assume \mathbf{A} to be countably infinite. Let A be a nonempty and finite subset of \mathbf{A} . For each agent $i \in A$ the set of potential mates of i equals A . The preference of $i \in A$ over A is a complete, transitive and antisymmetric binary relation P_i . Let \mathcal{P} denote the set of all possible preference profiles $P \equiv (P_i)_{i \in A}$.

A **matching** for a set of agents A is a function $\mu : A \rightarrow A$ such that for all $j, k \in A$, $\mu(j) = k$ implies $\mu(k) = j$. Here, $\mu(i)$ is the **mate of agent i** at μ . We say that agent i is self-matched at μ if $\mu(i) = i$. Let $\mathcal{M}(A)$ denote the set of all matchings for A .

A **(roommate) problem** is a pair $p \equiv (A, P)$, where A is an arbitrary set of agents and P is the profile of their preferences over potential mates in A . Let \mathbf{P} denote the set of all problems.

A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if and only if for all $i \in A$, $\mu(i) P_i i$ or $\mu(i) = i$. Let $\mathcal{I}(p)$ denote the set of all such matchings.

A pair of agents (i, j) with $i \neq j$, **blocks a matching** $\mu \in \mathcal{M}(A)$ if and only if $j P_i \mu(i)$ and $i P_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if and only if it is individually rational for p and there is no pair (i, j) blocking μ . Let $\mathcal{S}(p)$ denote the set of all such matchings. A matching $\mu \in \mathcal{M}(A)$ is **in the core** of p if there is no $A' \subseteq A$ such that $\bigcup_{i \in A'} \{\mu(i)\} = A'$ and $\mu' P_i \mu$ for all $i \in A'$. The **core** of p is the set $\mathcal{CO}(p)$ of all matchings which are in the core of p . As in marriage model, the core of p equals the set of stable matchings at p . As Gale and Shapley (1962) show the core may be empty.

A matching $\mu \in \mathcal{M}(A)$ **Pareto dominates** a matching $\mu' \in \mathcal{M}(A)$ if and only if for all $i \in A$, $\mu(i) R_i \mu'(i)$ and for some $j \in A$, $\mu(j) P_j \mu'(j)$. A matching $\mu \in \mathcal{M}(A)$ is **Pareto optimal** for p if and only if there is no $\mu' \in \mathcal{M}(A)$ Pareto dominating μ . Let $\mathcal{PO}(p)$ denote the set of all such matchings.

A **solution** is a correspondence φ that associates with each problem $p \equiv (A, P) \in \mathbf{P}$ a subset of $\mathcal{M}(A)$.²⁴ A **well-defined solution** is a correspondence φ that associates with each problem $p \equiv (A, P) \in \mathbf{P}$ a non-empty subset of $\mathcal{M}(A)$. A **core extension** is a solution φ such that for any $p \in \mathbf{P}$ with $\mathcal{S}(p) \neq \emptyset$, $\varphi(p) = \mathcal{S}(p)$.

Now, we define axioms for solutions:

First, at each problem the solution only recommends Pareto optimal matchings:

²⁴For some problems $p \equiv (A, P) \in \mathbf{P}$, $\varphi(p)$ can be empty set.

Pareto Optimality [PO]: For all $p \in \mathbf{P}$, $\varphi(p) \subseteq \mathcal{PO}(p)$.

To introduce the other axioms, we need some further definitions. Given a problem $p \equiv (A, P)$ and a subset of agents A' of the original set of agents A , for each agent $i \in A'$, define a preference relation over A' , denoted by P'_i , such that for all $i, j, k \in A'$, $j P'_i k$ if and only if $j P_i k$. Then, set $P_{|A'} \equiv (P'_i)_{i \in A'}$. For all $p \equiv (A, P) \in \mathbf{P}$ and all proper subsets $A' \subset A$, $p' \equiv (A', P_{|A'}) \in \mathbf{P}$ is the **reduced problem** of p with respect to A' . A **matched pair** under μ is simply $(i, \mu(i))$, and $\mu(i)$ may be equal to i . Writing $\mu(A') = \{\mu(i) : i \in A'\}$, next define $\mu_{|A'}$ by $\mu_{|A'} : A' \cup \mu(A') \longrightarrow A' \cup \mu(A')$ such that $\mu_{|A'}(i) = \mu(i)$ for all $i \in A' \cup \mu(A')$.

