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Chaotic n-Dimensional Euclidean and Hyperbolic Open Billiards and Chaotic Spinning Planar Billiards*

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Abstract. We propose a new method to handle the n-dimensional billiard problem in the exterior of a finite mutually disjoint union of convex (but not necessarily strictly convex) smooth obstacles without eclipse in the Euclidean or hyperbolic n-space, and we prove that there exist trajectories visiting the obstacles in any given doubly infinite prescribed order (with the obvious restriction of no consecutive repetition). As an interesting variant of planar billiards, we consider spinning obstacles and particles and prove that any forward sequence of obstacles has a trajectory that follows it.

Key words. chaos, billiards, dissipative, friction

AMS subject classifications. 37C, 37N15

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1. Introduction.

The billiard problems of various types have not only physical significance but also mathematical beauty. The long history of billiards is full of mathematical gems [10]. A less investigated class of dispersing billiards is the so-called open-billiards problem, where the reflections on the boundaries of some sorts of obstacles in an infinite affine space are producing the billiard map. An interesting theorem was proven by Morita [8] in the following set-up.

Let $O_1, O_2, \dots O_K$ ($K \geq 3$) be a finite number of mutually disjoint, closed, bounded, convex subsets of $\mathbb{R}^2$ (the so-called *obstacles*) with boundaries ∂O_j, which are simple smooth closed curves. Consider a particle in $\mathbb{R}^2$ outside $\bigcup_{j=1}^K O_j$ which moves along a straight line with unit speed and reflects at the boundary $\bigcup_{j=1}^K \partial O_j$ obeying the law of reflection; i.e., the angle of reflection coincides with the angle of incidence. Assume additionally the following two conditions.

Condition 1 (strict convexity). *The boundary curves all have nonvanishing curvature.*

Condition 2 (no eclipse). *For any triple (j_1, j_2, j_3) of distinct indices,*

$$\mathrm{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \emptyset,$$

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where conv$[A]$ *denotes the convex hull of the set* A.

Then the following theorem holds (in $\mathbb{R}^2$).

Theorem 1 (Morita [8]). *Given an itinerary* $(O_n)_{n\in\mathbb{Z}}$ *of obstacles without consecutive repetition, there exists a unique trajectory following this itinerary.*

The setting and two assumptions of Morita go back to Ikawa [6], who proved the existence and uniqueness of arbitrary periodic trajectories in the 3-dimensional case with strictly convex obstacles without eclipse. The n-dimensional version of this problem is considered in Stoyanov [9], where important and intricate estimates for the separation of nearby trajectories are given, and very recently the problem was solved by Blokh, Misiurewicz, and Simanyi for strictly convex obstacles without eclipse in $\mathbb{R}^n$ (Theorem 2.2 in [1]), the trajectories being unique by Chernov [2].

We propose another method to solve the problem, which can be applied to other similar billiard problems, and we exemplify this for the hyperbolic case. At the same time we weaken the strict convexity assumption for obstacles and allow their boundaries to have flat parts, as suggested by one of the referees. Throughout the literature it is assumed that the obstacles are strictly convex, so we hope this generalization will be of some value. Polyhedral obstacles smoothed along a narrow region of the edges might give interesting examples. This generalization works, however, at the price of uniqueness of trajectories with a fixed itinerary. It remains to be understood to what extent uniqueness is lost.

After proving the theorem for $\mathbb{R}^n$ (see Theorem 2), we replace $\mathbb{R}^n$ by $\mathbb{H}^n$ (the n-dimensional hyperbolic space) using the conformal unit disk model $\mathbb{B}^n \subset \mathbb{R}^n$ for $\mathbb{H}^n$. The rays along which the particle moves are no longer straight lines but the geodesics of $\mathbb{H}^n$, i.e., circle arcs orthogonal to the unit sphere S^{n-1} or Euclidean straight lines going through the origin of $\mathbb{B}^n$. Theorem 3 shows that the basic result (Theorem 2) also holds in this case, taking as obstacles hyperbolically convex smooth subsets which are diffeomorphic with balls inside $\mathbb{B}^n$.

Finally, we consider another generalization of Theorem 1 on the plane, where we assume the obstacles to be (geometric) disks but allow them to spin around their fixed centers. We also allow the moving particles to spin and collide with the spinning obstacles according to the laws of physics. Under some plausible assumptions, we show that there exists a trajectory following any given (forward) sequence of obstacles.

2. n-dimensional (Euclidean) open-billiards.

We define an obstacle in $\mathbb{R}^n$ to be a convex (not necessarily strictly convex) subset O (with boundary ∂O), which is diffeomorphic to the standard disk $\mathbb{D}^n$. Now, let us consider a finite number of mutually disjoint obstacles $O_1, O_2, \dots O_K$ ($K \geq 3$) for which Condition 2 holds.

Condition 2 (no eclipse). *For any triple* (j_1, j_2, j_3) *of distinct indices,*

$$\mathrm{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \emptyset.$$

To define the billiard map, we consider the space $\mathcal{L}$ of oriented lines in $\mathbb{R}^n$ (which can be identified with the total space of the tangent bundle of the unit sphere $S^{n-1} \subset \mathbb{R}^n$; see [10]). We denote the unit orientation vector of $L \in \mathcal{L}$ by $v(L)$ and the line going through a point p and having orientation vector v by $L_p v$. We denote the ray $\{x \in L_p v \mid x = p + tv \text{ with } t \geq 0\}$ by $L_p^+ v$.

The subspace $Q \subset \mathcal{L}$ of the oriented lines intersecting at least one obstacle is compact.

