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Hyper-stable collective rankings

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HIGHLIGHTS

- Hyper-stability is a new consistency property for social welfare functions.
- It relates ranking from preference orders and choosing from orders over orders.
- We provide sufficient conditions for hyper-stability.
- Hyper-stability can hold by using sequential application of tournament solutions.

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ABSTRACT

We introduce a new consistency property for social welfare functions (SWF), called hyper-stability. An SWF is hyper-stable if at any profile over finitely many alternatives where a weak order R is chosen, there exists a profile of linear orders over linear orders, called hyper-profile, at which only linearizations of R are ranked first by the SWF. Profiles induce hyper-profiles according to some minimal compatibility conditions. We provide sufficient conditions for hyper-stability, and we investigate hyper-stability for several Condorcet SWFs. An important conclusion is that there are non-dictatorial hyper-stable SWFs.

1. Introduction

Classical social choice theory studies how societies may in a satisfactory way aggregate a set of individual preferences over finitely many alternatives into the choice of one or several alternatives. Suppose that a society has to rank all alternatives instead of choosing one or several of them. If rankings (i.e. linear orders) of alternatives become resolute social outcomes, social choice theory calls for individuals reporting their preferences over all possible rankings. However, this seems hardly implementable in practice. Indeed, when facing m alternatives, voters would have to rank the $m!$ orders over these m alternatives. Assuming that preferences over rankings, called *hyper-preferences*, are themselves linear orders, a way out is to ask individuals to report only their first-preferred ranking over alternatives, and then to aggregate the resulting profile into a social ranking. A typical approach of this type consists in using a social welfare function (SWF), which maps every profile of rankings of alternatives to a weak order of alternatives, and in selecting from the set of all linearizations of this weak order.

While it overcomes the logistical constraint upon the size of ballots, the use of an SWF clearly entails a dramatic loss of information about how voters compare all possible rankings. In this paper, we introduce a new property for SWFs, called *hyper-stability*, which somehow reconciliates the quality of information about voters' preferences (about rankings) and the practicality of SWFs as voting procedures.

Loosely speaking, an SWF is hyper-stable if every profile of individual rankings of alternatives can be extended to a *consistent* profile of hyper-preferences such that both essentially lead to the same outcome. More formally, hyper-stability is defined as follows. We assume that individual linear orders over alternatives induce hyper-preferences, defined as linear order over linear orders, by means of some preference extension δ. Thus, any profile π over m alternatives generates a hyper-profile π^{δ} of hyper-preferences over the $m!$ orders over m alternatives. Now suppose that the society has adopted a way to rank *any* number of alternatives from any profile π. Formally, there is an SWF α, which selects a weak order at any profile of linear orders over *any* number of alternatives. We call an SWF α *hyper-stable for preference extension* δ if only linearizations of the weak order $\alpha(\pi)$ are ranked first by α when applied to hyper-profile π^{δ}. Note that SWFs are defined for

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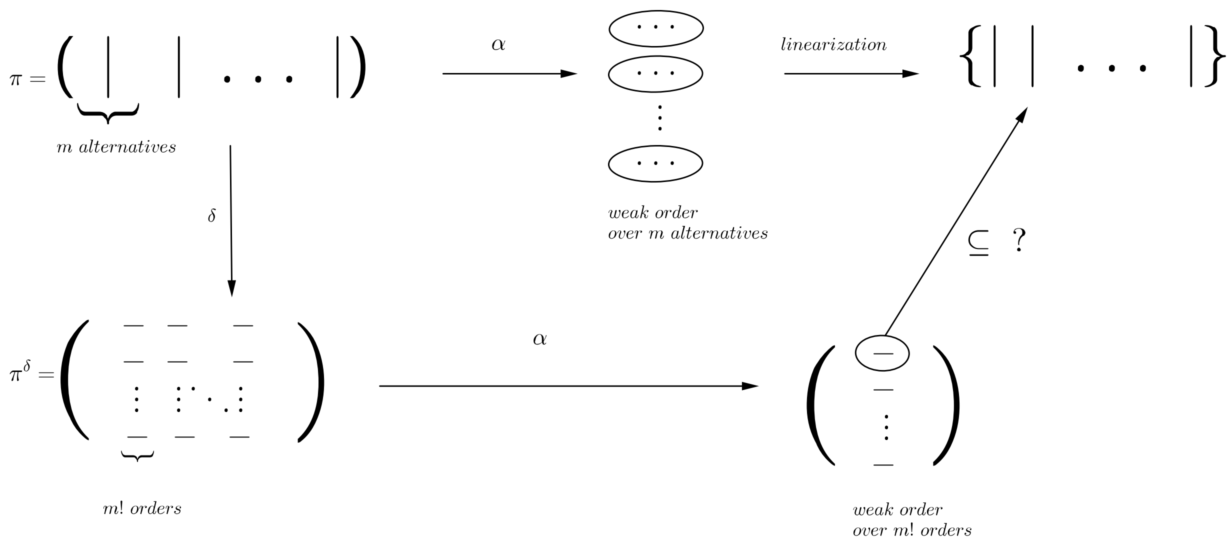

Fig. 1. The hyper-stability property.

a variable number of alternatives. This allows hyper-stability to be well-defined, since it requires SWFs to be defined for m and for $m!$ alternatives. Hyper-stability is illustrated in Fig. 1.

Hyper-stability clearly depends on which preference extension prevails. In this paper, we consider the class of preference extensions that share four properties: regularity, neutrality, clone-consistency, and independence. Regularity means that if an individual ranks alternatives according to linear order P, then P should be ranked itself above all other orders. Neutrality means that permuting the names of alternatives in P induces a hyper-preference that is consistent with this permutation. Clone-consistency holds if "exploding" one alternative into any finite number of clones does not reverse the ranking of two orders. Finally, independence means that completing any two orders by putting at top, or at bottom the same additional alternative does not reverse the way they were compared before completion.

While pertaining to the engineering of voting, hyper-stability can also be interpreted under the light of the classical revealed preference approach in choice theory. Although citizens' preferences over rankings are hardly observable, there nonetheless exists a profile of such preferences at which the society declares as first-best what is chosen from a very scarce information about individual wills, namely the knowledge of their most-preferred ranking. As a consequence, one can say that whenever an SWF α is hyper-stable for preference extension δ, the choice $\alpha(\pi)$ at some profile π over alternatives reveals a hyper-preference profile π^δ at which the choice $\alpha(\pi^\delta)$ confirms the choice $\alpha(\pi)$ made from partial observation of preferences.

To the best of our knowledge, hyper-stability is a new property, although a yeast of the present analysis can be found in Binmore (1975), who introduces hyper-preferences, at least implicitly, together with a consistency property close in spirit to hyper-stability. Binmore's study can be summarized as follows. A society intends to decide upon a social weak order of alternatives from citizen's preferences, and has agreed on using a neutral SWF (where neutrality means that the labelling of alternatives does not play any role). Binmore argues that "if a rational entity [individual] holds a certain preference preordering over a set $\mathcal{S}$ of alternatives, then that entity must also subscribe to a certain partial preordering of the set of all preorderings on $\mathcal{S}$. [...] If the society is partial to act as a rational entity, then the collective choice preference preordering on $\mathcal{P}$ (the set of all preorderings on $\mathcal{S}$) should be compatible with that induced on $\mathcal{P}$ by the collective preference preordering on $\mathcal{S}$. If this is *not* the case, we shall say that the society is inconsistent" (Binmore, 1975, p. 379).

Considering the case where $\mathcal{S}$ contains 3 alternatives, Binmore's definition of consistency works as follows. First, neutrality allows to define the outcome of the SWF at any profile over triples of weak orders. Moreover, every weak order R over $\mathcal{S}$ induces a hyper-preorder $\widetilde{R}$ (i.e. a reflexive and transitive but not necessarily complete binary relation) over the set of all weak orders, according to the following criterion: $\widetilde{R}$ will rank a weak order R' above another weak order R'' if the set of best elements in R' is rated at least as highly as the set of best elements in R'' relative to R.[2] Using hyper-preorders allows to rank certain (but not all) triples of preorders.[3] Denote by $\mathbb{T}$ the set of triples of weak orders that can be compared by all hyper-preorders. Now consider the weak order R chosen at some profile π over $\mathcal{S}$, and define $\widetilde{R}$ as the preorder over $\mathcal{P}$ that is induced by R. Moreover, denote by $\widetilde{R}\,|_T$ the weak order over triple T in $\mathbb{T}$. Furthermore, π also generates a profile $\widetilde{\pi}\,|_T$ of weak orders over each triple T in $\mathbb{T}$. Then, using neutrality of the SWF, we obtain from $\widetilde{\pi}\,|_T$ a weak order $Q\,|_T$ over T. Consistency holds if $Q\,|_T$ is equal to $\widetilde{R}\,|_T$ for all T in $\mathbb{T}$. Fig. 2 illustrates this notion of consistency. Binmore proves the very neat following result: all SWFs over 3 alternatives, other than a dictatorship, an anti-dictatorship or a collective apathy (general indifference), are inconsistent at some profile.

Binmore's paper clearly addresses as we do the issue of consistency between two levels of choices: Choice from preferences over alternatives and choice from hyper-preferences. However, our analysis contrasts with Binmore's analysis in several aspects. First, we assume that preferences over alternatives are linear rather than weak orders. Second, a preference extension generates a linear order over all rankings rather than over subsets of rankings. Third, and this is the main difference, instead of considering the linear order that δ generates from $\alpha(\pi)$, and comparing it to the outcome of α at π^δ, we just impose that, up to some tie-break rule, the initial weak order $\alpha(\pi)$ of alternatives is ranked first by α from the hyper-profile π^δ.

2 Since preferences are weak orders, there may exist more than one best element. Focusing on situations where there are at most two best elements, Binmore only assumes that the sets $\{x\}$, $\{y\}$, and $\{x, y\}$ are rated in the order $\{x\}$, $\{x, y\}$, $\{y\}$ if and only if alternative x is at least as good as alternative y according to R.

3 Let $\mathcal{S} = \{a, b, c\}$ and P be the linear order $cPbPa$. Moreover, let Q be the linear order $bQcQa$ and R be the weak order $bR^{=}cR^{+}a$, where R^{+} ($R^{=}$) stands for the asymmetric (resp. symmetric) part of R. Then we get $P\widetilde{P}R\widetilde{P}Q$. It is easy to check that P, Q, and R can be compared by the extensions of all 13 possible weak orders over $\mathcal{S}$. Furthermore, $\widetilde{P}$ cannot compare Q and the weak order $aZ^{=}cZ^{+}b$.

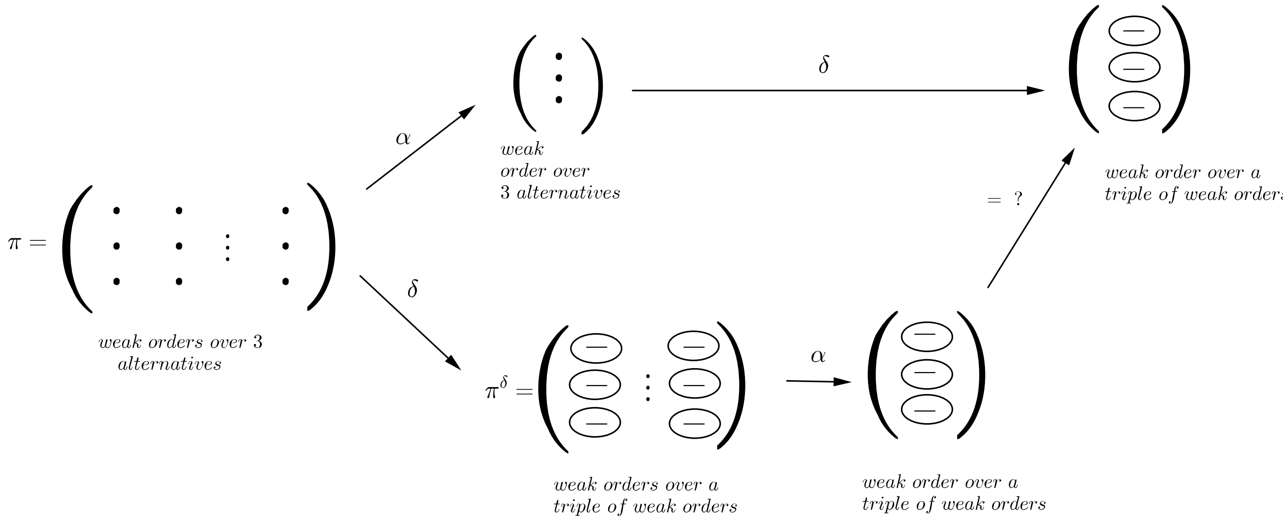

Fig. 2. Binmore's consistency property.