Consider some problem p and some solution φ . Let μ be a matching recommended by φ at p . We require that the restriction of μ to each subgroup of matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup:

Consistency: For all $p \equiv (A, P) \in \mathbf{P}$ with $\varphi(p) \neq \emptyset$ and all $\mu \in \varphi(p)$, $\mu_{|A'} \in \varphi(A' \cup \mu(A'), P_{|A' \cup \mu(A')})$ for all proper subsets $A' \subset A$.²⁵

Weak Consistency: For all $p \equiv (A, P) \in \mathbf{P}$ with $\varphi(p) \neq \emptyset$ and all $\mu \in \varphi(p)$, then either $\varphi(A' \cup \mu(A'), P_{|A' \cup \mu(A')}) = \emptyset$ or $\mu_{|A'} \in \varphi(A' \cup \mu(A'), P_{|A' \cup \mu(A')})$ for all proper subsets $A' \subset A$.²⁶

²⁵Note that, $(A' \cup \mu(A'), P_{|A' \cup \mu(A')})$ is simply the reduced problem of p with respect to $A' \cup \mu(A')$.

²⁶Note that, consistency implies weak consistency. But converse is not true. To see that

Now we define consistent enlargements and minimal consistent enlargements of a solution.

Given a solution φ , its **enlargement** E_φ is a well-defined solution such that for any $p \in \mathbf{P}$ we have $\varphi(p) \subseteq E_\varphi(p)$. Its **consistent enlargement** CE_φ is an enlargement which satisfies consistency. Its **minimal consistent enlargement** MCE_φ is an enlargement such that there is no consistent enlargement ψ of φ with $\varphi(p) \subseteq \psi(p) \subseteq MCE_\varphi(p)$ for any $p \in \mathbf{P}$.

In particular, a **consistent core enlargement** CE_S is a well-defined consistent solution which picks at least for some problem more than the core whenever it is non-empty and a **minimal consistent core enlargement** MCE_S is a minimal consistent well-defined solution with this property.

5.3 Results

We define the sub operation \vee on the set of solutions as a solution $\varphi_1 \vee \varphi_2$ that satisfies the following: If $\mu \in (\varphi_1 \vee \varphi_2)$ then either $\mu \in \varphi_1(p)$ or $\mu \in \varphi_2(p)$ for any problem $p = (A, P) \in \mathbf{P}$. The sub operator \vee is closed for consistent solutions. We state this result in the following proposition.

Proposition 5.1 *For any two consistent solutions φ_1 and φ_2 , the solution*

consider a rule that is defined as follows:

$$\varphi(A, P) = \begin{cases} PO(A, P) & \text{if } |A| > 4 \\ \emptyset & \text{otherwise.} \end{cases}$$

φ satisfies weak consistency but it is not consistent.

$\varphi_1 \vee \varphi_2$ is consistent.

Proof. Let φ_1 and φ_2 be two solutions that satisfy consistency and let $p = (A, P)$ be a problem with $(\varphi_1 \vee \varphi_2)(A, P) \neq \emptyset$. Take any $\mu \in (\varphi_1 \vee \varphi_2)(A, P)$. Then either $\mu \in \varphi_1(A, P)$ or $\mu \in \varphi_2(A, P)$. If $\mu \in \varphi_i(A, P)$ then by consistency of φ_i we have $\mu|_{A'} \in \varphi_i(A' \cup \mu(A'), P|_{A' \cup \mu(A)})$ for all proper subsets $A' \subset A$ for any $i \in \{1, 2\}$. Hence, $\mu|_{A'} \in (\varphi_1 \vee \varphi_2)(A' \cup \mu(A'), P|_{A' \cup \mu(A)})$ for all proper subsets $A' \subset A$, which means $\varphi_1 \vee \varphi_2$ is a consistent solution. ■

Remark 5.1 For any two weakly consistent solutions φ_1 and φ_2 , the solution $\varphi_1 \vee \varphi_2$ may not be weakly consistent. To show this, in example 5.1 we consider the following weakly consistent (indeed, well-defined and consistent) solution: the serial dictatorship solution based on the names (or indexes) of agents.

We define serial dictatorship solution as follows: For any problem $p \equiv (A, P)$ let the agent with the lowest index in A match with his best agent in A , if any. Remove the agents from the set A and let the agent with the lowest index in the remaining set match with his best agent in this set etc. More formally, fix a total order \succ of agents in A . Take any arbitrary finite set $A = \{a_1, \dots, a_{|A|}\}$ where, without loss of generality, $a_k \succ a_l$ holds for all $k, l \in \{1, \dots, |A|\}$ with $k < l$. For all $p \equiv (A, P)$, for all non-empty $A' \subseteq A$, and for all $a_n \in A$, we write $t(P_{a_n}; A')$ for the top-ranked agent

in A' with respect to the preference P_{a_n} of agent a_n , hence $t(P_{a_n}; A') R_{a_n} a_l$ for all $a_l \in A'$. The **serial dictatorship matching** μ^\succ (with respect to \succ) for $p \equiv (A, P)$ is recursively defined as follows: $\mu^\succ(a_1) = t(P_{a_1}; A)$. For any $n \in \{2, \dots, |A|\}$, write $A_n = A \setminus \bigcup_{k=1}^{n-1} \{a_k, \mu^\succ(a_k)\}$. Note that either $a_n \notin A_n$, hence $a_n = \mu^\succ(a_k)$ for some $k < n$ or $a_n \in A_n$, in which case we set $\mu^\succ(a_n) = t_n(P_{a_n}; A_n)$. Given a fixed order \succ on \mathbf{A} , the rule which assigns to each problem its serial dictatorship matching is called the **serial dictatorship solution** (with respect to \succ), and denoted by SD_\succ .