We define for $i = 1, 2, \dots, K$ the set $S_i \subset Q$ as the set of all oriented lines, which, in the direction of orientation, first hit the obstacle O_i and then, for some $j \neq i$, hit another obstacle O_j. (Thus by Condition 2, a line in S_i hits only these two obstacles O_i and O_j.) Note that the sets S_i are mutually disjoint.

Let $Q_0 = \bigcup_{i=1}^{K} S_i \subset Q$.

An oriented line L intersecting an obstacle O_i "enters" the obstacle at a point $p' \in \partial O_i$ with $\langle n_{p'}, v(L)\rangle \leq 0$ ($n_{p'}$ being the outward unit normal vector at p') and "leaves" the obstacle at a point $p \in \partial O_i$ with $\langle n_p, v(L)\rangle \geq 0$. Generically $L \cap \partial O_i$ consists of two points, but it can also be a singleton or the interval $[p', p]$.

Now we define the billiard map $f : Q_0 \to Q$.

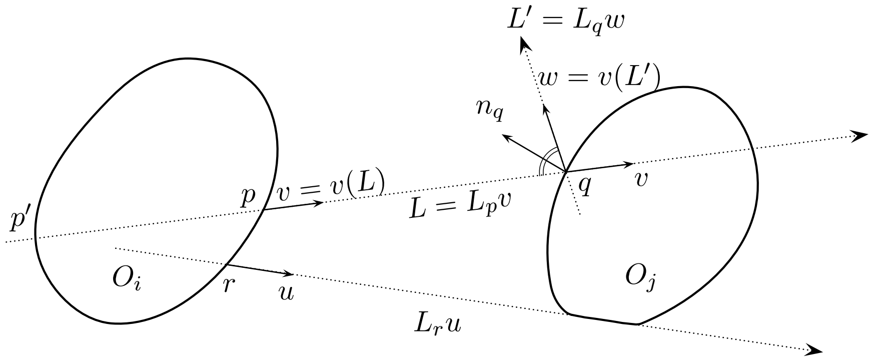

Figure 1. *The billiard map sends $L_p v$ to $L_q w$ with $w = v - 2\langle n_q, v\rangle n_q$. As a special case, $L_r u$ is sent to itself.*

Let $L \in S_i$, $L \cap O_j \neq \emptyset$, with $j \neq i$ and q the first-hit point of L on ∂O_j. We set $f(L) = L'$, where L' is the oriented line through q with the orientation vector $v(L') = v(L) - 2\langle n_q, v(L)\rangle n_q$ (see Figure 1).

We will prove the following.

Theorem 2. *Given an itinerary $(S_{i_n})_{n\in\mathbb{Z}}$ with $i_n \in \{1, 2, \dots, K\}$ and $i_n \neq i_{n+1}$, there exists a trajectory $(L_n)_{n\in\mathbb{Z}}$ following this itinerary; i.e., $f(L_n) = L_{n+1}$ and $L_n \in S_{i_n}$.*

This means obviously that given any sequence of obstacles in $\mathbb{R}^n$, there is a billiard trajectory in the usual sense hitting every obstacle in the given order.

To prove this theorem, we will use the following special case of the result in [7].

Theorem (the chaos lemma). *Let Q be a compact metric space, $Q_0 \subset Q$ a compact subset, $f : Q_0 \to Q$ a continuous map, and $S_i \subset Q_0$ ($i = 1, 2, \dots, K$) pairwise disjoint compact subsets of Q_0 with $\bigcup_{i=1}^{K} S_i = Q_0$. Assume there exists a collection $\{\mathcal{E}_i\}_{i=1}^{K}$, where each $\mathcal{E}_i$ is a nonempty family of nonempty compact subsets of Q, with the following property.*

Property 1. *If $E \in \mathcal{E}_i$ and $j \neq i$, there exists a set $E_j \subset E \cap S_i$ such that $f(E_j) \in \mathcal{E}_j$.*

Then, given any bi-infinite sequence $(S_{i_n})_{n\in\mathbb{Z}}$ of the sets $\{S_1, S_2, \dots, S_K\}$ with $i_n \in \{1, 2, \dots, K\}$ and $i_n \neq i_{n+1}$, there exists a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $x_n \in S_{i_n}$ and $f(x_n) = x_{n+1}$.

The sets S_i are called *symbol sets*, the sets $E \in \mathcal{E}_i$ are called *expanders*, and the sets $E_j \subset E \cap S_i$ with $f(E_j) \in \mathcal{E}_j$ are called *pre-expanders*.

3. Proof of Theorem 2.

We shall apply the chaos lemma to the billiards setting with the notation fixed above. To make the chaos lemma work, we have to define the expander sets $\mathcal{E}_i$. For this purpose we define the notion of a dispersive vector field on a nonempty, compact subset $D \subset \partial O_i$.

Let $\sigma : D \to S^{n-1}$ be a continuous outward unit vector field; i.e., $\langle \sigma(p), n_p \rangle \geq 0$ for all $p \in D$. We call σ *dispersive* if the condition

$$\langle \sigma(p_1) - \sigma(p_2), p_1 - p_2 \rangle \geq 0 \text{ for all } p_1, p_2 \in D$$

is satisfied. See Figure 4.

Lemma 1. *For a dispersive vector field σ, either the rays $L^+_{p_1}\sigma(p_1)$ and $L^+_{p_2}\sigma(p_2)$ are disjoint or one is contained in the other.*

We omit the proof.