The objectives of this paper are to provide a formal setting of the hyper-stability property, to establish sufficient conditions for hyper-stability, and to investigate whether several well-known Condorcet SWFs are hyper-stable. Being Condorcet means that whenever one alternative is preferred to all others by a majority of individuals, then this alternative is uniquely ranked first. Contrarily to Binmore, hyper-stability does not imply dictatorship, anti-dictatorship or collective apathy. Indeed, we show that any SWF which satisfies a weak version of independence of irrelevant alternatives together with another property called non-sensitivity to cloning, is hyper-stable. Furthermore, these two properties are satisfied by some Condorcet SWFs based on the sequential application of a tournament solution. Assuming an odd number of individuals, pairwise majority comparisons generate a complete and asymmetric binary relation called tournament. A tournament solution is a choice function S which maps every tournament T over m alternatives to a subset $S(T)$ of alternatives. Elements of $S(T)$ are called the S-winners of T. A sequential tournament solution *SeqS* is the SWF which ranks first all S-winners of T, second all S-winners of the tournament defined over the set of non S-winners, etc. until the set of alternatives is exhausted. We show that if S is either the top-cycle, the minimal covering set, or the bipartisan set, the resulting *SeqS* is hyper-stable, while it is not when S is either the uncovered set or the Banks set. We also consider Condorcet SWFs which rank all alternatives at once. The Copeland SWF, which ranks alternatives according to their outdegrees in the majority tournament, is not hyper-stable. Moreover, we prove that the Slater and Kemeny social preference functions satisfy (a stronger version of) hyper-stability.

The rest of the paper is organized as follows. Section 2 presents the model of collective choice, and formally defines hyper-stability. We derive in Section 3 a sufficient condition for hyper-stability. Results for Condorcet SWFs are presented in Section 4. Finally, several open problems are stated in Section 5.

2. Hyper-stable social welfare functions

2.1. Basic notions

$\mathbb{N}$ stands for the set of non-zero natural numbers, and $\mathcal{A}$ stands for the set of all finite non-empty subsets of $\mathbb{N}$. We consider a variable number m of social alternatives, interpreting each element A of $\mathcal{A}$ as the actual alternative set a society is confronting. The set of linear (resp. weak) orders over A is denoted by $\mathcal{L}(A)$ (resp. $\mathcal{R}(A)$). The asymmetric (resp. symmetric) part of a weak order R is denoted by R^+ (resp. $R^=$). A linear order $a_1Pa_2P\dots Pa_m$ over the m alternatives $a_1,\dots,a_n$, where $a_mPa_{m'}$ means that P ranks a_m above $a_{m'}$, is equivalently written $P=(a_1a_2\dots a_m)$. Given a non-empty subset B of A together with $R\in\mathcal{R}(A)$, we denote by $Max(R,B)$ (resp. $Min(R,B)$) the set of alternatives in B that are highest (resp. lowest) ranked in R (among all alternatives in B). If $B=A$, we write $Max(R,A)=Max(R)$. Moreover, given $B\subset A$ and $Q\in\mathcal{L}(B)$ we denote by $\mathcal{P}(Q\rightarrow)\subset\mathcal{L}(A)$ the subset of all orders in $\mathcal{L}(A)$ having the form $(Qb_1\dots b_{m'})$, where $|B|=m-m'$. Similarly, $\mathcal{P}(\rightarrow Q)$ is the subset of all orders with the form $(b_1\dots b_{m'}Q)$. A segment of P is an element T of $\mathcal{L}(B)$, where $B\subset A$, such that for all $a,a'\in A\backslash B$, and for all $b\in B$, $aPb\Leftrightarrow a'Pb$. Moreover, $P=(a_1a_2\dots a_m)$ is equivalently written $P=(a_1a_2\dots a_{m'}Ta_{m''+1}\dots a_m)$, where T is a segment of P involving subset $B=\{a_{m'+1},\dots,a_{m''}\}$. Given a permutation σ of A, the permutation P^σ of $P\in\mathcal{L}(A)$ is the element of $\mathcal{L}(A)$ defined by: $\forall x,y\in A, xP^\sigma y\Leftrightarrow\sigma(x)P\sigma(y)$.

A linear order $P\in\mathcal{L}(A)$ is a *linearization* of the weak order $R\in\mathcal{R}(A)$ if for any $a,b\in A$, $aR^+b\Rightarrow aPb$. The set of all linearizations of $R\in\mathcal{R}(A)$ is $\Delta(R)$.

A profile is an element $\pi=(P_1,\dots,P_n)$ of $\mathcal{L}(A)^n$, where $n\in\mathbb{N}$ is the number of individuals, and $P_i\in\mathcal{L}(A)$ is the linear order assigned to individual $i=1,\dots,n$. The set of all profiles over A is defined by $\Pi(A)=\cup_{n\in\mathbb{N}}\mathcal{L}(A)^n$.

A correspondence $F:\cup_{A\in\mathcal{A}}\Pi(A)\rightarrow\mathcal{A}$ is a *social choice correspondence* (SCC) if and only if $\forall A\in\mathcal{A}$, $\forall\pi\in\Pi(A)$, $F(\pi)$ is a non-empty subset of A. Moreover, if $F(\pi)$ is a singleton for all π, then F is a *social choice function* (SCF).

A function α from $\cup_{A\in\mathcal{A}}\Pi(A)$ to $\cup_{A\in\mathcal{A}}\mathcal{R}(A)$ is a *social welfare function* (SWF) if for any $A\in\mathcal{A}$, for any $\pi\in\Pi(A)$, $\alpha(\pi)\in\mathcal{R}(A)$. Given an SWF α, the *α-top choice correspondence* $F^\alpha:\cup_{A\in\mathcal{A}}\Pi(A)\rightarrow\mathcal{A}$ is the SCC defined by: $\forall A\in\mathcal{A}$, $\forall\pi\in\Pi(A)$, $\forall a\in A$, $a\in F^\alpha(\pi)=Max(\alpha(\pi))$. That is, $F^\alpha(\pi)$ is the set of all alternatives that α ranks first at profile π. Similarly, the *α-bottom choice correspondence* $F^{-\alpha}(\pi)$ returns to the set of all alternatives that α ranks last at profile π.

We now recall the notion of a Condorcet SWF. Given a profile π over A involving n individuals, an alternative $a^*\in A$ is the Condorcet winner of π if $|\{1\le i\le n: a^*P_ia\}|>\frac{n}{2}$ for all $a\in A\backslash\{a^*\}$.

Definition 1. An SWF α is Condorcet if for all $A\in\mathcal{A}$, $Max(\alpha(\pi))=\{a^*\}$ for all $\pi\in\Pi(A)$ having a^* as Condorcet winner.

The property of *composition-consistency* for SCCs, introduced in Laffond et al. (1996), is defined below. We first recall the notion of a composed tournament. A tournament over A is a complete and asymmetric binary relation T defined on $A \times A$. Note that T is a linear order in case where it is transitive. When asymmetry is dropped, T is called a weak tournament. Moreover, $a^* \in A$ is the Condorcet winner of T if a^*Tb for all $b \in A \backslash \{a^*\}$. Given a profile $\pi = (P_1, \ldots, P_n) \in \Pi(A)$, the weak majority tournament for π is the weak tournament T_π over A (with asymmetric part T_π^+) defined by: $\forall(a, b) \in A \times A, aT_\pi b \Leftrightarrow |\{1 \leq i \leq n : aP_ib\}| \geq \frac{n}{2}$, and $aT_\pi^+ b \Leftrightarrow |\{1 \leq i \leq n : aP_ib\}| > \frac{n}{2}$. If n is odd, then T_π is a tournament over A, called majority tournament for π. Clearly, alternative a^* is a Condorcet winner of T_π if and only if a^* is the Condorcet winner of π. A *component* of tournament T over A is a subset C of A such that, for all $a, a' \in C$, for all $b \in A \backslash C$, aTb if and only if $a'Tb$. A *decomposition* of T is a partition $\mathcal{P}(T)$ of A into components of T such that $\mathcal{P}(T) \notin \{\{\{x\}, a \in A\}, \{A\}\}$. Given $C, C' \in \mathcal{P}(T)$, we have by construction that if $c \in C$ and $c' \in C'$, then $\forall(a, a') \in C \times C', aTa' \Leftrightarrow cTc'$. Moreover, T is called *composed* if it admits a decomposition. Let T^* be a tournament over $\{1, \ldots, H\}$ and consider tournaments $T^1, \ldots, T^H$ respectively defined over $C^1, \ldots, C^H$, where $C^1, \ldots, C^H$ are non-empty finite disjoint sets. The composition product of T^* by $T^1, \ldots, T^H$ is the tournament $T = \otimes(T^*; T^1, \ldots, T^H)$ over $\cup_h C^h$ defined by $\forall(c, c') \in C^h \times C^{h'}, h \neq h' \Rightarrow [cTc' \Leftrightarrow hT^*h']$ and $h = h' \Rightarrow [cTc' \Leftrightarrow cT^hc']$. It is easily seen that a tournament T over A is composed if and only if it can be defined as a composition product $T = \otimes(T^*; T^1, \ldots, T^H)$ with $m > H > 1$, where:

- $\{C^1, \ldots, C^H\}$ is a decomposition of T
- $\forall 1 \leq h \leq H$, T^h is a tournament over C^h is such that $\forall(a_h, a_{h'}) \in C^h \times C^{h'}, a_hTa_{h'} \Leftrightarrow a_hT^ha_{h'}$
- T^* is the tournament over $\{1, \ldots, H\}$, called the *summary* of T, defined by: $\forall h, h' \in \{1, \ldots, H\}, hT^*h' \Leftrightarrow \exists(a_h, a_{h'}) \in C^h \times C^{h'}$ with $a_hTa_{h'}$.

The notion of a composed preference profile is defined in a similar way. A decomposition of $\pi = (P_1, \ldots, P_n) \in \Pi(A)$ is a partition $\mathcal{P}(\pi) = \{C^1, \ldots, C^H\}$ of A which is a decomposition of P_i for all $1 \leq i \leq n$. Then π is composed if, for all $1 \leq i \leq n$, $P_i = \otimes(P_i^*; P_1^1, \ldots, P_n^H)$ where

- $\forall 1 \leq h \leq H, P_i^h \in \mathcal{L}(C^h)$ is such that $\forall(a_h, a_{h'}) \in C^h \times C^{h'}$, $a_hP_ia_{h'} \Leftrightarrow a_hP_i^ha_{h'}$
- $P_i^* \in \mathcal{L}(\{1, \ldots, H\})$ and $\forall h, h' \in \{1, \ldots, H\}, hP_i^*h' \Leftrightarrow \exists(a_h, a_{h'}) \in C^h \times C^{h'}$ with $a_hP_ia_{h'}$.

A composed profile π will be denoted by $\pi = \otimes(\pi^*; \pi^1, \ldots, \pi^H)$, where $\pi^* = (P_1^*, \ldots, P_n^*)$ is called the summary of π, and where $\pi^h = (P_1^h, \ldots, P_n^h)$ for all $1 \leq h \leq H$.

Definition 2. An SCC F is composition-consistent if for any $A \in \mathcal{A}$ and for any composed profile $\pi = \otimes(\pi^*; \pi^1, \ldots, \pi^H)$ over A, we have $F(\pi) = \cup_{h \in F(\pi^*)} F(\pi^h)$.

Composition-consistency is interpreted as follows. Suppose that alternatives can be classified into groups, each group having several representatives. Moreover, for all individuals, if some representative of one group is judged better than some representative of another group, then this comparison also holds for any two representatives of the two groups. Then composition-consistency requires that chosen alternatives are the best representatives of the best groups.