Example 5.1 Let φ_1 and φ_2 be two solutions defined as follows:

$$\varphi_1(A, P) = \begin{cases} \mathcal{PO}(A, P) & \text{if } |A| > 4 \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\varphi_2(A, P) = \bigcup_{\succ} SD_\succ(A, P)$$

Consider the problem $p = (A, P)$, where $A = \{a, b, c, d, e\}$ and P as follows:

P_a	P_b	P_c	P_d	P_e
c	c	b	b	e
b	a	d	c	d
d	d	a	a	b
a	b	c	d	c
e	e	e	e	a

Let $\mu = \{(a, b), (c, d), (e)\}$. Clearly $\mu \in (\varphi_1 \vee \varphi_2)(A, P)$. Then consider the subproblem $p' = (A' \cup \mu(A'), P_{|A' \cup \mu(A')})$ where $A' = \{a, c\}$. Since $\mu_{|A'} \notin SD_{\succ}$ for any order \succ we have $\mu_{|A'} \notin (\varphi_1 \vee \varphi_2)(p')$. Hence $\varphi_1 \vee \varphi_2$ does not satisfy weak consistency.

Proposition 5.2 *Let φ_1 be a consistent solution and let $\varphi = \varphi_1 \vee \varphi_2$. The solution φ is a consistent enlargement of φ_1 if and only if φ_2 satisfies the followings:*

- i) For any problem $p = (A, P)$ if $\varphi_1(p) = \emptyset$ then $\varphi_2(p) \neq \emptyset$*
- ii) For any $\mu \in \varphi(p)$, if or any proper subset A' of A , $\mu_{|A'} \notin \varphi_1(A', P_{|A'})$ then $\mu_{|A'} \in \varphi_2(A', P_{|A'})$.*

Proof. Let φ_1 be a consistent solution and let $\varphi = \varphi_1 \vee \varphi_2$.

First, assume that φ is consistent enlargement of φ_1 . Take any $p = (A, P)$. Suppose $\varphi_1(p) = \emptyset$. Since φ is a well-defined solution this implies $\varphi_2(p) \neq \emptyset$. Take any $\mu \in \varphi(p)$ and suppose $\mu_{|A'} \notin \varphi_1(A', P_{|A'})$ for some proper subset A' of A . Since φ is consistent $\mu_{|A'} \in \varphi(A', P_{|A'})$. Hence, $\mu_{|A'} \in \varphi_2(A', P_{|A'})$.

Now, for the other direction assume that φ_2 satisfies (i) and (ii). By (i), for any problem $p = (A, P)$ we have $(\varphi_1 \vee \varphi_2)(p) \neq \emptyset$. Take any $\mu \in \varphi(p)$. Then we have either $\mu \in \varphi_1(p)$ or $\mu \in \varphi_2(p)$. By consistency of φ and by (ii), φ satisfies consistency. ■

Now, we are ready to prove our main result in which we compute a minimal consistent enlargement of the core.

Proposition 5.3 *For any fixed order \succ on the set of agents a minimal consistent enlargement of the core $MCE_{\mathcal{S}}$ is $\mathcal{S} \vee SD_{\succ}$.*

Proof. Let \succ be a fixed order.

Note that, SD_{\succ} is well-defined and consistent solution.²⁷ Since \mathcal{S} satisfies consistency then $\mathcal{S} \vee SD_{\succ}$ will be a consistent well-defined solution by Proposition 4.3 .

To show $\mathcal{S} \vee SD_{\succ}$ is minimal suppose for a contradiction $\mathcal{S} \vee SD_{\succ}$ is not minimal. Then there is a well-defined consistent solution ψ such that $\mathcal{S} \subsetneq \psi \subsetneq \mathcal{S} \vee SD_{\succ}$. Hence, there is a problem $p' = (A', P')$ such that $\mu \in \mathcal{S} \vee SD_{\succ}(p')$ but $\mu \notin \psi(p')$. Note that if $\mathcal{S}(p) = \emptyset$ for any $p = (A, P)$ then we have $\psi(p) = SD_{\succ}(p)$. This means that $\mathcal{S}(p') \neq \emptyset$ and μ is the serial dictatorship matching, that is $\mu = \mu^{\succ}$ for the order \succ .