We call a dispersive vector field $\sigma : D \to S^{n-1}$ *exhaustive* (or call σ an exhaustively dispersive vector field) if there exists a continuous extension $\sigma^* : \partial O_i \to S^{n-1}$ of σ such that

$$\langle n_p, \sigma^*(p) \rangle \geq 0 \text{ for all } p \in \partial O_i, \text{ and}$$
$$L^+_p \sigma^*(p) \cap \partial O_j = \emptyset \text{ for all } p \in \partial O_i \backslash D,\ j \neq i.$$

(That is, σ can be extended to an outward unit vector field on ∂O_i in such a way that the rays along the new vectors outside D do not hit any of the obstacles.)

We can associate to every vector field $\sigma : D_\sigma \subset \partial O_i \to S^{n-1}$ the set $E(\sigma)$ of oriented lines defined as $E(\sigma) = \{L_p \sigma(p) \mid p \in D_\sigma\} \subset Q$.

Now we define our expanders (recall that expanders are sets of sets):

$$\mathcal{E}_i = \{E \subset Q \mid \text{there exists an exhaustively dispersive vector field}$$
$$\sigma : D_\sigma \subset \partial O_i \to S^{n-1} \text{ such that } E = E(\sigma)\}.$$

$\mathcal{E}_i$ is nonempty because the outward unit normal vector field $\mathcal{N}$ is exhaustively dispersive, and thus $E(\mathcal{N}) \in \mathcal{E}_i$.

To obtain Theorem 2 from the chaos lemma, we have to verify Property 1: For any $E \in \mathcal{E}_i$ ($E = E(\sigma)$ for some $\sigma : D_\sigma \to S^{n-1}$) and for all $j \neq i$, there exists a subset $E_j \subset E \cap S_i$ such that $f(E_j) \in \mathcal{E}_j$.

We define as a pre-expander

$$E_j = \{L_p \sigma(p) \mid p \in D_\sigma \text{ and } L^+_p \sigma(p) \cap \partial O_j \neq \emptyset\}$$

and are going to show that $f(E_j) \in \mathcal{E}_j$. To this end we first note some well-known facts.

$\sigma^* : \partial O_i \to S^{n-1}$ has degree 1 (because it is homotopic to the normal vector field) and thus it is onto. In other words, given any unit vector in $\mathbb{R}^n$, there is a point in ∂O_i at which this vector is attached. Moreover, we can state the following lemma.

Lemma 2. *Given any point $q \in \mathbb{R}^n \backslash O_i$, there exists a point $p \in \partial O_i$ such that $q \in L^+_p \sigma^*(p)$.*

Proof. Let $q \in \mathbb{R}^n \backslash O_i$. Define the map

$$\omega_q : \partial O_i \to S^{n-1},$$
$$\omega_q(p) = \frac{q - p}{\|q - p\|},$$

which assigns a unit vector in direction q for each $p \in \partial O_i$. ω_q has degree 0 because it is not onto.

Now, suppose that $\sigma^*(p) \neq \omega_q(p)$ for all $p \in \partial O_i$. Then the map

$$H : \partial O_i \times [0,1] \to S^{n-1},$$
$$H(p,t) = \frac{t\sigma^*(p) - (1-t)\omega_q(p)}{\|t\sigma^*(p) - (1-t)\omega_q(p)\|}$$

is a homotopy between $-\omega_q(p) = H(p,0)$ and $\sigma^*(p) = H(p,1)$. To see this, it is enough to show that $\|t\sigma^*(p) - (1-t)\omega_q(p)\| \neq 0$ for all $p \in \partial O_i$ and $t \in [0,1]$. If we had $\|t\sigma^*(p) - (1-t)\omega_q(p)\| = 0$ for some t, this would give $t\sigma^*(p) = (1-t)\omega_q(p)$ and thus $t = \frac{1}{2}$ (by $\|\sigma^*(p)\| = \|\omega_q(p)\| = 1$), contradicting $\sigma^*(p) \neq \omega_q(p)$.

This homotopy implies that $degree(\sigma^*(p)) = degree(-\omega_q(p))$, which is impossible, because $degree(-\omega_q(p)) = 0$ but $degree(\sigma^*(p)) = 1$. ∎

As a consequence, we have $\partial O_j \subset \bigcup_{L \in E_j} L_p^+ \sigma(p)$ for any $j \neq i$.

Let $\varphi : \partial O_j \to S^{n-1}$ be defined as follows: Given $q \in \partial O_j$ there exists $p \in D_\sigma$ with $q \in L_p^+ \sigma(p)$. A point with this property might not be uniquely defined, but $\sigma(p)$ is well defined by dispersivity. So we set $\varphi(q) = \sigma(p)$. φ can be seen to be continuous.

We can now express $f(E_j)$ as $E(\tau)$ for the function

$$\tau : D_\tau \to S^{n-1},$$
$$\tau(q) = \varphi(q) - 2\langle n_q, \varphi(q)\rangle n_q,$$

where $D_\tau = \{q \in \partial O_j \mid \langle n_q, \varphi(q)\rangle \leq 0\} \subset \partial O_j$ (see Figure 2).