2.2. Preference extensions

We assume that individuals have preferences over all possible orders, called hyper-preferences, which are represented by linear orders over orders. Hyper-preferences are related to initial preferences over alternatives by means of a preference extension. A *preference extension* is a function δ from $\cup_{A \in \mathcal{A}} \mathcal{L}(A)$ to $\cup_{A \in \mathcal{A}} \mathcal{L}(\mathcal{L}(A))$ such that, for any $A \in \mathcal{A}$ and for any $P \in \mathcal{L}(A)$, $\delta(P) \in \mathcal{L}(\mathcal{L}(A))$. Given a profile $\pi = (P_1, \ldots, P_n) \in \diamond(A)$ together with a preference extension δ, the *hyper-profile* π^δ of π is defined by $\pi^\delta = (\delta(P_1), \ldots, \delta(P_n)) \in [\mathcal{L}(\mathcal{L}(A))]^n$. We impose four properties to preference extensions.

- *Regularity*: δ is regular if $\forall A \in \mathcal{A}, \forall P, Q \in \mathcal{L}(A), P\delta(P)Q$: individuals rank first among orders their own order of alternatives.
- *Neutrality*: δ is neutral if $\forall A \in \mathcal{A}$, $\forall \sigma$ a permutation of A, $\forall P, Q, Q' \in \mathcal{L}(A), Q\delta(P)Q' \Leftrightarrow Q^\sigma\delta(P^\sigma)Q'^\sigma$: any reshuffling of the names of alternatives leads to the same reshuffling of the ranks of alternatives.
- *Clone-consistency*: Given $A \in \mathcal{A}$ together with $a \in A$, define the set $\mu_t(a) = \{a^1, \ldots, a^t\}$ as the set of alternatives obtained by creating $t-1$ copies, or clones, of $a = a^1$, where $\{a^2, \ldots, a^t\} \cap A = \varnothing$. Given $P \in \mathcal{L}(A)$, a $\mu_t(a)$-extension of P is an element P^{μ_t} of $\mathcal{L}(A \cup \mu(a))$ such that $\forall k = 1, \ldots, t, \forall b \in A \backslash \{a\}, aPb \Leftrightarrow a^kP^{\mu_t}b$, and $\forall b, c \neq a, bPc \Leftrightarrow bP^{\mu_t}c$. It follows that if $a = a_h$ and if $P = (a_1 \ldots a_m)$, then $P^{\mu_t} = (a_1 \ldots a_{h-1}Qa_{h+1} \ldots a_m)$, where $Q \in \mathcal{L}(\mu_t(a_h))$. Thus P^{μ_t} is obtained from P by replacing one alternative a by a segment involving a and $t-1$ copies of a. We write $P^{\mu_t} = P(\rightarrow Q \rightarrow)$. A preference extension δ is clone-consistent if $\forall A \in \mathcal{A}, \forall P, P', P'' \in \mathcal{L}(A), \forall a \in A, \forall t \geq 1$, $\forall Q, Q' \in \mathcal{L}(\mu_t(a))$, we have
$$P'\delta(P)P'' \Leftrightarrow P'(\rightarrow Q \rightarrow)\delta(P(\rightarrow Q' \rightarrow))P''(\rightarrow Q \rightarrow).$$
Clone-consistency holds if whenever one alternative is replicated $t-1$ times, any two orders completed by the same order Q of those t clones will be compared as initially when the initial hyper-preference is completed by some (maybe different) order Q' of the clones.
- *Independence*: A preference extension δ is independent if $\forall A \in \mathcal{A}, \forall P, P', P'' \in \mathcal{L}(A)$ with $P'\delta(P)P''$, $\forall a \notin A$, $\forall \widetilde{P} \in \mathcal{L}(A \cup \{a\})$ such that $\widetilde{P}|_A = P$, we have $(aP')\delta(\widetilde{P})(aP'')$ and $(P'a)\delta(\widetilde{P})(P''a)$: any two orders completed by putting either at top or at bottom the same additional alternative will be compared as before completion.

Below are three examples of regular, neutral, clone-consistent and independent preference extensions.

- The *lexicographic* preference extension δ_L is defined as follows: given any $P, Q = (a_1a_2 \ldots a_m), Q' = (b_1b_2 \ldots b_m) \in \mathcal{L}(A)$, one has $Q\delta_L(P)Q'$ if and only if there exists $k^* \in \{1, \ldots, m-1\}$ such that $[k < k^* \Rightarrow a_k = b_k$, and $a_{k^*}Pb_{k^*}]$. Thus, given an order P, $\delta_L(P)$ compares two orders according to the position in P of the best-ranked alternative on which they differ.
- The *inverse lexicographic* preference extension δ_{IL} is defined as follows: given any $P, Q = (a_1a_2 \ldots a_m), Q' = (b_1b_2 \ldots b_m) \in \mathcal{L}(A)$, one has $Q\delta_{IL}(P)Q'$ if and only if there exists $k^* \in \{1, \ldots, m-1\}$ such that $[k > k^* \Rightarrow a_k = b_k$, and $b_{k^*}Pa_{k^*}]$. Given an order P, $\delta_{IL}(P)$ compares two orders according to the position in P of the lowest-ranked alternative on which they differ.
- The *Kemeny distance* between two tournaments T, T' over A is defined by $d_K(T, T') = \left|\{(a, b) \in A \times A : aTb \text{ and } bT'a\}\right|$, that is the number of pairs of alternatives T and T' disagree on. The Kemeny distance allows to define a class of preference extensions. Given $P \in \mathcal{L}(A)$, define $R_K(P)$ as the weak order over $\mathcal{L}(\mathcal{L}(A))$ such that for any $P', P'' \in \mathcal{L}(A), P'R_K^+(P)P''$ if and only if $d_K(P, P') < d_K(P, P'')$. δ is a *Kemeny preference extension* if for all $A \in \mathcal{A}$ and all $P \in \mathcal{L}(A)$, $\delta(P)$ is a linearization of $R_K(P)$.

2.3. Hyper-stability

We turn now to the key concept of hyper-stability for SWFs.

Definition 3. An SWF α is *hyper-stable for the preference extension* δ if $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A)$, we have $\Delta(\alpha(\pi)) \supseteq F^{\alpha}(\pi^{\delta})$. Moreover, α is *hyper-stable* if it is hyper-stable for some δ.

An SWF α is hyper-stable for the preference extension δ if at any profile π, only linearizations of the weak order $\alpha(\pi)$ are ranked first by α when applied to the hyper-profile π^{δ} built from π by means of δ. The set $\Delta(\alpha(\pi))$ of all linearizations of $\alpha(\pi)$ is interpreted as the set of potentially chosen rankings (from the profile over alternatives), each resulting from a specific tie-breaking rule. Requiring $\Delta(\alpha(\pi))$ to contain $F^{\alpha}(\pi^{\delta})$ means that no best order chosen at hyper-profile π^{δ} contradicts the ranking of indifference classes in the weak order $\alpha(\pi)$ chosen from the initial profile over alternatives. However, since we do not require the two sets $\Delta(\alpha(\pi))$ and $F^{\alpha}(\pi^{\delta})$ to coincide, the way ties are broken in $\alpha(\pi)$ may matter. Hence, hyper-stability holds when α gives mutually consistent outcomes at two different levels of choice, 'consistent' meaning that disagreement between outcomes can exist only about the way social indifference is broken.

Since we consider regular preference extensions, an SWF can be conceived as a tops-only choice function, which obviously involves a loss of information about individual preferences. What hyper-stability states is that there exists at least one underlying hyper-profile at which there is no such loss.

For an illustration, consider the following example.

Example 1. Given any m, a score vector is defined by $S^m = (s^{1,m}, s^{2,m}, \ldots, s^{m,m}) \in \mathbb{R}^m$, where (1) $s^{m,m} = 0$, (2) $s^{1,m} \geq s^{2,m} \geq \cdots \geq s^{m,m}$, and (3) $s^{1,m} > 0$. Given profile $\pi = (P_1, \ldots, P_n)$ over A with $|A| = m$ together with a score vector S^m, the score of $a \in A$ in π is $S^m(a, \pi) = \sum_{i=1}^{n} s^{r_i(a,\pi),m}$, where $r_i(a, \pi)$ is the rank of a in P_i. An SWF α is a scoring rule if at any profile alternatives are ranked according to their scores. Formally, there exists a sequence $\{S^m_\alpha\}_{m\geq 1} = \{S^1_\alpha, S^2_\alpha, S^3_\alpha, \ldots\}$ of score vectors such that, for all A, for all $\pi \in \Pi(A)$, for all $a, b \in A$, $a\alpha(\pi)b \iff S^m_\alpha(a, \pi) \geq S^m_\alpha(b, \pi)$. The *anti-plurality rule* is the scoring rule α_{ant} based on the score vector giving score 0 to the least-preferred alternative, and score 1 to all other alternatives. Consider the following profile π involving 3 alternatives and 15 individuals

$\pi =$

P_1, P_2, P_3	P_4, P_5	P_6, P_7, P_8	P_9, P_{10}, P_{11}	$P_{12}, P_{13}, P_{14}, P_{15}$
a	a	b	c	c
b	c	a	a	b
c	b	c	b	a

where $a\alpha^{+}_{ant}(\pi)b\alpha^{+}_{ant}(\pi)c$, so that $\Delta(\alpha_{ant}(\pi)) = \{P_1\}$. Consider the lexicographic preference extension δ_L. We get the following hyper-profile π^{δ_L} of π

$\pi^{\delta_L} =$

$\delta_L(P_i)$ $i = 1, 2, 3$	$\delta_L(P_i)$ $i = 4, 5$	$\delta_L(P_i)$ $i = 6, 7, 8$	$\delta_L(P_i)$ $i = 9, 10, 11$	$\delta_L(P_i)$ $i = 12, 13, 14, 15$
abc	acb	bac	cab	cba
acb	abc	bca	cba	cab
bac	cab	abc	acb	bca
bca	cba	acb	abc	bac
cab	bac	cba	bca	acb
cba	bca	cab	bac	abc

where $acb\alpha^{+}_{ant}(\pi^{\delta_L})bca\alpha^{+}_{ant}(\pi^{\delta_L})bac\alpha^{=}_{ant}(\pi^{\delta_L})cba\alpha^{=}_{ant}(\pi^{\delta_L})bca\alpha^{+}_{ant}(\pi^{\delta_L})abc$, and thus $F^{\alpha_{ant}}(\pi^{\delta_L}) \nsubseteq \Delta(\alpha_{ant}(\pi))$: the anti-plurality rule is not hyper-stable for δ_L.

3. Sufficient condition for hyper-stability

We first introduce the property of *non-sensitivity to cloning* for SWFs, which describes how the comparison between alternatives is partially preserved when replacing these alternatives by the same number of perfect copies. Given profile $\pi = (P_1, \ldots, P_n)$ over A, a profile $\pi^t = (P^t_1, \ldots, P^t_n) \in \mathcal{L}(A_t)^n$ is called a t-replica of π if there exists a mapping μ from A to A_t such that:

(1) $\forall a \in A, \mu(a) = \{a^t_1, \ldots, a^t_t\}$
(2) $\forall a, b \in A, \mu(a) \cap \mu(b) = \varnothing$
(3) $\forall i = 1, \ldots, n, \forall a, b \in A, aP_ib \Rightarrow a^t_jP^t_ib^t_{j'}$ for all $j, j' = 1, \ldots, t$.

Hence, a t-replica of a profile π is obtained by cloning t times each of the alternatives, so that clones of two different alternatives are compared as their original copies. Non-sensitivity to cloning (NSC) holds for an SWF α when top-ranked clones in $\alpha(\pi^t)$ are clones of top-ranked alternatives in $\alpha(\pi)$, and each top-ranked alternative in $\alpha(\pi)$ has at least one top-ranked clone in $\alpha(\pi^t)$: $\forall A \in \mathcal{A}, \forall t \in \mathbb{N}, \forall \pi \in \Pi(A), \forall a \in A, a \in F^{\alpha}(\pi)$ if and only if $\mu(a) \cap F^{\alpha}(\pi^t) \neq \varnothing$.

Next, we consider two weak versions of the well-known property of independence of irrelevant alternatives. Suppose that some subset of top-ranked (resp. bottom-ranked) alternatives is deleted, and consider the residual profile, that is the restriction of the initial profile to the set of remaining alternatives. The two properties below impose some logical link between alternatives in the residual set that are top-ranked (or bottom-ranked) at the initial profile on the one hand, and top-ranked (or bottom-ranked) alternatives in the residual profile on the other hand. Given two disjoint subsets $B, C \subseteq A$, we write $B\alpha(\pi)C$ if $b\alpha(\pi)c$ for all $(b, c) \in B \times C$.