Let $A' = \{a_1, a_2, \dots, a_n\}$ be the agents in A' . Assume that $a_1 \succ a_2 \succ \dots \succ a_n$. We add three new agents x, y, z into the society with an order $a_1 \succ a_2 \succ \dots \succ a_n \succ x \succ y \succ z$. We extend the preferences of the agents of A' to the larger set $A^* = A' \cup \{x, y, z\}$ of agents in a way that $P_{|A'}^* = P'$ and new agents are unacceptable to the old agents and vice versa. That is:

²⁷Indeed, SD_{\succ} satisfies the property that is stated in Proposition 5.2.

...	$P_{a_i}^*$...	P_x^*	P_y^*	P_z^*
....		y	z	x
			z	x	y
	a_i		x	y	z
	x		.	.	.
	y		.	.	.
	z		.	.	.

Note that $\mathcal{S}(A^*, P^*) = \emptyset$. Then $\psi(A^*, P^*) = SD_{\succ}(A^*, P^*)$ which means that the matching $\mu^* \in \psi(A^*, P^*)$ where $\mu^*(a_i) = \mu(a_i)$ for any $a_i \in A'$, $\mu^*(x) = y$ and $\mu^*(z) = z$. But $\mu^*_{|A'} = \mu$ and $\mu \notin \psi(A', P')$, hence $\mu^*_{|A'} \notin \psi(A' \cup \mu^*(A'), P_{|A' \cup \mu^*(A')})$. Since ψ satisfies consistency this gives a contradiction.

■

Remark 5.2 *Since the solution $\mathcal{S} \vee SD_{\succ}$ depends on the orders on the set of agents in the society, there is no unique minimal consistent core enlargement.*

5.4 Concluding Remarks

Due to Özkal-Sanver (2010), we know that there exists no consistent core extension in roommate problems. Here, we study consistent enlargements of the core. As no core extension is consistent, a consistent enlargement of

the core must pick at leasts for some problem more than the core whenever it is not empty. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. Next, we compute minimal consistent core enlargements as a solution which picks all stable matchings and serial dictatorship matching for a fixed order on the set of agents.

6 Conclusion

Consistency and converse consistency are two well-known axioms that have recently played fundamental role in axiomatic analysis. These axioms impose invariance on the solution for the population changes. Consistency is a kind of independence of irrelevant alternatives axiom. Converse consistency is a kind of two-agent decentralization axiom. First we survey applications of consistency and converse consistency in the literature. These axioms have been applied to number of models such as bargaining, taxation, allocation problems, apportionment, matching theory. We shortly survey applications of these axioms for different economics problems. We present the results for matching problems in a detail way. There are several related open questions to be investigated further. In this thesis, we deal with some of them. First, we characterize the core of marriage problems in general domains. We extend the characterization result of Sasaki and Toda (1992) for general domains. That paper is the first study which introduces consistency and converse consistency to the matching literature. Then, we compute a maximal conversely consistent subsolution of the Pareto optimal solution in marriage problems. Finally, we study on consistent enlargements of the core in roommate problems. There are still open questions to be investigated. For instance, for many to one matching problems (college admission problems) characteriza-

tion of the core by using converse consistency is still open question worth to be investigated. There are some studies for college admission problems about these axioms. But in those studies, for instance Toda (2006) and Klaus and Klijn (2013), there are different definitions of reduced problem which lead to different definitions of consistency and converse consistency. Therefore, to study these axioms for college admission problems, first of all there should be a study about these different definitions. Then by deciding on a more reasonable and admissible version of the axiom for real life applications, similar analysis which are done throughout this thesis can be adapted to college admissions problems.

7 Appendices

Appendix I

Lemma 7.1 *The solution φ^* defined for each $p = (A, P)$ as $\varphi^*(p) = \{\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$ such that there is no blocking pair (m, w) with $\mu(m) \neq m\}$ satisfies CCON.*

Proof. Take any arbitrary problem $p = (A, P)$ and any matching $\mu \in \mathcal{M}(A)$ such that $\mu|_N \in \varphi^*(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subset A$ with $\mu(i) \neq j$. We want to show that $\mu \in \varphi^*(p)$. Suppose, by contrary, $\mu \notin \varphi^*(p)$.

First we will show that $\mu \in \mathcal{IR}(p)$. Suppose $\mu \notin \mathcal{IR}(p)$, then wlog we may assume that there exists m such that $m P_m \mu(m)$. Let $\mu(m) = w$. Taking some man $\hat{m} \in M \setminus \{m\}$, let $N = \{m, \hat{m}\}$. Either we have $w P_w m$ or $m P_w w$. In both cases, by individual rationality of φ^* , we have $\mu|_N \notin \varphi^*(N \cup \mu(N), P|_{N \cup \mu(N)})$, a contradiction.