Lemma 3. *τ, as defined above, is exhaustively dispersive.*

Proof (τ is dispersive). We must show $\langle \tau(q_1) - \tau(q_2),\ q_1 - q_2\rangle \geq 0$ for all $q_1, q_2 \in D_\tau$. Let $q_1 = p_1 + t_1\sigma(p_1)$, $q_2 = p_2 + t_2\sigma(p_2)$ and assume $t_1 \geq t_2 > 0$. We thus have, inserting $\tau(q_\alpha) = \sigma(p_\alpha) - 2\langle\sigma(p_\alpha), n_{q_\alpha}\rangle n_{q_\alpha}$ for $\alpha = 1, 2$,

$$\begin{aligned}\langle \tau(q_1) - \tau(q_2), q_1 - q_2\rangle &= \langle \sigma(p_1) - \sigma(p_2), q_1 - q_2\rangle \\ &\quad + 2\langle[\langle\sigma(p_2), n_{q_2}\rangle n_{q_2} - \langle\sigma(p_1), n_{q_1}\rangle n_{q_1}], q_1 - q_2\rangle.\end{aligned} \tag{1}$$

We will show that both terms on the right-hand side of (1) are nonnegative. The first term satisfies

$$\begin{aligned}\langle \sigma(p_1) - \sigma(p_2), q_1 - q_2\rangle &= \langle \sigma(p_1) - \sigma(p_2), p_1 - p_2\rangle + t_2\|\sigma(p_1) - \sigma(p_2)\|^2 \\ &\quad + (t_1 - t_2)\langle\sigma(p_1) - \sigma(p_2), \sigma(p_1)\rangle.\end{aligned} \tag{2}$$

The first term on the right-hand side of (2) is nonnegative, because σ is dispersive. The other two terms are nonnegative for obvious reasons.

The second term of the right-hand side of (1),

$$\langle\sigma(p_2), n_{q_2}\rangle\langle n_{q_2}, q_1 - q_2\rangle - \langle\sigma(p_1), n_{q_1}\rangle\langle n_{q_1}, q_1 - q_2\rangle,$$

is also nonnegative because

$$\langle n_{q_2}, q_1 - q_2\rangle \leq 0 \text{ and } \langle n_{q_1}, q_1 - q_2\rangle \geq 0$$

by convexity of O_j. (The other two factors are ≤ 0 by construction.) ■

Proof (τ is exhaustive). To define the extension

$$\tau^* : \partial O_j \to S^{n-1},$$

we first note that $D_\tau = \{q \in \partial O_j \mid \langle n_q, \varphi(q)\rangle \leq 0\}$ and $D'_\tau = \{q \in \partial O_j \mid \langle n_q, \varphi(q)\rangle \geq 0\}$ are both closed.

Let

$$\tau^*(q) = \begin{cases} \varphi(q) - 2\langle \varphi(q), n_q\rangle n_q & \text{for} \quad q \in D_\tau \\ \varphi(q) & \text{for} \quad q \in D'_\tau \end{cases}$$

(see Figure 2).

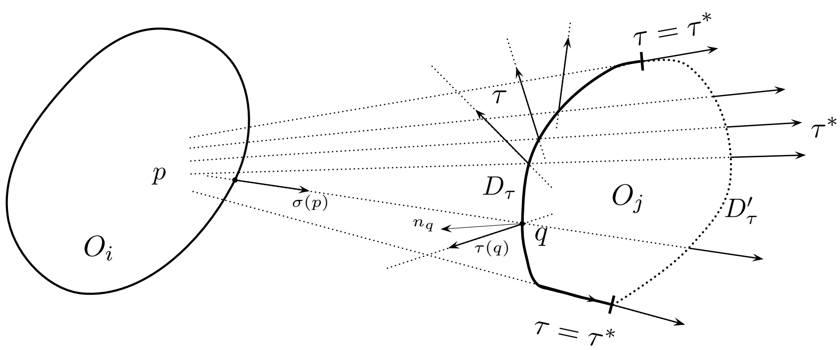

Figure 2. *The extension of τ.*

τ^* is well defined on $D_\tau \cup D'_\tau = \partial O_j$ since $\langle \varphi(q), n_q\rangle = 0$ for $q \in D_\tau \cap D'_\tau$, and consequently τ^* is continuous. τ^* is outward on D'_τ ($\langle n_q, \tau^*(q)\rangle \geq 0$ for $q \in D'_\tau$), and $L_q^+\tau^*(q) \cap \partial O_k = \emptyset$ for $k \neq j$ by Condition 2. Thus τ is exhaustive. ■

The chaos lemma now verifies Theorem 2.

4. Open billiards in hyperbolic n-space.

We consider the open-billiards problem in the n-dimensional hyperbolic space $\mathbb{H}^n$ using the conformal unit-disk model $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{E}} < 1\}$ with the Riemannian metric

$$g_p(u_p, v_p) = \langle u_p, v_p\rangle_{\mathbb{H}} = \frac{4}{(1 - \|p\|^2)^2}\langle u_p, v_p\rangle_{\mathbb{E}},$$

where u_p, v_p are vectors at the point $p \in \mathbb{B}^n$ and the subscripts $\mathbb{E}$ and $\mathbb{H}$ denote the Euclidean and hyperbolic metrics, respectively. In this section we will assume the hyperbolic metric applies when the subscript is dropped. The angles between (hyperbolic) vectors are the same as in the Euclidean case; only the lengths are affected.

Given any point $p \in \mathbb{B}^n$ and any unit vector v_p at p, the geodesic going through p in the direction of v_p is a Euclidean circle-arc perpendicular to the unit sphere $S^{n-1} \subset \mathbb{R}^n$. (In the case when it goes through the origin of $\mathbb{R}^n$, it is a straight line segment.) We denote the oriented geodesic through p in the direction of v_p by $L_p v_p$, and we call the geodesic part starting at p in the direction of v_p a hyperbolic ray and denote it by $L_p^+ v_p$. As obstacles we consider hyperbolically convex (the geodesic segment connecting any two points is contained in the obstacle) smooth subsets of $\mathbb{B}^n$ which are diffeomorphic to the standard closed disk $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{E}} \leq 1\}$.