An SWF α is *Top-IIA* if when removing any subset of first-best alternatives, any first-best alternative at the residual profile was already first-best within the residual set: $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A)$, $\forall B \subset A$ such that $B\alpha(\pi)A\backslash B$, $F^{\alpha}(\pi\,|_{A\backslash B}) \subseteq Max(\alpha(\pi), A\backslash B)$.

Similarly, α is *Bottom-IIA* if, when removing any subset of bottom-ranked candidates, any bottom-ranked alternative at the residual profile was initially bottom-ranked in the residual set: $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \forall B \subset A$ with $(A\backslash B)\alpha(\pi)B$, $F^{-\alpha}(\pi\,|_{A\backslash B}) \subseteq Min(\alpha(\pi), A\backslash B)$.

For an illustration, consider the following weak order $\alpha(\pi)$ over $A = \{a_1, \ldots, a_6\}$

$$a_1\alpha^{=}(\pi)a_2\alpha^{+}(\pi)a_3\alpha^{+}(\pi)a_4\alpha^{=}(\pi)a_5\alpha^{+}(\pi)a_6.$$

Suppose that α is Top-IIA and delete alternatives a_1 and a_2. Then a_3 is the unique best alternative for α in the residual set $A\backslash\{a_1, a_2\}$. Top-IIA says that a_3 must be the unique alternative that is top-ranked by α at the restriction of π to $A\backslash\{a_1, a_2\}$. If a_1, a_2 and a_3 are removed, then either a_4 or a_5, or both, are top-ranked in the restriction of π to $A\backslash\{a_1, a_2, a_3\}$. However, a_6 cannot be top-ranked by α in this restriction of π.

Combining NSC with either Top-IIA or Bottom-IIA suffices to ensure hyper-stability.

Proposition 1. *Any SWF that is non-sensitive to cloning and satisfies either Top − IIA or Bottom − IIA is hyper-stable.*

Proof. Let α be an SWF satisfying Top-IIA and NSC. Pick any profile $\pi = (P_1, \ldots, P_n)$ over A. First note that $\forall a \in F^{\alpha}(\pi)$, $\mathcal{P}(a \rightarrow) \cap \Delta(\alpha(\pi)) \neq \varnothing$. Let $\pi^{\delta_L} = (P^{\delta_L}_1, \ldots, P^{\delta_L}_n) \in \mathcal{L}(\mathcal{L}(A))^n$ be the hyper-profile built from π by means of the lexicographic preference extension δ_L. It follows from the definition of δ_L that $\forall 1 \leq i \leq n$, $\forall 1 \leq h, k \leq m$, $a_hP_ia_k \Rightarrow \widetilde{P}P^{\delta_L}_i\widehat{P}$ for all $(\widetilde{P}, \widehat{P}) \in \mathcal{P}(a_h \rightarrow) \times \mathcal{P}(a_k \rightarrow)$. Since $\{\mathcal{P}(a_1 \rightarrow), \ldots, \mathcal{P}(a_m \rightarrow)\}$ is a partition of $\mathcal{L}(A)$ into subsets with equal cardinality $(m-1)!$, then π^{δ_L} is a $(m-1)!$-replica of π. Now choose any $P = (b_1 \ldots b_m) \in F^{\alpha}(\pi^{\delta_L})$. NSC implies that $b_1 \in F^{\alpha}(\pi)$. Define $A^1 = A\backslash\{b_1\}$. From the definition of δ_L, we get that $\forall i, \forall b, c \in A^1$, $bP_ic \Rightarrow \widetilde{P}P^{\delta_L}_i\widehat{P}$ for all $(\widetilde{P}, \widehat{P}) \in \mathcal{P}(b_1b \rightarrow) \times \mathcal{P}(b_1c \rightarrow)$. Hence, $\mathcal{P}(b_1b \rightarrow) \cap \Delta(\alpha(\pi)) \neq \varnothing$ for any $b \in F^{\alpha}(\pi\,|_{A^1})$. Furthermore, $\cup_{a\in A^1} \mathcal{P}(a_1a)$ forms a partition of $\mathcal{L}(A^1)$ into subsets with equal cardinality $(m-2)!$. Thus $\pi^{\delta_L}\,|_{\mathcal{L}(A^1)}$ is a $(m-2)!$-replica of $\pi\,|_{A^1}$, and NSC implies that $b_2 \in F^{\alpha}(\pi\,|_{A^1})$. Moreover, Top-IIA implies that $b_2 \in F^{\alpha}(\pi\,|_{A^1})$. Hence $\mathcal{P}(b_1b_2 \rightarrow) \cap \Delta(\alpha(\pi)) \neq \varnothing$. Let $A^2 = A^1\backslash\{b_2\}$. Using again the definition of δ_L, we get that $\mathcal{P}(b_1b_2b \rightarrow) \cap \Delta(\alpha(\pi)) \neq \varnothing$ for any $b \in F^{\alpha}(\pi\,|_{A^2})$, while NSC ensures that $b_3 \in F^{\alpha}(\pi\,|_{A^2})$. Again, it follows from Top-IIA that $\mathcal{P}(b_1b_2b_3\ \rightarrow) \cap \Delta(\alpha(\pi)) \neq \varnothing$. By iterating the same argument as the one above to $A^3 = A^2\backslash\{b_3\}$, A^4, ..., A^m, we get $P = (b_1b_2b_3\dots b_m) \in \Delta(\alpha(\pi))$, and thus $\Delta(\alpha(\pi)) \supseteq F^{\alpha}(\pi^{\delta_L})$, which proves that α is hyper-stable for δ_L. If α satisfies Bottom-IIA and NSC, one proves by following a similar construction that α is hyper-stable for the inverse lexicographic extension δ_{IL}. □

There exist Condorcet SWFs that satisfy NSC and Top-IIA. Indeed, define the SWF α_{cond} by: $\alpha_{cond}(\pi) = T_\pi$ if T_π is transitive, while $a\alpha^{=}_{cond}(\pi)b$ for all $a, b \in A$ otherwise. Since α_{cond} clearly satisfies Top-IIA and NSC, then it is hyper-stable for δ_L. We show below that hyper-stability also holds for other Condorcet SWFs that are somehow less degenerated than α_{cond}.

4. Hyper-stability of Condorcet social welfare functions

We restrict in the sequel the analysis to the case where the number n of individuals is odd.[4] We distinguish two types of Condorcet SWFs. The first type consists in ranking alternatives by successively choosing the best ones according to some Condorcet SCC, while the second ranks all alternatives at once.

4.1. Sequential Condorcet social welfare functions

A rather intuitive way to select a ranking is to successively apply a choice correspondence. Think about a situation where alternatives are athletes involved in a sport competition. In order to rank them, one may first decide about which athlete(s) are the best, putting them on top of the social ranking. The next step is to decide who should be ranked second, that is who should be declared the best athletes among the remaining ones…An so on until all athletes are ranked. At each step is applied a specific choice correspondence, called a tournament solution, based on pairwise majority comparisons between alternatives.[5]

Formally, let $\mathcal{T}(A)$ stand for the set of all tournaments over A. Given a subset $B \subset A$, the restriction of T to B is denoted by $T\,|_B$. A *tournament solution* is a function $\mathbf{S}$ from $\cup_{A\in\mathcal{A}}\mathcal{T}(A)$ to $\cup_{A\in\mathcal{A}} 2^A\backslash\varnothing$ such that, for all $A \in \mathcal{A}$ and all $\forall T \in \mathcal{T}(A)$,

- $\mathbf{S}(T) \in 2^A\backslash\varnothing$
- $\forall a \in A$, $\mathbf{S}(T) = \{a\}$ whenever a is a Condorcet winner of T.

We briefly recall the definition of several well-known tournament solutions.[6] Let T be a tournament over A.

- The *top-cycle* of T is the set $\boldsymbol{TC}(T) = \{a \in A : \forall b \in A, \exists(a_1,\dots,a_K) \in A^K$ such that $a_1 = a, a_K = b$, and $1 \le h < k \le K \Rightarrow a_hTa_k\}$.
- The *uncovered set* of T is the set $\boldsymbol{UC}(T)$ of maximal elements of A for the covering relation $\gg_T$ defined on A by: $\forall a, b \in A, a \gg_T b$ if aTb and for all $c \in A, bTc \Rightarrow aTc$.
- The *minimal covering set* of T is the (unique) minimal (for inclusion) set $\boldsymbol{MC}(T)$ such that $\forall b \notin \boldsymbol{MC}(T), b \notin \boldsymbol{UC}(T\,|_{\boldsymbol{MC}(T)\cup\{b\}})$.
- The *Banks set* of T is the set $\boldsymbol{BA}(T) = \{a \in A : \exists$ a maximal transitive path $\widehat{T}$ in T with a as top element$\}$.[7]

[4] Assuming an odd number of voters precludes ties in pairwise majority comparisons. However, there may be ties in the ranking, since several candidates may be declared best. As an example, no tennis game can end up without a winner, whereas the ranking of tennis players may allow for ties.

[5] The reader may refer to Bouyssou (2004) for a study of the monotonicity properties of sequential Condorcet social welfare functions. More references about properties of ranking methods based on successive choices can be found there.

[6] A detailed review of tournament solutions can be found in Laslier (1997).

[7] A transitive path in T is a transitive sub-tournament $\widehat{T} \in \mathcal{T}\,|_B$, where $B \subseteq A$, and $\widehat{T}$ is maximal if $\nexists a \in A$ such that aTb for all $b \in B$.

- The *bipartisan set* of T is the support $\boldsymbol{BP}(T)$ of the (unique and symmetric) equilibrium in mixed strategies of the 2-player game $\mathcal{G}(T)$ where each of the players 1, 2 have A as set S_i of pure strategies, and payoffs are defined by: $p_1(a, b)$ is 1 if aTb, -1 if bTa, and 0 if $a = b$, where $(a, b) \in S_1 \times S_2$.

Given a tournament T together with a tournament solution $\mathbf{S}$, we write $\mathbf{S}^1(T) = \mathbf{S}(T\,|_{A\backslash\mathbf{S}(T)}), \dots, \mathbf{S}^k(T) = \mathbf{S}(T\,|_{A\backslash\cup_{z\le k-1}\mathbf{S}^z(T)})$, with the convention $\mathbf{S}^0(T) = \mathbf{S}(T)$.

Definition 4. An SWF α is a sequential tournament solution if there exists a tournament solution $\mathbf{S}$ such that, for all $A \in \mathcal{A}$, for all $\pi \in \Pi(A)$, for all $a, b \in A$,

- $a\alpha^{+}(\pi)b$ if and only if there exists $k \in \mathbb{N}$ such that $a \in \mathbf{S}^k(T_\pi)$ and $b \notin \cup_{z\le k}\mathbf{S}^z(T_\pi)$
- $a\alpha^{=}(\pi)b$ if and only if there exists $k \in \mathbb{N}$ such that $a, b \in \mathbf{S}^k(T_\pi)$.

If α is a sequential tournament solution associated with $\mathbf{S}$, we write $\alpha = Seq\mathbf{S}$. Composition-consistency for tournament solutions easily derives from Definition 2:

Definition 5. A tournament solution $\mathbf{S}$ is composition-consistent (CC) if for all $A \in \mathcal{A}$, and for any composed tournament $T = \otimes(T^*; T^1, \dots, T^H) \in \mathcal{L}(A)$, we have $\mathbf{S}(T) = \cup\{\mathbf{S}(T^h), h \in \mathbf{S}(T^*)\}$.

Another interesting property of tournament solutions is the strong superset property, which holds when the set of chosen alternatives is invariant under the removal of non-chosen candidates.

Definition 6. A tournament solution $\mathbf{S}$ satisfies the strong superset Property (SSP) if $\forall A \in \mathcal{A}$, $\forall T \in \mathcal{T}(A)$, $\forall B \subset A$, $\mathbf{S}(T) \subseteq B$ implies $\mathbf{S}(T\,|_B) = \mathbf{S}(T)$.

Composition-consistency together with the strong superset property ensure the hyper-stability of sequential tournament solutions.