Next we will show that $\mu \in \mathcal{PO}(p)$. Suppose $\mu \notin \mathcal{PO}(p)$, then there exists $\mu^* \in \mathcal{M}(A)$ Pareto dominating μ . There exists at least an agent $i \in A$ such that $\mu^*(i) P_i \mu(i)$. Since $\mu \in \mathcal{IR}(p)$, $\mu^*(i) \in A(i) \setminus \{i\}$, call it j , and we have $\mu^*(j) P_j \mu(j)$. Note that if $i \in M$, then $j \in W$ and if $j \in M$, then $i \in W$. So there exists w such that $\mu^*(w) P_w \mu(w)$. There are two cases to consider. Either $\mu(w) = w$ or $\mu(w) \in M$.

Case 1: Suppose $\mu(w) = w$. Let $\mu^*(w) = m$. Either $\mu(m) = m$ and $\mu(m) \in W$.

Subcase 1.1: Let $\mu(m) = m$. Let $N = \{m, w\}$. Since $\mu \in IR(p)$, $w P_m m$ and $m P_w w$. Then, $\varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)}) = \{\tilde{\mu}\}$ where $\tilde{\mu}(m) = w$. So $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Subcase 1.2: Next suppose that $\mu(m) \neq m$, say $\mu(m) = w'$. As μ^* Pareto dominates μ and $\mu \in IR(p)$, we have $\mu^*(w') \neq w'$, call $\mu^*(w') = m'$. Let $N = \{m, w\}$. Recall that $P_{|N \cup \mu(N)}$ is as follows:

P_m	P_w	$P_{w'}$
w	m	m
w'	w	w'
m		

Since (m, w) blocks $\mu|_N$ and $\mu(m) \neq m$, we have $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Case 2: Suppose $\mu(w) \in M$. Let $\mu(w) = m''$. As μ^* Pareto dominates μ and $\mu \in IR(p)$, we have $\mu^*(w) \neq w$ and $\mu^*(m'') \neq m''$. Let $\mu^*(w) = m$ and $\mu^*(m'') = w''$. There are two subcases to consider. Either $\mu(w'') = w''$ or $\mu(w'') \in M$.

Subcase 2.1: First suppose that $\mu(w'') = w''$. Let $N = \{m'', w''\}$. As μ^* Pareto dominates μ and $\mu \in IR(p)$, $P_{|N \cup \mu(N)}$ is as follows:

$P_{m''}$	P_w	$P_{w''}$
w''	m''	m''
w	w	w''
m''		

Since (m'', w'') blocks $\mu|_N$ and $\mu(m'') \neq m''$, we have $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Subcase 2.2: Next, suppose that $\mu(w'') \in M$. Let $\mu(w'') = m'''$.²⁸ Let $N = \{m'', m'''\}$ Recall that $P_{|N \cup \mu(N)}$ is as follows:

$P_{m''}$	$P_{m'''}$	P_w	$P_{w''}$
w''	m''
w	m'''
m'''	w''

Regardless of the preferences of m''' and w , we have $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, since (m'', w'') is a blocking pair with $\mu(m'') \neq m''$, a contradiction.

Since $\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$, but $\mu \notin \varphi^*(p)$, there exists a pair (m^*, w^*) such that $\mu(m^*) \neq m^*$ blocking μ . Let $N = \{m^*, w^*\}$. Either $\mu(w^*) = w^*$ or $\mu(w^*) \in M$. In both cases, we have $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, since (m^*, w^*) blocks $\mu|_N$ at the reduced problem, completing the proof. ■

²⁸If we are in four agent set up, subcase 2.2 does not hold.

Lemma 7.2 *The solution φ be defined for each $p = (A, P)$ as $\varphi(p) = \{\mu \in \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w)\}$ satisfies CCON.*

Proof. Take any arbitrary problem $p = (A, P)$ and any matching $\mu \in \mathcal{M}(A)$ such that $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for each $N = \{i, j\} \subset A$ with $\mu(i) \neq j$. We want to show that $\mu \in \varphi(p)$. Suppose, by contrary, $\mu \notin \varphi(p)$.

Next we will show that $\mu \in \mathcal{PO}(p)$. Suppose $\mu \notin \mathcal{PO}(p)$, then there exists $\mu^* \in \mathcal{M}(A)$ Pareto dominating μ . There exists at least an agent $i \in A$ such that $\mu^*(i) P_i \mu(i)$. Wlog assume that $i = m$. There are two cases to consider. Either $\mu^*(m) = m$ or $\mu^*(m) \in W$.

Case 1: Suppose $\mu^*(m) = m$. Let $\mu(m) = w$. Either $\mu^*(w) = w$ or $\mu^*(w) \in M$.

Subcase 1.1: Let $\mu^*(w) = w$. Let $N = \{m, w\}$. Since φ satisfies *PO* we have $\varphi(N \cup \mu(N), P|_{N \cup \mu(N)}) = \{\tilde{\mu}\}$ where $\tilde{\mu}(m) = m$ and $\tilde{\mu}(w) = w$. So $\mu|_N \notin \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$, a contradiction.