Let $\mathcal{L}$ denote the space of oriented geodesics, and let $Q \subset \mathcal{L}$ be the compact subspace of oriented geodesics intersecting at least one obstacle. As before, let S_i be the set of all oriented geodesics which in the direction of orientation first hit the obstacle O_i and then, for some $j \neq i$, another obstacle O_j. We set $Q_0 = \bigcup_{i=1}^K S_i \subset Q$.

For an oriented geodesic L intersecting an obstacle O_i, there is a first-hit point $p' \in \partial O_i$ with $\langle n_{p'}, v_{p'} \rangle \leq 0$ (note that the hyperbolic normal vector is in the direction of the Euclidean normal vector, but possibly of different length) and a last-hit point $p \in \partial O_i$ with $\langle n_p, v_p \rangle \geq 0$.

The billiard map $f : Q_0 \to Q$ can be defined as before: For $L \in S_i$, $L \cap O_j \neq \emptyset$ with $j \neq i$, p the last-hit point of L on ∂O_i, q the first-hit point of L on ∂O_j, we set $f(L) = L'$, where L' is the oriented geodesic through q with the vector $w_q = P_p^q v_p - 2\langle n_q, P_p^q v_p \rangle n_q$, P_p^q denoting the parallel transportation of a vector at p, along the geodesic, to a vector at q (see Figure 3).

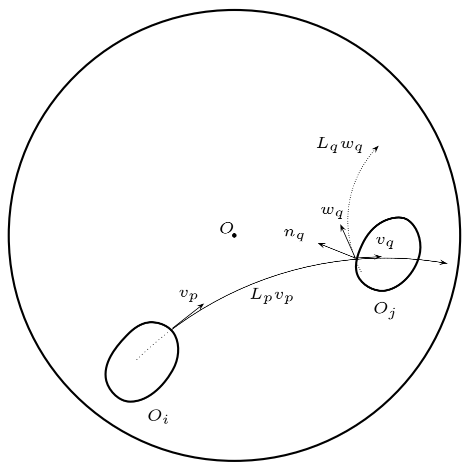

Figure 3. *The billiard map sends $L_p v_p$ to the geodesic $L_q w_q$ with $w_q = v_q - 2\langle n_q, v_q \rangle n_q$, where $v_q = P_p^q v_p$.*

Now let us consider a finite number of mutually disjoint obstacles $O_1, O_2, \ldots, O_K$ ($K \geq 3$). We again assume Condition 2.

Condition 2 (no eclipse). *For any triple (j_1, j_2, j_3) of distinct indices,*

$$\mathrm{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \emptyset.$$

Theorem 3. *Given an itinerary $(S_{i_n})_{n \in \mathbb{Z}}$ with $i_n \in \{1, 2, \ldots, K\}$ and $i_n \neq i_{n+1}$, there exists a trajectory $(L_n)_{n \in \mathbb{Z}}$ following this itinerary; i.e., $f(L_n) = L_{n+1}$ and $L_n \in S_{i_n}$.*

For proof, we again apply the chaos lemma. We will outline those points where there are slight modifications in comparison to the Euclidean case.

We denote the restriction of the hyperbolic unit tangent bundle $T_1(\mathbb{B}^n)$ to ∂O_i by Σ_i. Let $D \subset \partial O_i$ be a nonempty, compact subset of ∂O_i and $\sigma : D \to \Sigma_i$ a continuous outward vector field, i.e., $\sigma(p) = (p, \sigma_2(p))$ with $\langle n_p, \sigma_2(p) \rangle \geq 0$ for $p \in D$. Let $p_1, p_2 \in D$ and α_i denote the angle between $\sigma_2(p_i)$ and the hyperbolic segment $[p_1 p_2]$ for $i = 1, 2$. We call σ a dispersive vector field if $\alpha_1 + \alpha_2 \geq \pi$ for all $p_1, p_2 \in D$. (This condition is equivalent to the one we used in the Euclidean case, but this formulation is more convenient for the hyperbolic setting, especially for checking the dispersiveness of reflected rays.)

Lemma 4. *For a dispersive vector field σ, either the hyperbolic rays $L^+_{p_1}\sigma_2(p_1)$ and $L^+_{p_2}\sigma_2(p_2)$ are disjoint or one is contained in the other.*

We omit the proof.

The definition of exhaustiveness remains the same. As for expanders, we take again sets of oriented geodesics determined by exhaustively dispersive vector fields. If $\sigma : D_\sigma \to \Sigma_i$, $D_\sigma \subset \partial O_i$, we set $E(\sigma) = \{L_p\sigma_2(p) \mid p \in D_\sigma\} \subset Q$ and define

$$\mathcal{E}_i = \{E(\sigma) \mid \sigma \text{ is an exhaustively dispersive vector field on } D_\sigma \subset \partial O_i\}.$$

$\mathcal{E}_i$ is nonempty because by hyperbolic convexity the segment $[p_1p_2]$ lies inside O_i for $p_1, p_2 \in \partial O_i$, making the angle with the normals n_{p_1} and n_{p_2} greater than (or equal to) $\frac{\pi}{2}$. This shows that the normal outward vector field on ∂O_i is exhaustively dispersive.

We define the pre-expanders as before: given $E \in \mathcal{E}_i$, we set

$$E_j = \{L_p\sigma_2(p) \mid p \in D_\sigma \text{ and } L^+_p\sigma_2(p) \cap \partial O_j \neq \emptyset\}.$$

We have to show that $f(E_j) \in \mathcal{E}_j$. In the Euclidean case, we used degree theory to see this. In the hyperbolic setting, degree theory can still be used with suitable modifications.