Proposition 2. *If a tournament solution* $\mathbf{S}$ *is composition-consistent and satisfies the strong superset property, then Seq*$\mathbf{S}$ *is hyper-stable.*

Proof. We prove that SSP of $\mathbf{S}$ implies Bottom-IIA of $Seq\mathbf{S}$, while CC of $\mathbf{S}$ implies NSC of $Seq\mathbf{S}$. Choose any $A \in \mathcal{A}$ and any profile $\pi \in \Pi(A)$. From the definition of $Seq\mathbf{S}$, $\{\mathbf{S}^0(T_\pi), \dots, \mathbf{S}^K(T_\pi)\}$ forms a partition of A into K non-empty sets, such that: $\forall h, k \in \{0, \dots, K\}$, $\forall(a, b) \in \mathbf{S}^h(T_\pi) \times \mathbf{S}^k(T_\pi)$, $h < k \Rightarrow a\alpha^{+}(\pi)b$, and $h = k \Rightarrow a\alpha^{=}(\pi)b$.

In order to show that SSP of $\mathbf{S}$ implies Bottom-IIA of $Seq\mathbf{S}$, consider any $B \subset A$ such that $A\backslash B\alpha^{+}(\pi)B$. Now choose any $a, b \in A\backslash B$ such that $a\alpha^{+}(\pi)b$. Let $h \in \{0, \dots, K\}$ be such that $a \in \mathbf{S}^h(T_\pi)$. Thus, $B \subseteq \cup_{h+1\le z\le K}\mathbf{S}^z(T_\pi)$. It follows from the definition of $Seq\mathbf{S}$ that $b \notin \mathbf{S}^h(T_\pi)$ and $d \notin \mathbf{S}^h(T_\pi)$ for all $d \in B$. Hence, $\mathbf{S}^h(T_\pi) \subseteq A\backslash B$. Since $b \notin \mathbf{S}^h(T_\pi)$, one has $\mathbf{S}^h(T_\pi) \neq A\backslash B$, and therefore $\mathbf{S}^h(T_\pi) \subset A\backslash B$. Since $\mathbf{S}$ satisfies SSP, then $\mathbf{S}^h(T_\pi\,|_{A\backslash B}) = \mathbf{S}^h(T_\pi)$. Thus, $b \notin \mathbf{S}^h(T_\pi\,|_{A\backslash B})$ and $a \in \mathbf{S}^h(T_\pi\,|_{A\backslash B})$, which implies that $a\alpha^{+}(\pi\,|_{A\backslash B})b$. This shows that $a\alpha^{+}(\pi)b \Rightarrow a\alpha^{+}(\pi\,|_{A\backslash B})b$.

Suppose now that $a\alpha^{=}(\pi)b$, where $a, b \in \mathbf{S}^h(T_\pi)$, and $a, b\alpha^{+}(\pi)d$ for any $d \in B$. Then $\mathbf{S}^h(T_\pi\,|_{A\backslash B}) = \mathbf{S}^h(T_\pi) \Rightarrow a, b \in \mathbf{S}^h(T_\pi\,|_{A\backslash B})$. Hence, $a\alpha^{=}(\pi\,|_{A\backslash B})b$. Thus, we have for any $a, b \in (A\backslash B)$, $a\alpha^{+}(\pi)b \Rightarrow a\alpha^{+}(\pi\,|_{A\backslash B})b$ and $a\alpha^{=}(\pi)b\alpha^{+}(\pi)d$ for all $d \in B \Rightarrow a\alpha^{=}(\pi\,|_{A\backslash B})b$. This clearly ensures that $F^{-\alpha}(\pi)\,|_{A\backslash B} = F^{-\alpha}(\pi\,|_{A\backslash B})$, therefore $Seq\mathbf{S}$ satisfies Bottom-IIA.

Finally, in order to show that CC of $\mathbf{S}$ implies NSC of $Seq\mathbf{S}$, let π^t be a t-replica of π. It follows from the definition of π^t that $T_{\pi^t} = \otimes(T^*; T^1, \dots, T^K)$, where $T^* \in \mathcal{T}(\{1, \dots, m\})$, and T^k is a tournament over $\mu(a_k)$ for all $1 \le k \le m$. Consider any $a \in Max(\alpha(\pi))$. From the definition of $Seq\mathbf{S}$, we have that $a \in \mathbf{S}(T_\pi)$. Then CC of $\mathbf{S}$ implies that $\mathbf{S}(T_{\pi^t}) = \cup_{a_k\in\mathbf{S}(T_\pi)}\{\mathbf{S}(T^k)\}$, then $\mu(a) \cap \mathbf{S}(T_{\pi^t}) \neq \varnothing$. Hence, one clone of a is ranked first by α from π^t, and thus $Seq\mathbf{S}$ satisfies NSC. We conclude by means of Proposition 1. □

Given a profile π together with a sequential tournament solution $\alpha = Seq\mathbf{S}$, define $K(\pi, \mathbf{S})$ as the number of indifference classes in $\alpha(\pi)$. Fix $\bar{k} \in \mathbb{N}$, and define the $\bar{k}$-sequential tournament solution $\alpha_{\bar{k}}$ as follows: $\forall a, b \in A$,

- $a\alpha^{+}(\pi)b$ if and only if there exists $k \leq \bar{k}$ such that $a \in \mathbf{S}^{k}(T_\pi)$ and $b \notin \cup_{z \leq k} \mathbf{S}^{z}(T_\pi)$
- $a\alpha^{=}(\pi)b$ if and only if either there exists $k \leq \bar{k}$ such that $a, b \in \mathbf{S}^{k}(T_\pi)$, or $a, b \notin \cup_{z \leq \bar{k}} \mathbf{S}^{z}(T_\pi)$.

It is straightforward to see that the proof of Proposition 2 allows for stating the following more general result: If $\mathbf{S}$ satisfies SSP and CC, then for any $\bar{k} \in \mathbb{N}$, $\alpha_{\bar{k}}$ is hyper-stable.

Since ***MC*** and ***BP*** satisfy CC and SSP, an immediate consequence of Proposition 2 is,

Proposition 3. *Seq**MC** and Seq**BP** are hyper-stable (for δ_{IL}).*

Another hyper-stable sequential tournament solution is *Seq**TC***:

Proposition 4. *Seq**TC** is hyper-stable for both δ_L and δ_{IL}.*

Proof. Note first that ***TC*** satisfies SSP. Using the proof of Proposition 3, we get that *Seq**TC*** satisfies Bottom-IIA. Therefore, Proposition 2 ensures that *Seq***TC** is hyper-stable for δ_{IL} if it satisfies NSC. Let π be a profile over A and consider any t-replica π^t of π. Then $T_{\pi^t} = \otimes(T_\pi; T^1, \ldots, T^m)$, where $T^* \in \mathcal{L}(\{1, \ldots, m\})$, and $T^h \in \mathcal{L}(\mu(a_h))$, $1 \leq h \leq m$. By definition of the composition product, one has $\forall(a, b) \in A \times A, \forall(a_h^t, b_{h'}^t) \in \mu(a) \times \mu(b), aT_\pi b \Leftrightarrow a_h^t T_{\pi^t} b_{h'}^t$. Thus, if b does not defeat directly or indirectly a in T_{π^t}, no $b_{h'}^t$ in $\mu(b)$ can defeat in T_{π^t} directly or indirectly some a_h^t in $\mu(a)$. Moreover, if $a_h \in \boldsymbol{TC}(T)$, at least one a_h^t in $\boldsymbol{TC}(T^h)$ defeats directly or indirectly in T_{π^t} all other alternatives in T. Thus, $a \in \boldsymbol{TC}(T_\pi) \Leftrightarrow \mu(a) \cap \boldsymbol{TC}(T_{\pi^t}) \neq \varnothing$, and NSC holds. □

We turn now to *Seq**UC*** and *Seq**BA***. We first state two useful results.

Lemma 1 (McGarvey, 1953)**.** *For any $A \in \mathcal{A}$, for any $T \in \mathcal{T}(A)$, there exists $\pi \in \Pi(A)$ such that $T = T_\pi$.*

A preference extension δ is said to be *tournament-consistent* if at any profile having a linear order P as majority tournament, δ generates a hyper-profile having P as Condorcet winner. Formally, δ is tournament-consistent if for any $A \in \mathcal{A}$, for any $\pi \in \Pi(A)$, $T_\pi \in \mathcal{L}(A)$ only if T_π is a Condorcet winner of T_{π^δ}.

Lemma 2 (Laffond and Lainé (2000, Theorem 2))**.** *A neutral, regular and independent preference extension δ is tournament-consistent if and only if $\delta \in \{\delta_L, \delta_{IL}\}$.*

The proof of Lemma 2 is by induction over $m \geq 3$, and strongly rests upon the independence property. Note that δ_L and δ_{IL} are Kemeny preference extensions only in the 3-alternative case. It is rather straightforward to see that there are Kemeny preference extensions which are not tournament-consistent. Indeed, consider the following profile π over $A = \{a_1, a_2, a_3\}$ such that $T_\pi = P_1$:

$$\pi = \begin{pmatrix} P_1 & P_2 & P_3 \\ \hline a_1 & a_3 & a_2 \\ a_2 & a_1 & a_1 \\ a_3 & a_2 & a_3 \end{pmatrix}$$

Consider the Kemeny preference extension $\widetilde{\delta}$ such that $P_1\widetilde{\delta}(P_1)$ $(a_1a_3a_2)\widetilde{\delta}(P_1)(a_2a_1a_3)\widetilde{\delta}(P_1)(a_3a_1a_2)\widetilde{\delta}(P_1)(a_2a_3a_1)\widetilde{\delta}(P_1)(a_3a_2a_1)$. It follows from neutrality that $\widetilde{\delta}(P)$ is well-defined for all $P \in \mathcal{L}(A)$. In particular, one has $(a_2a_3a_1)\widetilde{\delta}(P_2)P_1$ and $(a_2a_3a_1)\widetilde{\delta}(P_3)P_1$. Therefore, $\widetilde{\delta}$ is not tournament-consistent.

Proposition 5. *Neither Seq**UC** nor Seq**BA** is hyper-stable.*

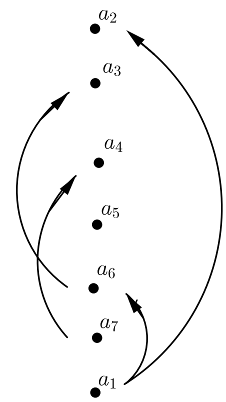

Fig. 3. Neither *Seq**UC*** nor *Seq**BA*** is hyper-stable for δ_L.

Proof. Note first that both ***UC*** and ***BA*** are composition-consistent. Hence, using the proof of Proposition 2, *Seq**UC*** and *Seq**B*** satisfy NSC.

The proof is organized in 3 claims.

Claim 1: If *Seq***UC** and *Seq**BA*** are hyper-stable for δ, then $\delta \in \{\delta_L, \delta_{IL}\}$.

Proof of Claim 1: We first show that if *Seq***UC** and *Seq**BA*** are hyper-stable for δ, then δ is tournament-consistent. Let π be any profile over A such that $T_\pi \in \mathcal{L}(A)$. For any tournament solution S, we have $SeqS(\pi) = \{T_\pi\}$. Therefore $\Delta(SeqS(\pi)) = \{T_\pi\}$. If $SeqS$ is hyper-stable for δ, then $F^{SeqS}(\pi^\delta) \subseteq \Delta(SeqS(\pi))$. Thus $F^{SeqS}(\pi^\delta) = \{T_\pi\}$. Since $F^{SeqS}(\pi^\delta) = S(T_{\pi^\delta})$, then $S(T_{\pi^\delta}) = \{T_\pi\}$. Furthermore, it is obviously checked that for any tournament T, $\mathbf{UC}(T) = \{a\}$ if and only if a is a Condorcet winner of T. Since $\boldsymbol{BA}(T) \subseteq \boldsymbol{UC}(T)$, then the same result holds for ***BA***. Hence, T_π is a Condorcet winner of T_{π^δ} whenever $S \in \{\boldsymbol{UC}, \boldsymbol{BA}\}$. Therefore δ is tournament-consistent, and Claim 1 follows from Lemma 2.

Claim 2: Neither *Seq**UC*** nor *Seq**BA*** is hyper-stable for δ_L.