Subcase 1.2: Next suppose that $\mu^*(w) \neq w$, say $\mu^*(w) = m'$. As μ^* Pareto dominates μ , we have $\mu^*(m') P_{m'} \mu(m')$, call $\mu^*(w') = m'$. Let $N = \{m, m'\}$. Recall that $P|_{N \cup \mu(N)}$ is as follows:

P_m	$P_{m'}$	P_w	$P_{\mu(m')}$
m	w	m'	\dots
w	$\mu(m')$	m	\dots
		w	

Since (m', w) blocks $\mu|_N$, we have $\mu|_N \notin \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Case 2: Suppose $\mu^*(m) \in W$. Let $\mu^*(m) = w'$. As μ^* Pareto dominates μ we have $\mu^*(w)P_w\mu(w)$. When we take $N = \{m, w'\}$, (m', w) blocks $\mu|_N$, we have $\mu|_N \notin \varphi(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Since $\mu \in \mathcal{PO}(p)$, but $\mu \notin \varphi(p)$, there exists a pair (m^*, w^*) blocking μ . Let $N = \{m^*, w^*\}$. We have $\mu|_N \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, since (m^*, w^*) blocks $\mu|_N$ at the reduced problem, completing the proof. ■

Appendix II

Continuation of Proof of Theorem 4.1:

Proof. Assume we are in the second case. For this preference profile $MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_2), (m_2, w_1)\}$ then we have $\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_1	w_4	w_2	m_2	m_4	m_1	m_3
w_1	w_2	w_3	w_4	m_3	m_2	m_3	m_2
w_2	w_3	w_1	w_3	m_1	m_1	m_4	m_4
w_4	w_4	w_2	w_1	m_4	m_3	m_2	m_1

Let $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_1	w_1	m_1	m_3
m_1	m_3	w_1	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_1$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ .

Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_4	m_1	m_4
w_4	w_1	m_4	m_1
m_1	m_4	w_1	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_2	w_3	m_2	m_3
w_3	w_2	m_3	m_2
m_2	m_3	m_2	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_2	w_2	m_4	m_2
w_4	w_4	m_2	m_4
m_2	m_4	w_2	w_4

Since $m_4 \succ m_2$ then we have $w_2 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_2$ and $t(P_{m_4}; M) = w_2$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\}$ and P_{w_2} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the third case. For this preference profile $MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_2), (m_2, w_1)\}$ then we have $\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of

$M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_1	w_4	w_2	m_2	m_4	m_1	m_3
w_2	w_2	w_3	w_4	m_3	m_2	m_3	m_2
w_1	w_3	w_1	w_3	m_1	m_1	m_4	m_4
w_4	w_4	w_2	w_1	m_4	m_3	m_2	m_1

Let $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_1	w_1	m_1	m_3
m_1	m_3	w_1	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_1$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCSP_{\mathcal{O}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_4	m_1	m_4
w_4	w_1	m_4	m_1
m_1	m_4	w_1	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_2	w_3	m_2	m_3
w_3	w_2	m_3	m_2
m_2	m_3	w_2	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_2	w_2	m_4	m_2
w_4	w_4	m_2	m_4
m_2	m_4	w_2	w_4

Since $m_4 \succ m_2$ then we have $w_2 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_2$ and $t(P_{m_4}; M) = w_2$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\}$ and P_{w_2} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the fourth case. For this preference profile $MXCCS_{PO}(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_2	w_4	w_1	m_4	m_2	m_1	m_3
w_2	w_1	w_3	w_4	m_1	m_3	m_3	m_2
w_1	w_3	w_2	w_3	m_2	m_1	m_4	m_4
w_4	w_4	w_1	w_2	m_3	m_4	m_2	m_1

Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_2	w_2	m_1	m_3
m_1	m_3	w_2	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_2$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_2	w_4	m_1	m_4
w_4	w_2	m_4	m_1
m_1	m_4	w_2	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_1	w_3	m_2	m_3
w_3	w_1	m_3	m_2
m_2	m_3	w_1	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_1	m_4	m_2
w_4	w_4	m_2	m_4
m_2	m_4	w_1	w_4

Since $m_4 \succ m_2$ then we have $w_1 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_1$ and $t(P_{m_4}; M) = w_1$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\}$ and P_{w_1} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the fifth case. For this preference profile $MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_2	w_4	w_1	m_4	m_2	m_1	m_3
w_1	w_1	w_3	w_4	m_2	m_3	m_3	m_2
w_2	w_3	w_2	w_3	m_1	m_1	m_4	m_4
w_4	w_4	w_1	w_2	m_3	m_4	m_2	m_1

Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin$

$\psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_2	w_2	m_1	m_3
m_1	m_3	w_2	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_2$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$. Hence $\mu_{|N'}^* \in MXCCSP_{\mathcal{O}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_2	w_4	m_1	m_4
w_4	w_2	m_4	m_1
m_1	m_4	w_2	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_1	w_3	m_2	m_3
w_3	w_1	m_3	m_2
m_2	m_3	w_1	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_1	m_4	m_2
w_4	w_4	m_2	m_4
m_2	m_4	w_1	w_4

Since $m_4 \succ m_2$ then we have $w_1 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_1$ and $t(P_{m_4}; M) = w_1$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\}$ and P_{w_1} is coherent to \succ .

Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the sixth case. For this preference profile $MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p)$. We introduce four new agents m_3, m_4, w_3 and w_4 to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

$P_{m_1}^*$	$P_{m_2}^*$	$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_2}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_3	w_2	w_4	w_1	m_4	m_2	m_1	m_3
w_2	w_1	w_3	w_4	m_2	m_3	m_3	m_2
w_1	w_3	w_2	w_3	m_1	m_1	m_4	m_4
w_4	w_4	w_1	w_2	m_3	m_4	m_2	m_1

Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since μ^* is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^*)$ for each $N' = \{i, j\} \subset M^* \cup W^*$

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P_{|N' \cup \mu^*(N')}^* = P_{|N \cup \mu(N)}$ and $\mu_{|N'}^* = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_3}^*$	$P_{w_2}^*$	$P_{w_3}^*$
w_3	w_3	m_3	m_1
w_2	w_2	m_1	m_3
m_1	m_3	w_2	w_3

Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_2$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and P_{w_3} is coherent to \succ .

Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCS_{\mathcal{PO}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_2, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_1}^*$	$P_{m_4}^*$	$P_{w_2}^*$	$P_{w_4}^*$
w_2	w_4	m_1	m_4
w_4	w_2	m_4	m_1
m_1	m_4	w_2	w_4

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_3}^*$	$P_{w_1}^*$	$P_{w_3}^*$
w_1	w_3	m_2	m_3
w_3	w_1	m_3	m_2
m_2	m_3	w_1	w_3

Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_2}^*$	$P_{m_4}^*$	$P_{w_1}^*$	$P_{w_4}^*$
w_1	w_1	m_4	m_2
w_4	w_4	m_2	m_4
m_2	m_4	w_1	w_4

Since $m_4 \succ m_2$ then we have $w_1 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_1$ and $t(P_{m_4}; M) = w_1$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\}$ and P_{w_1} is coherent to \succ . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCSP_{\mathcal{O}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

$P_{m_3}^*$	$P_{m_4}^*$	$P_{w_3}^*$	$P_{w_4}^*$
w_4	w_4	m_3	m_3
w_3	w_3	m_4	m_4
m_3	m_4	w_3	w_4

Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and P_{w_4} is coherent to \succ and P_{m_3} is coherent to \succ_{μ^*} . Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu_{|N'}^* \in MXCCSP_{\mathcal{O}}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction. ■

8 References

- Abraham, D.J., Biró, P., and Manlove, D.F., (2006), “ “Almost Stable” Matchings in the Roommate Problem” , In T. Erlebach and G. Persiano, (Eds.), Proceedings of WAOA 2005, Lecture Notes in Computer Science Vol. 3879, 1 – 14. Springer, Berlin/Heidelberg.
- Balinski, M. and Young, P., (1982), "Fair Representation", Yale University Press.
- Bevia C., (2006), “Identical Preferences Lower Bound Solution and Consistency in Economies with Indivisible Goods”, *Social Choice and Welfare*, 13, 113 – 126.
- Can, B. and Klaus, B., (2012), "Consistency and Population Sensitivity Properties in Marriage and Roommate Markets," *Social Choice and Welfare*, in press.
- Chun, Y., (1988), "The Proportional Solution for Rights Problems", *Mathematical Social Sciences*, 15, 231 – 246.
- Chung, K.S., (2000), “On the Existence of Stable Roommate Matchings”, *Games and Economic Behavior*, 53, 248 – 261.
- Gale D. and Shapley, L., (1962), “College Admissions and the Stability of Marriage”, *American Mathematical Monthly* 69, 9 – 15.

- Hart, S. and Mas-Colell, A.,(1989), "Potential, Value and Consistency", *Econometrica*, 57, 589 – 614.
- Inarra, E., Larea, C., and Molis, E., (2008a), “The Stability of the Roommate Problem Revisited”, unpublished paper.
- Inarra, E., Larea, C., and Molis, E., (2008b), “Random Paths to P-stability in the Roommate Problem”, *International Journal of Game Theory*, 461 – 471.
- Klaus, B. (2013), "Consistency and its Converse for Roommate problems", working paper.
- Klaus, B., and Klijn, F., (2009), “Smith and Rawls Share a Room: Stability and Medians”, Harvard Business School Working Paper \neq 09 – 111.
- Klaus, B., and Klijn, F., (2013), “Local and Global Consistency Properties for Student Placement”, *Journal of Mathematical Economics*, 49, 222 – 229.
- Klaus, B., Klijn, F., and Walzl, M., (2008), “Stochastic Stability for Roommate Markets”, Meteor RM/08 – 010, Maastricht University.
- Lensberg, T., (1987), "Stability and Collective Rationality", *Econometrica*, 55, 935 – 961.
- Lensberg, T., (1988), "Stability and the Nash Solution", *Journal of Economic Theory*, 45, 330 – 341.
- Korthues, B., (2000), “Characterization of an Extended Walrasian Concept for