Given any section $\sigma : \partial O_i \to \Sigma_i$, for any $p \in \partial O_i$ we can parallel-transport the vector $\sigma_2(p)$ to the origin along the geodesic between p and the origin (which is a Euclidean straight line segment). We thus obtain a map $\tilde{\sigma} : \partial O_i \to S^{n-1}_{\mathbb{H}}$ (the hyperbolic unit sphere is also a Euclidean sphere) and define *degree*(σ) to be *degree*$(\tilde{\sigma})$.

If $\sigma : D_\sigma \to \Sigma_i$ is any exhaustively dispersive vector field, then we get as before *degree*$(\sigma^*) = 1$. As a consequence, given any point $q \in \mathbb{B}^n \backslash O_i$, there exists a point $p \in \partial O_i$ with $q \in L^+_p\sigma^*_2(p)$. Then we get $\partial O_j \subset \bigcup_{L \in E_j} L^+_p\sigma_2(p)$. We define as before $\varphi : \partial O_j \to \Sigma_j$ $\varphi(q) = P^q_p\sigma_2(p)$ for $q \in L^+_p\sigma_2(p)$ and $\tau : D_\tau \to \Sigma_j$ with $D_\tau = \{q \in \partial O_j \mid \langle n_q, \varphi(q)\rangle \leq 0\} \subset \partial O_j$ and

$$\tau(q) = (q, \varphi(q) - 2\langle n_q, \varphi(q)\rangle n_q).$$

With this τ it holds that $f(E_j) = E(\tau)$, and to show $f(E_j) \in \mathcal{E}_j$ we must prove that τ is continuous, dispersive, and exhaustive. Continuity and exhaustiveness go almost verbatim as in the Euclidean case, but dispersiveness requires a separate argument. We have to show that for $q_1, q_2 \in D_\tau$, the angle β_1 between $\tau_2(q_1)$ and the hyperbolic segment $[q_1q_2]$ and the angle β_2 between $\tau_2(q_2)$ and $[q_1q_2]$ satisfy the inequality $\beta_1 + \beta_2 \geq \pi$.

We first show that the angles θ_1, θ_2 between $P^{q_1}_{p_1}\sigma_2(p_1), P^{q_2}_{p_2}\sigma_2(p_2)$ and the hyperbolic segment $[q_1q_2]$ satisfy $\theta_1 + \theta_2 \geq \pi$. Then we will show $\beta_1 \geq \theta_1$ and $\beta_2 \geq \theta_2$. Now let $\measuredangle p_1p_2q_1 = \alpha'_2$, $\measuredangle q_1p_2q_2 = \alpha''_2$, $\measuredangle q_2q_1p_2 = \gamma'$, and γ be the angle between $P^{q_1}_{p_1}\sigma_2(p_1)$ and $[p_2q_1]$. By the triangle inequality for angles, we have

$$\alpha_2 \leq \alpha'_2 + \alpha''_2 \text{ and } \gamma \leq \gamma' + \theta_1.$$

On the other hand, for the hyperbolic triangle $p_1p_2q_1$, $\gamma > \alpha_1 + \alpha'_2$ (because the sum of the inner angles of a hyperbolic triangle is less than π). We thus get

$$\begin{aligned} \gamma + \alpha'_2 + \alpha''_2 &\geq \alpha_1 + \alpha'_2 + \alpha_2, \\ \gamma + \alpha''_2 &\geq \alpha_1 + \alpha_2 \geq \pi. \end{aligned}$$

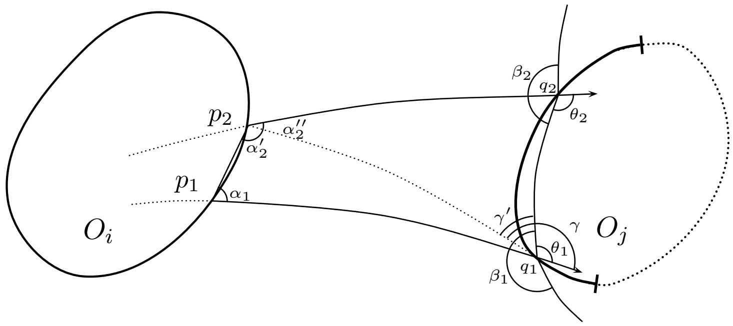

Figure 4. *τ is a dispersive vector field.*

From the hyperbolic triangle $p_2q_1q_2$ we have $\theta_2 > \gamma' + \alpha_2''$. Hence, $\theta_1 + \theta_2 \geq \theta_1 + \gamma' + \alpha_2'' \geq \gamma + \alpha_2'' \geq \alpha_1 + \alpha_2 \geq \pi$.

We now show $\beta_1 \geq \theta_1$:

$$\tau_2(q_1) = P_{p_1}^{q_1}\sigma_2(p_1) - 2\langle n_{q_1}, P_{p_1}^{q_1}\sigma_2(p_1)\rangle n_{q_1} \text{ by definition of } \tau.$$

As the scalar product on the right-hand side is negative, we have

$$\tau_2(q_1) = P_{p_1}^{q_1}\sigma_2(p_1) + \lambda n_{q_1} \text{ with } \lambda \geq 0.$$

Denote the unit vector at q_1 in the direction of q_1q_2 by u, and multiply the above equation by u:

$$\langle \tau_2(q_1), u\rangle = \langle P_{p_1}^{q_1}\sigma_2(p_1), u\rangle + \lambda\langle n_{q_1}, u\rangle,$$
$$\cos\beta_1 = \cos\theta_1 + \lambda\langle n_{q_1}, u\rangle.$$

As $\langle n_{q_1}, u\rangle \leq 0$ we get $\cos\beta_1 \leq \cos\theta_1$, and thus $\beta_1 \geq \theta_1$. Similarly, $\beta_2 \geq \theta_2$, giving the dispersiveness and completing the proof.