Proof of Claim 2: Consider the tournament T over $A = \{a_1, \ldots, a_7\}$ described in Fig. 3 (where an edge from a to b means aTb and where only edges pointing upwards are drawn):

The reader will check that $\boldsymbol{UC}(T) = \{a_1, \ldots, a_5\}$. From Lemma 1, there exists $\pi = (P_1, \ldots, P_n) \in \Pi(A)$ such that $T = T_\pi$. Then *Seq**UC***(π) defines the following weak order $\succsim$ over A:

$$a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5 \succ a_6 \succ a_7.$$

It is easily checked that $\boldsymbol{BA}(T) = \{a_1, \ldots, a_4\}$, $\boldsymbol{BA}(T\mid_{\{a_5,a_6,a_7\}}) = \{a_5\}$, and $\boldsymbol{BA}(T\mid_{\{a_6,a_7\}}) = \{a_6\}$. Thus, *Seq**BA***$(\pi)$ defines the following weak order $\succsim^*$ over A:

$$a_1 {\sim^*} a_2 {\sim^*} a_3 {\sim^*} a_4 \succ^* a_5 \succ^* a_6 \succ^* a_7.$$

One gets that $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap \Delta(Seq\boldsymbol{BA}(T_\pi)) = \varnothing$. Since $\boldsymbol{BA}(T_\pi) \subseteq \boldsymbol{UC}(T_\pi)$, then $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap \Delta(Seq\boldsymbol{UC}(T_\pi)) = \varnothing$.

It follows from the definition of δ_L that for all individuals i, for all $h, k \in \{1, \ldots, 7\}$, $a_hP_ia_k$ if and only if $QP_i^{\delta_L}Q'$ for all $(Q, Q') \in \mathcal{P}(a_h \rightarrow) \times \mathcal{P}(a_k \rightarrow)$. Thus $\{\mathcal{P}(a_1 \rightarrow), \ldots, \mathcal{P}(a_7 \rightarrow)\}$ forms a partition of $\mathcal{L}(\mathcal{L}(A))$ into components of π^{δ_L}, each with cardinality 6!. Moreover, by defining for all $1 \leq h \leq 7$, $\mu(a_h) = \mathcal{P}(a_h \rightarrow)$, one gets that π^{δ_L} is a (6!)-replica of π. Since $a_1 \in \boldsymbol{BA}(T_\pi)$, then NSC implies that $\mathcal{P}(a_1 \rightarrow) \cap \boldsymbol{BA}(T_{\pi^{\delta_L}}) \neq \varnothing$. Now consider $\pi^{\delta_L}\mid_{\mathcal{P}(a_1\rightarrow)}$. Using again the definition of δ_L, one has for all individuals i, for all $2 \leq h, k \leq 7$, $a_hP_ia_k$ if and only if $QP_i^{\delta_L}Q'$ for all $(Q, Q') \in \mathcal{P}(a_1a_h \rightarrow) \times \mathcal{P}(a_1a_k \rightarrow)$. Hence $\{\mathcal{P}(a_1a_2 \rightarrow), \ldots, \mathcal{P}(a_1a_7 \rightarrow)\}$ forms a partition of $\mathcal{L}(\mathcal{L}(A\backslash\{a_1\}))$ into components of $\pi^{\delta_L}_{\mathcal{P}(a_1\rightarrow)}$, each with cardinality 5!. Therefore $\pi^{\delta_L}\pi^{\delta_L}_{\mathcal{P}(a_1\rightarrow)}$ is a (5!)-replica of $\pi\mid_{A\backslash\{a_1\}}$. Since a_2 is a Condorcet winner of $T\mid_{A\backslash\{a_1\}}$, then $\boldsymbol{BA}(T_\pi\mid_{A\backslash\{a_1\}}) = \{a_2\}$. Thus NSC ensures that $\mathcal{P}(a_1a_2 \rightarrow) \cap \boldsymbol{BA}(T_{\pi^{\delta_L}}) \neq \varnothing$. The construction carries on along the same lines as above. One gets that $\pi^{\delta_L}\mid_{\mathcal{P}(a_1a_2\rightarrow)}$ is a (4!)-replica of $\pi\mid_{A_7\backslash\{a_1,a_2\}}$ by observing that for all i, for all $3 \leq$

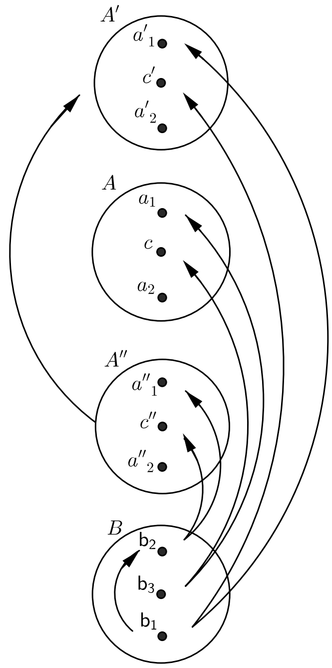

Fig. 4. Neither *Seq**UC*** nor *Seq**BA*** is hyper-stable for δ_{IL}.

$h, k \leq 6$, $a_hP_ia_k$ if and only if $QP_i^{\delta_L}Q'$ for all $(Q, Q') \in \mathcal{P}(a_1a_2a_h \rightarrow) \times \mathcal{P}(a_1a_2a_k \rightarrow)$. Moreover, since $a_6 \in \boldsymbol{BA}(T_\pi|_{A\setminus\{a_1,a_2\}})$, NSC implies that $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap \boldsymbol{BA}(T_{\pi^{\delta_L}}) \neq \varnothing$. It follows that $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap F^{Seq\boldsymbol{BA}}(\pi^{\delta_L}) \neq \varnothing$. Hence, since $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap \Delta(Seq\boldsymbol{BA}(T_\pi)) = \varnothing$, then $\Delta(Seq\boldsymbol{BA}(T_\pi)) \not\supseteq F^{Seq\boldsymbol{BA}}(\pi^{\delta_L})$, and thus *Seq**BA*** is not hyper-stable for δ_L. Finally, since $\boldsymbol{BA}(T_{\pi^{\delta_L}}) \subseteq \boldsymbol{UC}(T_{\pi^{\delta_L}})$, then $F^{Seq\boldsymbol{BA}}(\pi^{\delta_L}) \subseteq F^{Seq\boldsymbol{UC}}(\pi^{\delta_L})$. Thus $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap F^{Seq\boldsymbol{UC}}(\pi^{\delta_L}) \neq \varnothing$ while $\mathcal{P}(a_1a_2a_6 \rightarrow) \cap \Delta(Seq\boldsymbol{UC}(T_\pi)) = \varnothing$. Therefore *Seq**UC*** is not hyper-stable for δ_L.

Claim 3: Neither *Seq**UC*** nor *Seq**BA*** is hyper-stable for δ_{IL}.

Proof of Claim 3: Define $\overline{A} = A \cup A' \cup A'' \cup B$ where $A = \{a_1, a_2, c\}$, $A' = \{a_1', a_2', c'\}$, $A'' = \{a_1'', a_2'', c''\}$, and $B = \{b_1, b_2, b_3\}$. Moreover, consider the tournament T over $\overline{A}$ described in Fig. 4 (where an edge from a set to another means that all elements in the former defeat all elements of the latter in T):

It is easy to check that $\boldsymbol{UC}(T) = \{a_1, a_2, a_1', a_2', a_1'', a_2''\}$. Moreover, we have

- $a_1 \gg_T c \gg_T b_1$
- $a_1' \gg_T c' \gg_T b_2$
- $a_1'' \gg_T c'' \gg_T b_3$.

Using Lemma 1, $Seq\boldsymbol{UC}(\pi)$, where $T_\pi = T$, defines the following weak order $\succsim$ over $\overline{A}$:

$$a_1 \sim a_2 \sim a_1' \sim a_2' \sim a_1'' \sim a_2'' \succ c \sim c' \sim c'' \succ b_1 \sim b_2 \sim b_3.$$

Thus $\Delta(Seq\boldsymbol{UC}(\pi))$ is the set of all orders $(QQ'Q'')$, where $Q \in \mathcal{L}(\{a_1, a_2, a_1', a_2', a_1'', a_2''\})$, $Q' \in \mathcal{L}(\{c, c', c''\})$, and $Q'' \in \mathcal{L}(\{b_1, b_2, b_3\})$. Hyper-stability of *Seq**UC*** clearly requires that any order $P \in F^{Seq\boldsymbol{UC}}(\pi^{\delta_{IL}})$ has also the form $P = (QQ'Q'')$.

Suppose that $F^{Seq\boldsymbol{UC}}(\pi^{\delta_{IL}}) \subseteq \mathcal{P}(\rightarrow Q'')$, where Q'' is an element of $\mathcal{L}(\{b_1, b_2, b_3\})$. We use an argument similar to the one developed for Claim 2. From the definition of δ_{IL}, one has for all $x, y \in \overline{A}\setminus B$ that xP_iy if and only if $QP_i^{\delta_L}Q'$ for all $(Q, Q') \in \mathcal{P}(\rightarrow yQ'') \times \mathcal{P}(\rightarrow xQ'')$ and for all individuals i. Thus $\{\mathcal{P}(\rightarrow xQ''), x \in \overline{A}\setminus B\}$ forms a partition of $\mathcal{L}(\mathcal{P}(\rightarrow Q''))$ into components of $\pi^{\delta_{IL}}|_{\mathcal{P}(\rightarrow Q'')}$, each with cardinality 9!. Now consider $T^* = T|_{\overline{A}\setminus B}$. One easily checks that:

$$c \gg_{T^*} a_2, \qquad c' \gg_{T^*} a_2', \quad \text{and} \quad c'' \gg_{T^*} a_2''.$$

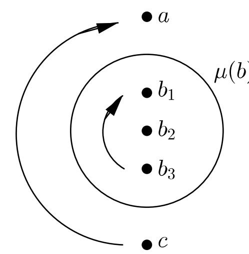

Fig. 5. α_{co} is not hyper-stable (case 1).

Then $Seq\boldsymbol{UC}(\pi)$ induces over $\pi|_{\overline{A}\setminus B}$ the following weak order $\succsim^*$:

$$a_1 \sim^* a_1' \sim^* a_1'' \succ^* c \sim^* c' \sim^* c'' \succ^* a_2 \sim^* a_2' \sim^* a_2''.$$

It follows from NSC that $F^{Seq\boldsymbol{UC}}(\pi^{\delta_{IL}}) \cap \mathcal{P}(\rightarrow xQ'') \neq \varnothing$ for some $x \in \{a_2, a_2', a_2''\}$. Thus $F^{Seq\boldsymbol{UC}}(\pi^{\delta_{IL}}) \not\subseteq \mathcal{P}(\rightarrow Q'Q'')$, and therefore $Seq\boldsymbol{UC}(\pi)$ is not hyper-stable for δ_{IL}.

Finally, one easily check that $\boldsymbol{BA}(T) = \boldsymbol{UC}(T)$, and that $\boldsymbol{BA}(T\setminus(A_{12} - \boldsymbol{BA}(T))) = \{c, c', c''\}$. Hence, $Seq\boldsymbol{BA}(\pi)$ induces the same weak order over A_{12} as $Seq\boldsymbol{UC}(\pi)$. Since ***BA*** always gives a subset of ***UC***, we conclude by the above argument that *Seq**BA*** is not hyper-stable for δ_{IL}.

Combining Claims 1–3 completes the proof. □

4.2. Other Condorcet social welfare functions

We turn now to one Condorcet SWF and two Condorcet social preference functions that rank all alternatives at once. The SWF is based on the Copeland scores assigned to alternatives in majority tournament. Define the Copeland score of alternative a in $T \in \mathcal{T}(A)$ as the number $s(a, T) = |\{b \in A : aTb\}|$ of alternatives in A defeated by $a \in A$ in T. The *Copeland SWF* α_{co} ranks alternatives according to their respective Copeland score: given any $A \in \mathcal{A}$ and any profile π over A, given any $(a, b) \in A \times A$, $a\alpha_{co}^{+}(\pi)b \Leftrightarrow s(a, T_\pi) > s(b, T_\pi)$, and $a\alpha_{co}^{=}(\pi)b \Leftrightarrow s(a, T_\pi) = s(b, T_\pi)$. The α_{co}-top choice correspondence $F^{\alpha_{co}}$ is the Copeland tournament solution ***CO***.