- Open Economies", *Journal of Mathematical Economics*, 33, 449 – 461.
- McLean, R.P., Pazgal, A., Sharkey, W., (2004), "Potential, Consistency and Cost Allocation Prices", *Mathematics of Operations Research*, 29, 602 – 62.
- Moulin, H., (1985), "The Separability Axiom and Equal Sharing Methods", *Journal of Economic Theory*, 36, 120 – 148.
- Moulin, H. and Shenker, S., (1994), "Average Cost Pricing versus Serial Cost Sharing: An Axiomatic Comparison", *Journal of Economic Theory*, 64, 178 – 201.
- Nizamogullari, D. and Özkal-Sanver, İ., (2013), "Characterization of the Core in Full Domain Marriage Problems", unpublished monograph.
- Özkal-Sanver, İ., (2004), "A Note on Gender Fairness in Matching Problems", *Mathematical Social Sciences*, 47, (2), 211 – 217.
- Özkal-Sanver, İ., (2010), "Impossibilities for Roommate Problems", *Mathematical Social Sciences*, 59, 360 – 363.
- Özkal-Sanver, İ., (2013), "Minimal Conversely Consistent Extension of the Men-Optimal Solution", *Social Choice and Welfare*, 40, (11), 89 – 99.
- Peleg, B., (1986), "On the Reduced Game Property and its Converse", *International Journal of Game Theory*, 15, 187 – 200.
- Peleg, B., (1989), "An axiomatization of the Core of Market Games", *Mathe-*

- matics of Operations Research*, 14, 448 – 456.
- Peleg, B., (1992), "Axiomatizations of the Core", Chapter 13 in the Handbook of Game Theory, (R. Aumann and S. Hart, eds) North Holland, 397 – 412.
- Peleg, B. and Tijs, S., (1996), "The Consistency Principle for Games in Strategic Form", *International Journal of Game Theory*, 25, 13 – 34.
- Roth, A. and Sotomayor, M., (1990), "Two-sided Matching: A Study in Game Theoretic Modelling and Analysis", Cambridge University Press, Cambridge.
- Sasaki, H., (1995), "Consistency and Monotonicity in Assignment Problems", *International Journal of Game Theory*, 24, 373 – 397.
- Sasaki, H. and Toda, M., (1992), "Consistency and Characterization of the Core of Two-sided Matching problems", *Journal of Economic Theory*, 56, 218 – 237.
- Sethuraman, J., and Teo, C. P., (2001), "A Polynomial-Time Algorithm for the Bistable Roommates Problem", *Journal of Computer and Systems Sciences*, 63, 486 – 497.
- Sobolev, A.I., (1975), "The Characterization of Optimality Principles in Cooperative Games by Functional Equations", *Mathematical Methods in the Social Sciences*, 6, 150 – 165.
- Tadenuma, K., (1992), "Reduced Games, Consistency and the Core", *Interna-*

- tional Journal of Game Theory*, 20, 325 – 334.
- Tan, J. J. M., (1990), “A Maximum Stable Matching for the Roommate Problems”, *BIT*, 29, 631 – 640.
- Thomson, W., (1994a), “Consistent Solutions to the Problem of Fair Division When Preferences are Single-peaked”, *Journal of Economic Theory*, 63, 219–245.
- Thomson, W., (1994b), “Consistent Extensions”, *Mathematical Social Sciences*, 28, 35 – 49.
- Thomson, W., (1996), “Consistent Allocation Rules”, unpublished monograph, University of Rochester.
- Thomson, W.,(1990),The Consistency Principle. In: Ichiishi, T., Neyman A., Tauman, Y., (Eds.), *Game Theory and Applications*. Proceedings of the 1987 International Conference held at Ohio State University, Columbus, Ohio. Academic Press, New York, 187 – 215.
- Thomson, W., (1999), “Monotonic Extensions on Economic Domains”, *Review of Economic Design*, 4, 13 – 33.
- Thomson, W., (2004), “Consistency and its Converse: an Introduction”, unpublished monograph, University of Rochester.
- Thomson, W., (2009), “Consistent Allocation Rules”, unpublished monograph,

University of Rochester.

Toda M., (1993), "Characterization of the Core of Two-sided Matching Problems which Allow Self-matchings", working paper.

Toda M., (2006), "Monotonicity and Consistency in Matching Markets", *International Journal of Game Theory*, 34, 13 – 31.

Young, P., (1987), "On Dividing an Amount According to Individual Claims or Liabilities", *Mathematics of Operations Research*, 12, 398 – 414.