5. Spinning planar billiards.

5.1. Introduction. Billiard dynamics has been a standard dynamics model for many years, but the model is far from a complete description of reality. Real billiard balls have spin, which is utilized by billiards players. Here we extend the model to permit the billiard to spin. See [3, 4, 5] for a discussion of the physics. We call the particle a "puck"; it is analogous to a hockey puck that can spin on the plane. In contrast, a billiard ball on a table spins in three dimensions, a case we do not consider here.

We find the new mathematics that result intriguing. The state space describing a puck is larger since it must include speed and spin rate as well as position and direction—we can no longer assume the puck's speed is constant. Our puck is assumed to have no friction between collisions, but there is friction in the collision. Friction in the collision introduces an interaction among the spin of the obstacle, the spin of the puck, and the incoming velocity of the puck. Most people have observed that a thrown rubber ball has a bounce that is affected by the spin of the ball. A rapidly spinning ball can gain speed in the collision, translating spin energy into velocity. So-called *super balls* are resilient (i.e., they retain most of their energy in a bounce) and have a high friction coefficient. Their bounces are especially influenced by spin.

In the appendix, we discuss the equations for a collision between the puck and an obstacle, using the most commonly used rules of friction, namely, stick-slip friction. A block sliding on a surface slows down, and its deceleration is independent of its velocity, provided that the velocity is positive. The block therefore reaches 0 speed in finite time. We must compress this phenomenon into an instantaneous bounce. A key factor is the relative speeds of the points on the puck and obstacle that are in contact during the collision. For the incoming trajectory, they are unlikely to be equal, but for the outgoing trajectory, equality is quite likely if the incoming difference is small.

The stick-slip dynamics of a collision with a fixed obstacle (described in the appendix) have some easily derived properties that can be useful in modeling, though they are not necessary for the results in this section. Let Δs denote the absolute value of the difference between the incoming spin rate of a puck (just before a collision) and its spin immediately after. Similarly, let Δv denote the norm of the difference in the incoming and outgoing velocities. Let w be the norm of the normal component of the incoming velocity. Then there are constants $C_1 > 0$ and $C_2 > 0$ (that depend only on the coefficient of friction and moment of inertia of the puck) such that

$$\Delta s \leq C_1 w \quad \text{and} \quad C_2 w \leq \Delta v \leq C_1 w.$$

In particular, in a tangential collision, $w = 0$, so the change in velocity and spin are 0.

We also note that the outgoing velocity and spin are continuous functions of initial position, velocity, and spin. Additionally, the change in the normal component of velocity depends only on w, and the outgoing velocity must be nonzero.

5.2. Spinning billiards.

In our result we assume the following properties, which are much less specific than would be required by the stick-slip friction model.

Condition 2 (no eclipse). *For any triple (j_1, j_2, j_3) of distinct indices,*

$$\operatorname{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \emptyset.$$

Condition 3 (physical properties). *The obstacles are fixed disks with fixed spin rates, all having the same fixed coefficients of friction. The puck is a disk and has a fixed coefficient of restitution $e > 0$, so its outgoing speed is always positive (the puck cannot stop at a collision). The velocity and spin rate of the puck are constant between collisions.*

Condition 4 (continutity). *For each obstacle, following a collision, the puck's outward velocity and rate of spin are continuous functions of the inward velocity, rate of spin, direction of motion, and point of contact with the obstacle. Furthermore, if the puck hits an obstacle tangentially, then its velocity and rate of spin do not change.*

The ambient phase space of the dynamics (at instants of collisions) will be

$$Q = \bigcup_{i=1}^{K} (S_i^1 \times S^1 \times (0, \infty) \times \mathbb{R}),$$

where the first factor S_i^1 is the boundary of the disk O_i, the second factor S^1 codes the direction of the puck, the third factor $(0,\infty)$ codes its linear speed, and the last factor $\mathbb{R}$ codes its angular velocity at the moment of leaving O_i.

The symbol sets will be

$$S_i = \{(p,v,s,r) \mid p \in S_i^1,\ \langle n_p, v\rangle \geq 0, \text{ and } \exists j \text{ such that } L_p^+ v \cap S_j^1 \neq \emptyset\} \subset Q,$$

where n_p and $L_p^+ v$ denote as usual the outward unit normal at p and the (Euclidean) ray starting at p in direction v. Let

$$Q_0 = \bigcup_{i=1}^{K} S_i.$$

The billiard map $f : Q_0 \to Q$ is continuous by Condition 4. See Figure 5.

Theorem 4. *Assume Conditions 2, 3, and 4. Given a forward itinerary $(S_{i_n})_{n\in\mathbb{N}}$ with $i_n \in \{1,2,\dots,K\}$ and $i_n \neq i_{n+1}$, there exists a trajectory $(L_n)_{n\in\mathbb{N}}$ following this itinerary; i.e., $f(L_n) = L_{n+1}$ and $L_n \in S_{i_n}$.*

Since this theorem discusses only forward itineraries, the corresponding trajectories are not unique. Even if the forward and backward trajectories were specified, there is no guarantee that they would correspond to a *unique* trajectory. The approach we use is general enough that it can be applied to situations where uniqueness of trajectories does not hold.