Proposition 6. *α_{co} is hyper-stable for no clone-consistent preference extension.*

Proof. Consider $A = \{a, b, c\}$ together with a profile $\pi = (P_1, \ldots, P_n)$ over A such that $aT_\pi bT_\pi cT_\pi a$. Since $\boldsymbol{CO}(T_\pi) = \{a, b, c\}$, then $\Delta(\alpha_{co}(\pi)) = \mathcal{L}(A)$. Given any preference extension δ, $\boldsymbol{CO}(T_{\pi^\delta}) \neq \mathcal{L}(\{a, b, c\})$. Indeed, since $\sum_{P\in\mathcal{L}(A)} s(P, T_{\pi^\delta}) = 15$ is not divisible by 6, at least one order over $\{a, b, c\}$ defeats less orders than another one. Thus, for any profile π with $T_\pi = T$, $F^{\alpha_{co}}(\pi^\delta) \neq \mathcal{L}(\{a, b, c\})$. This leaves 6 possible cases:

Case 1: $(abc) \notin F^{\alpha_{co}}(\pi^\delta)$.

Define $A' = \{a, c\} \cup \mu(b)$, where $\mu(b) = \{b_1, b_2, b_3\}$, and consider the $\overline{T}$ over A' defined in Fig. 5.

From Lemma 1, there exists a profile $\widetilde{\pi} = (\widetilde{P}_1, \ldots, \widetilde{P}_n)$ over A' such that $\overline{T} = T_{\widetilde{\pi}}$. Moreover, since b is replicated into 3 clones, $\widetilde{\pi}$ can obviously be chosen such that, for all $1 \leq i \leq n$, $\widetilde{P}_i = \otimes(P_i; \{a\}, Q_i, \{c\})$, where $Q_i \in \mathcal{L}(\mu(b))$. Furthermore, $s(a, \overline{T}) = 3$, $s(b_1, \overline{T}) = s(b_2, \overline{T}) = s(b_3, \overline{T}) = 2$ and $s(c, \overline{T}) = 1$. Hence, $\Delta(\alpha_{co}(\widetilde{\pi})) = \cup_{Q\in\mathcal{L}(\mu(b))}(aQc)$. Using clone-consistency of δ, we have $(abc) \notin F^{\alpha_{co}}(\pi^\delta) \Rightarrow (aQc) \notin F^{\alpha_{co}}(\widetilde{\pi}^\delta)$ for all $Q \in \mathcal{L}(\mu(b))$. Thus, $\Delta(\alpha_{co}(\widetilde{\pi})) \not\supseteq F^{\alpha_{co}}(\widetilde{\pi}^\delta)$ and therefore α_{co} is not hyper-stable for δ.

Case 2: $(acb) \notin F^{\alpha_{co}}(\pi^\delta)$.

Define $A'' = \mu(a) \cup \mu(b) \cup \{c\}$, where $\mu(a) = \{a_1, \ldots, a_7\}$ and $\mu(b) = \{b_1, \ldots, b_5\}$. Consider the composed tournament $\widehat{T}$ over A'' defined in Fig. 6, where the components $\widehat{T}|_{\mu(a)}$ and $\widehat{T}|_{\mu(b)}$ are regular.

Since $\widehat{T}$ is composed, there exists a profile $\widehat{\pi} = (\widehat{P}_1, \ldots, \widehat{P}_n)$ over A'' such that:

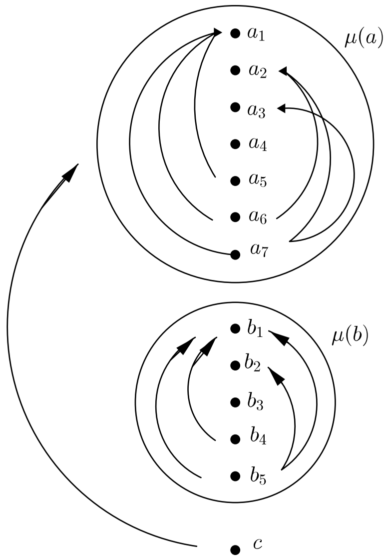

Fig. 6. α_{co} is hyper-stable (case 2).

- $\widehat{T} = T_{\widehat{\pi}}$
- $\forall 1 \le i \le n, \widehat{P}_1 = \otimes(P_i; Q_a, Q_b, \{c\})$, where $Q_a \in \mathcal{L}(\mu(a))$ and $Q_b \in \mathcal{L}(\mu(b))$.

We get that $\Delta(\alpha_{co}(\widetilde{\pi})) = \cup_{Q_a \in \mathcal{L}(\mu(a))} \cup_{Q_b \in \mathcal{L}(\mu(b))} (Q_a c Q_b)$. Using the clone-consistency of δ, we get that $(acb) \notin F^{\alpha_{co}}(\pi^\delta) \Rightarrow (Q_a c Q_b) \notin F^{\alpha_{co}}(\widehat{\pi}^\delta)$ for all $(Q_a, Q_b) \in \mathcal{L}(\mu(a)) \times \mathcal{L}(\mu(b))$. Since $\Delta(\alpha_{co}(\widehat{\pi})) \cap F^{\alpha_{co}}(\widehat{\pi}^\delta) = \varnothing$, then α_{co} is not hyper-stable for δ.

In all other cases, the same argument works, provided that $\widetilde{\pi}$ and $\widehat{\pi}$ are modified according to an appropriate reshuffling of the alternatives. □

A linear order $K \in \mathcal{L}(A)$ is a *Kemeny order* for profile $\pi = (P_1, \ldots, P_n)$ over A if $\sum_{i=1}^n d_K(P_i, K) \le \sum_{i=1}^n d_K(P_i, P)$ for any $P \in \mathcal{L}(A)$. A linear order $S \in \mathcal{L}(A)$ is a *Slater order* for a tournament T on A if $d_K(T, S) \le d_K(T, P)$ for any $P \in \mathcal{L}(A)$.

A profile may admit more than one Kemeny or Slater order. Hence choosing at any profile π the set of Kemeny (resp. Slater) orders for π defines a social preference function (SPF). Formally, an SPF is a function $\beta : \cup_{A \in \mathcal{A}} \Pi(A) \to \cup_{A \in \mathcal{A}} 2^{\mathcal{L}(A)} \backslash \varnothing$ such that $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \beta(\pi) \in 2^{\mathcal{L}(A)} \backslash \varnothing$. Furthermore, given an SPF β, the *β-top choice correspondence* $F^\beta : \cup_{A \in \mathcal{A}} \Pi(A) \to \cup_{A \in \mathcal{A}} 2^A \backslash \varnothing$ is the SCC defined by: $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \forall a \in A, a \in F^\beta(\pi)$ if and only if $\mathcal{P}(a \to) \cap \beta(\pi) \neq \varnothing$. Hence, F^β selects at profile π all alternatives which are top-ranked by some order in $\beta(\pi)$.

The *Kemeny SPF*, denoted by β_{ke}, is defined by $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \beta_{ke}(\pi) = \{P \in \mathcal{L}(A) : P \text{ is a Kemeny order for } \pi\}$. Similarly, the *Slater SPF* is the SPF β_{sl} defined by $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \beta_{sl}(\pi) = \{P \in \mathcal{L}(A) : P \text{ is a Slater order for } T_\pi\}$.

Hyper-stability for an SPF is defined as follows. An SPF β is *hyper-stable for the preference extension* δ if $\forall A \in \mathcal{A}, \forall \pi \in \Pi(A), \beta(\pi) = F^\beta(\pi^\delta)$: when applied β to hyper-profile π^δ built from π by means of extension δ, the SPF β ranks first *all* orders chosen from the initial profile π over alternatives. Hence, hyper-stability holds when consistency is ensured between the two levels of choice regardless the way to select one order within the set $\beta(\pi)$. Note that this concept of hyper-stability is somehow more demanding than the one for SWFs, which allows to select at the upper level a proper subset of all linearizations of the weak order initially chosen at the lower level.

Proposition 7. *Both α_{ke} and α_{ke} are hyper-stable for δ_L.*

Proof. The proof is organized as follows. We first show in Claims 1 and 2 that if T in a composed tournament with equally-sized components, any Slater (resp. Kemeny) order for T is the composition of component-wise Slater (resp. Kemeny) orders with a Slater (resp. Kemeny) order of the summary of T. Then we combine these claims with the fact that δ_L generates a decomposition of any hyper-profile into equally-sized components in order to conclude. Consider any composed profile $\pi = (P_1, \ldots, P_n)$ having a decomposition $\{C^1, \ldots, C^H\}$ in equally sized components, with $|C^h| = C$ for all $1 \le h \le H$. Hence $\pi = \otimes(\pi^*; \pi^1, \ldots, \pi^H)$ where $\pi^* = (P_1^*, \ldots, P_n^*) \in \mathcal{L}(\{1, \ldots, H\}^n)$ and $\pi^h = (P_1^h, \ldots, P_n^h) \in \mathcal{L}(C^h)$ for all $1 \le h \le H$. Clearly, the majority tournament T_π for π is composed, and $T_\pi = \otimes(T^*; T^1, \ldots, T^H)$, where $T^* \in \mathcal{T}(\{1, \ldots, H\})$, and $T^h \in \mathcal{T}(C^h)$ for all h.

It is already known that, if $S \in \mathcal{L}(A)$ is a Slater order for T_π, then for all $1 \le h \le H$, $S|_{C^h}$ is a Slater order for T^h (Lainé et al., 2013, Proposition 3.4.3. p. 65). Furthermore, there exists a Slater order $S = \otimes(S^*; S^1, \ldots, S^H)$, where S^* is a Slater order for T^* and, for all $1 \le h \le H$, S^h is a Slater order for T^h (see Lainé et al., 2013, Proposition 4.3.4. p. 66). We first prove the following claim, which establishes a stronger property of Slater orders for composed tournament with equally sized components.

Claim 1: $S \in \mathcal{L}(A)$ is a Slater order for T_π if and only if $S = \otimes(S^*; S^1, \ldots, S^H)$, where S^* is a Slater order for T^* and, for all $1 \le h \le H$, S^h is a Slater order for T^h.

Proof of Claim 1: Given $P \in \mathcal{L}(A)$ together with $a, b \in A$, define $\theta_P(a, b) = 1$ if aPb and $\theta_P(a, b) = 0$ otherwise. Then the Kemeny distance between P and tournament T is equal to

$$d_K(P, T) = \sum_{1 \le h \le H} d_K(P|_{C^h}, T^h) + \sum_{1 \le h < h' \le H} \sum_{(a,b) \in C^h \times C^{h'}} [\theta_P(b, a)\theta_T(a, b) + \theta_P(a, b)\theta_T(b, a)]. \quad (1)$$

The first sum in (1) provides the total "intra-component" Kemeny distance between P and T. Denote by $\mathcal{S}(C^h)$ the set of Slater orders for T^h, and let $C^h = \{a_1^h, \ldots, a_C^h\}$, where $1 \le h \le H$. It follows from the definition of a Slater order that

$$\sum_{1 \le h \le H} d_K(P|_{C^h}, T^h) \ge \sum_{1 \le h \le H} d_K(S^h, T^h)$$
$$\text{for all } (S^1, \ldots, S^H) \in \mathcal{S}(C^1) \times \cdots \times \mathcal{S}(C^H). \quad (2)$$

The second sum in (1) provides the total "inter-component" Kemeny distance $V(P)$ between P and T. Since $T = \otimes(T^*; T^1, \ldots, T^H)$, then for all $1 \le h < h' \le H$ and for all $(a, b) \in C^h \times C^{h'}$, $\theta_T(a, b) = 1 \Leftrightarrow hT^*h'$. Thus

$$V(P) = \sum_{h<h':hT^*h'} \sum_{(a,b) \in C^h \times C^{h'}} \theta_P(b, a) + \sum_{h<h':h'T^*h} \sum_{(a,b) \in C^h \times C^{h'}} \theta_P(a, b).$$

Consider the following constrained minimization program:

Minimize V over $\mathcal{L}(A)$ under the constraints

(i) $\forall a \neq b \in A$, $\theta_P(a, b) \in \{0, 1\}$ and $\theta_P(a, b) + \theta_P(b, a) = 1$

(ii) $\forall a, b, c \in A$ with $a \neq b$, $a \neq c$ and $b \neq c$,

$\theta_P(a, b) + \theta_P(b, c) + \theta_P(c, a) \le 2.$

Constraint (ii) ensures the transitivity of the solutions.[8] Consider any Slater order S^* for T^*, and consider any order $\widetilde{P}$ over A which ranks alternatives consistently with S^*: $\forall 1 \le h < h' \le H$, $\forall (a, b) \in C^h \times C^{h'}$, $a\widetilde{P}b \Leftrightarrow hS^*h'$. Then $V(\widetilde{P}) \le V(P)$ for all

[8] Note that one does not need to preclude the existence of cycles with length greater than 3, since any complete digraph containing a cycle with length L greater than 3 contains a cycle with length 3.

$P \in \mathcal{L}(A)$. In order to see why, choose any partition Θ of A into H-tuples $(a_1, a_2, \ldots, a_H) \in C^1 \times C^2 \times \cdots \times C^H$ and weaken constraint (ii) by imposing transitivity only within each of the H-tuples in Θ. Since S^* is a Slater order for T^*, then an obvious solution of the modified program is $\widetilde{P}$. Since $\widetilde{P}$ is transitive, then $\widetilde{P}$ also solves the initial program, and hence $V(\widetilde{P}) \leq V(P)$ for all $P \in \mathcal{L}(A)$. Combining this result with (1) and (2) ensures that $d_K(P, T) \geq d_K(\otimes(S^*; S^1, \ldots, S^H), T)$ for all $P \in \mathcal{L}(A)$, which proves Claim 1.

Claim 2 states a similar property for Kemeny orders. Given a composed profile with equally sized components, then every Kemeny order for this profile is the product of a Kemeny order for the profile over components induced by π with component-wise Kemeny orders.

Claim 2: $K \in \mathcal{L}(A)$ is a Kemeny order for π if and only if $K = \otimes(K^*; K^1, \ldots, K^H)$, where K^* is a Kemeny order for π^* and, for all $1 \leq h \leq H$, K^h is a Kemeny order for π^h.

Proof of Claim 2: The proof is similar to the one of Claim 1. Consider any order $Q \in \mathcal{L}(A)$. Then the total Kemeny distance between Q and π is

$$d_K(\pi, Q) = \sum_{1 \leq i \leq n} \sum_{1 \leq h \leq H} d_K(P_i^h, Q\mid_{C^h}) + \sum_{1 \leq h < h' \leq H} \sum_{(a,b) \in C^h \times C^{h'}} \sum_{1 \leq i \leq n} [\theta_{P_i}(a, b)\theta_Q(b, a) + \theta_{P_i}(b, a)\theta_Q(a, b)]. \quad (3)$$

By definition of K^h, $1 \leq h \leq H$, we have

$$\sum_{1 \leq i \leq n} \sum_{1 \leq h \leq H} d_K(P_i^h, Q\mid_{C^h}) \geq \sum_{1 \leq i \leq n} \sum_{1 \leq h \leq H} d_K(P_i^h, K^h). \quad (4)$$

For all $1 \leq h < h' \leq H$, denote by $n_{hh'}$ the number of orders P_i^* in π^* such that hP_i^*h', and define $w_{hh'} = n_{hh'} - n_{h'h}$. Using the fact that π is composed, the last sum $G(Q)$ in (3) can be equivalently written

$$G(Q) = \sum_{1 \leq h < h' \leq H} \left\{ n_{hh'} \sum_{(a,b) \in C^h \times C^{h'}} \theta_Q(b, a) + n_{h'h} \sum_{(a,b) \in C^h \times C^{h'}} \theta_Q(a, b) \right\}.$$

Moreover, since all components have the same size C, then

$$G(Q) = \sum_{1 \leq h < h' \leq H} \left\{ n_{hh'} \sum_{(a,b) \in C^h \times C^{h'}} (C^2 - \theta_Q(a, b)) + n_{h'h} \sum_{(a,b) \in C^h \times C^{h'}} \theta_Q(a, b) \right\}$$

and thus

$$G(Q) = C^2 \sum_{1 \leq h < h' \leq H} n_{hh'} + \sum_{1 \leq h < h' \leq H} w_{h'h} \sum_{(a,b) \in C^h \times C^{h'}} \theta_Q(a, b).$$

Obviously, minimizing $G(Q)$ over $\mathcal{L}(A)$ is equivalent to solving

$$\text{Maximize} \left[\sum_{1 \leq h < h' \leq H} w_{h'h} \sum_{(a,b) \in C^h \times C^{h'}} \theta_Q(a, b) \right]$$

over $\mathcal{L}(A)$ under the constraints

(i) $\forall a \neq b \in A$, $\theta_P(a, b) \in \{0, 1\}$ and $\theta_P(a, b) + \theta_P(b, a) = 1$

(ii) $\forall a, b, c \in A$ with $a \neq b$, $a \neq c$ and $b \neq c$,
$\theta_P(a, b) + \theta_P(b, c) + \theta_P(c, a) \leq 2$.

Consider any Kemeny order K^* for T^*, and consider any order $\widetilde{Q}$ over A which ranks alternatives consistently with S^*: $\forall 1 \leq h < h' \leq H, \forall (a, b) \in C^h \times C^{h'}, a\widetilde{Q}b \Leftrightarrow hK^*h'$. Then, using the definition of a Kemeny order, one get by using the same argument as in the proof of Claim 1 that $V(\widetilde{Q}) \leq V(Q)$ for all $P \in \mathcal{L}(A)$. Combining this result with (3) and (4) ensures that $d_K(\pi, Q) \geq d_K(\pi, \otimes(K^*; K^1, \ldots, K^H))$ for all $Q \in \mathcal{L}(A)$, which proves Claim 2.

Now, choose any profile π over A, and consider β_{sl}. Note first that $P = (b_1 b_2 \ldots b_m) \in \beta_{sl}(\pi)$ if and only if $\forall k \in \{1, \ldots, m\}$, $P\mid_{\{b_k \ldots, b_m\}} \in \beta_{sl}(\pi\mid_{\{b_k \ldots, b_m\}})$. Since this is true for any $m \in \mathbb{N}$, it follows that $P \in F^{\beta_{sl}}(\pi^{\delta_L})$ if only if $P\mid_{\{b_k \ldots, b_m\}} \in F^{\beta_{sl}}(\pi^{\delta_L}\mid_{\mathcal{L}(\{b_k \ldots, b_m\})})$ for all $1 \leq k \leq m$.

From the definition of δ_L, π^{δ_L} admits a decomposition $\{\mathcal{P}(a_1 \rightarrow), \ldots, \mathcal{P}(a_m \rightarrow)\}$ into components with equal size $(m-1)!$. Hence $\mathcal{P}(b_1 \rightarrow) \cap \beta_{sl}(\pi) \neq \varnothing$ if and only if $\mathcal{P}(b_1 \rightarrow) \cap F^{\beta_{sl}}(\pi^{\delta_L}) \neq \varnothing$. Indeed, it follows from Claim 1 that $S \in F^{\beta_{sl}}(\pi^{\delta_L})$ if and only if $S = \otimes(S^*; S^1, \ldots, S^H)$, where $S^* \in \mathcal{L}(\{1, \ldots, m\})$ is a Slater order for T^*, and S^h is a Slater order for T^h for all $1 \leq h \leq H$, with $T^h = T_{\pi^{\delta_L}}\mid_{\mathcal{P}(b_h \rightarrow)}$ for all $1 \leq h \leq m$. Since $\forall 1 \leq h, h' \leq m, hT^*h'$ if and only if QTQ' for all $(Q, Q') \in \mathcal{P}(b_h \rightarrow) \times \mathcal{P}(b_{h'} \rightarrow)$, then b_1 is a top-alternative of a Slater order for T_π if and only if $\mathcal{P}(b_1 \rightarrow)$ contains an order ranked first by a Slater order for $T_{\pi^{\delta_L}}$. Similarly, for all $k \in \{1, \ldots, m\}$, $\pi^{\delta_L}\mid_{\mathcal{L}(\{b_k, \ldots, b_m\})}$ admits a decomposition $\{\mathcal{P}(a_k \rightarrow), \ldots, \mathcal{P}(a_m \rightarrow)\}$ into components with equal size $(m-k)!$. Using Claim 1, $\mathcal{P}(b_k \rightarrow) \cap \beta_{sl}(\pi\mid_{\{b_k, \ldots, b_m\}}) \neq \varnothing$ if and only if $\mathcal{P}(b_k \rightarrow) \cap F^{\beta_{sl}}(\pi^{\delta_L}\mid_{\mathcal{L}(\{b_k, \ldots, b_m\})}) \neq \varnothing$. Thus $P = (b_1 \ldots b_m) \in \beta_{sl}(\pi)$ if and only if $\mathcal{P}(b_1 \ldots b_k \rightarrow) \cap F^{\beta_{sl}}(\pi^{\delta_L}) \neq \varnothing$ for all $1 \leq k \leq m$. It follows that $\beta_{sl}(\pi) = F^{\beta_{sl}}(\pi^{\delta_L})$ and therefore β_{sl} is hyper-stable for δ_L.

The proof for β_{ke} is essentially the same, using Claim 2 instead of Claim 1. □

Note that the proof for β_{ke} does not require an odd number of individuals.

5. Further comments

An SWF is hyper-stable if, given any initial profile π over alternatives, there exists a hyper-profile of linear orders over all rankings of alternatives at which all chosen rankings are linearizations of the weak order chosen from π. Hyper-profiles are generated from initial profiles by means of a preference extension which fulfils several properties (neutrality, regularity, independence and clone-consistency). Hence, when hyper-stability holds for an SWF, there exists an underlying profile over the set of all outcomes of that SWF which makes knowing each individual's most preferred ranking enough to rank alternatives consistently with what would prevail under the complete knowledge of preferences.

The picture of hyper-stability shows three components: the domain of preference over alternatives, the way to extend it to a hyper-preference domain, and the prevailing SWF. Playing with components brings several interesting questions for further research.

(1) All results in this paper are obtained for a universal preference domain over alternatives. Is it true that any SWF is hyper-stable for some reasonable domain restriction? The answer is positive for Condorcet SWFs. Indeed, Lemma 2 implies that any Condorcet SWF is hyper-stable in restriction to any domain leading to a transitive majority tournament. We find worthwhile to seek whether similar results hold for other classes of SWFs, such as the class of scoring rules.

(2) Each SWF α can be associated with a set $\Omega(\alpha)$ of preference extensions for which it is hyper-stable (within the universal domain). The set $\Omega(\alpha)$ gives insights to the type of choice problem where α is consistent in the hyper-stability sense. For instance, if $\Omega(\alpha)$ contains only lexicographic extensions δ_L and δ_{IL}, then using α as SWF may put consistency at risk if neither δ_L nor δ_{IL} induces appealing hyper-preferences for the choice problem at

stake. Characterizing $\Omega(\alpha)$ for SWFs such as the Borda rule and other scoring rules, or each of the Condorcet SWFs considered above, is left as an open problem.

(3) A dual approach of (2) is to fix a preference extension, and to characterize the associated class of hyper-stable SWFs. An interesting problem is to characterize the class of Condorcet SWFs that are hyper-stable for the lexicographic preference extension, or the Kemeny preference extension. A similar problem is open for scoring rules. A first analysis can be found in Lainé et al. (2013), who show that no scoring rule is Kemeny-stable, where Kemeny stability holds if the choice made from *every* hyper-profile of linear orders built by means of the Kemeny rule contains at least one linearization of the weak order obtained from preferences over alternatives. Is it possible to establish general properties of SWFs that are hyper-stable for either lexicographic or Kemeny preference extensions?

Finally, this paper focuses on Condorcet SWFs, and in particular those based on successive tournament solutions. Note however that we impose an odd number of individuals. Obviously, all negative results remain valid without this assumption. If majority ties are allowed, investigating hyper-stability for SWFs such as *Seq**TC***, *Seq**MC***, *Seq**BP***, α_{co} and for the SPF β_{sl} can be undertaken provided that the corresponding tournament solutions are extended to weak tournament solutions along the lines drawn by Peris and Subiza (1999) and by Brandt et al. (2014).

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