To define the expanders in the present case, we first consider continuous maps

$$g : J \to S_i^1 \times S^1 \times (0,\infty) \times \mathbb{R},$$
$$t \mapsto (p(t), v(t), s(t), r(t)),$$

where J is a compact interval in $\mathbb{R}$ or $J \subset S_i^1$ for some i and the second coordinate $v(t)$ is an outward direction at the point $p(t)$. We call such a map g *exhaustive* if it can be extended to a map g^* on a circle such that, on the complementary closed arc J', the following conditions hold:

1. For $t \in J'$, $v(t)$ is an outward direction at $p(t)$, and the ray $L_{p(t)}^+ v(t)$ does not hit any disk O_j for $j \neq i$.
2. The degree of the map given by the first coordinate of g^* is ± 1. (That is, $p(t)$ winds around S_i^1 once.)

The expanders are

$$\mathcal{E}_i = \{\text{Image}(g) \mid g : J \to S_i^1 \times S^1 \times (0,\infty) \times \mathbb{R} \text{ is an exhaustive map}\}.$$

Each $E \in \mathcal{E}_i$ is a subset of $S_i^1 \times S^1 \times (0,\infty) \times \mathbb{R} \subset Q$.

We must prove the existence of pre-expanders: Given $E = \text{Image}(g) \in \mathcal{E}_i$ and $j \neq i$, there must exist some $E_j \subset E \cap S_i$ such that $f(E_j) \in \mathcal{E}_j$.

As t traverses the interval $J = [a,b]$ from a to b, we start with the ray $L_{p(a)}^+ v(a)$ which does not hit O_j. There will be t with $L_{p(t)}^+ v(t)$ hitting O_j by the degree condition, and so there will be tangencies. Finally, the last ray $L_{p(b)}^+ v(b)$ does not hit O_j again. This implies the existence of a subinterval $[a_j, b_j] \subset [a,b]$ sweeping O_j: $L_{p(a_j)}^+ v(a_j)$ and $L_{p(b_j)}^+ v(b_j)$ are tangent to O_j, $L_{p(t)}^+ v(t)$ hits O_j for $t \in [a_j, b_j]$, and $O_j \subset \bigcup_{t\in[a_j,b_j]} L_{p(t)}^+ v(t)$.

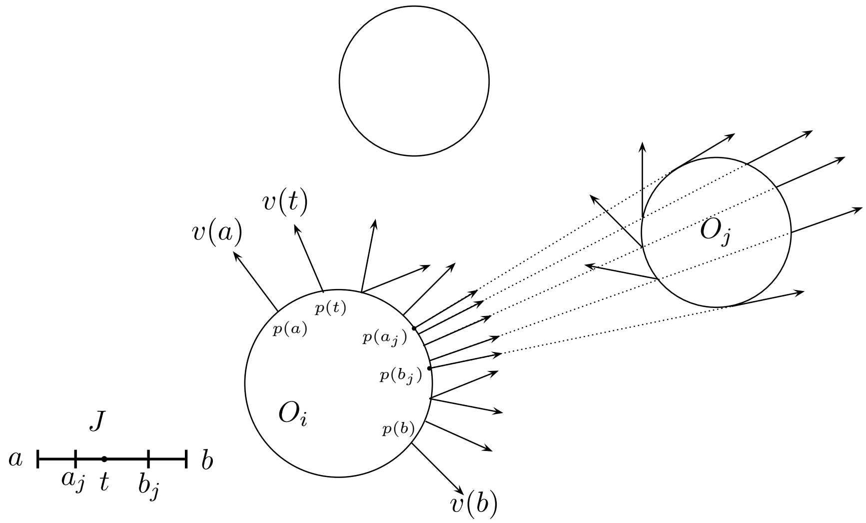

Figure 5. *Spinning billiards. Three obstacles are shown, with trajectories leaving obstacle O_i. If O_j has a strong clockwise spin, the puck's outward trajectories might be as shown. Independent of the spin of the obstacles, trajectories like $p(a_j)$ and $p(b_j)$ hit O_j tangentially and continue without deviating.*

If we set $E_j = \text{Image}\left(g|_{[a_j,b_j]}\right)$, then $E_j \subset E \cap S_i$. It can be seen by the same reasoning as at the end of the proof of Theorem 2 that $f(E_j)$ is an expander again, completing the proof that in spinning billiards in the Euclidean plane, any nonrepeating but otherwise arbitrary infinite sequence of obstacles has a trajectory that follows it.

Appendix. In this section, we develop a model for the scattering of a circular puck from an immovable circular obstacle, where the interaction between the two bodies involves friction. The obstacles' centers do not move, and each spins at a constant rate unaffected by collision with the puck.

Consider the situation shown in Figure 6. At time $t = 0$, a puck starts out at initial position $\mathbf{x_i}$ and with initial velocity $\mathbf{v}$ directed toward an obstacle situated at $\mathbf{x_o}$. The initial angular velocity of the puck is $\boldsymbol{\omega_p}$, and the obstacle is given to be always rotating with an angular velocity of $\boldsymbol{\omega_o}$. The goal is to find the final state given by $\mathbf{v}'$ and $\boldsymbol{\omega_p'}$ immediately after the collision at $\mathbf{x_c}$, the point of impact. With the exception of the angular velocities $\boldsymbol{\omega_p}$ and $\boldsymbol{\omega_o}$ which are strictly in the z direction, all vectorial quantities in this discussion are (and hence all motion is) confined to the $x-y$ plane. The dynamics are independent of the puck's mass m, which we can set to 1.

The first step is to determine if and when a collision occurs. For times t before the collision, the position vector of the puck is given by $\mathbf{x} = \mathbf{x_i} + \mathbf{v}\,t$. The collision takes place at some time t_c, when the puck is at $\mathbf{x_c}$:

$$\mathbf{x_c} = \mathbf{x_i} + \mathbf{v}\,t_c. \tag{3}$$

We assume that the collision deforms neither the puck nor the obstacle. This means that at

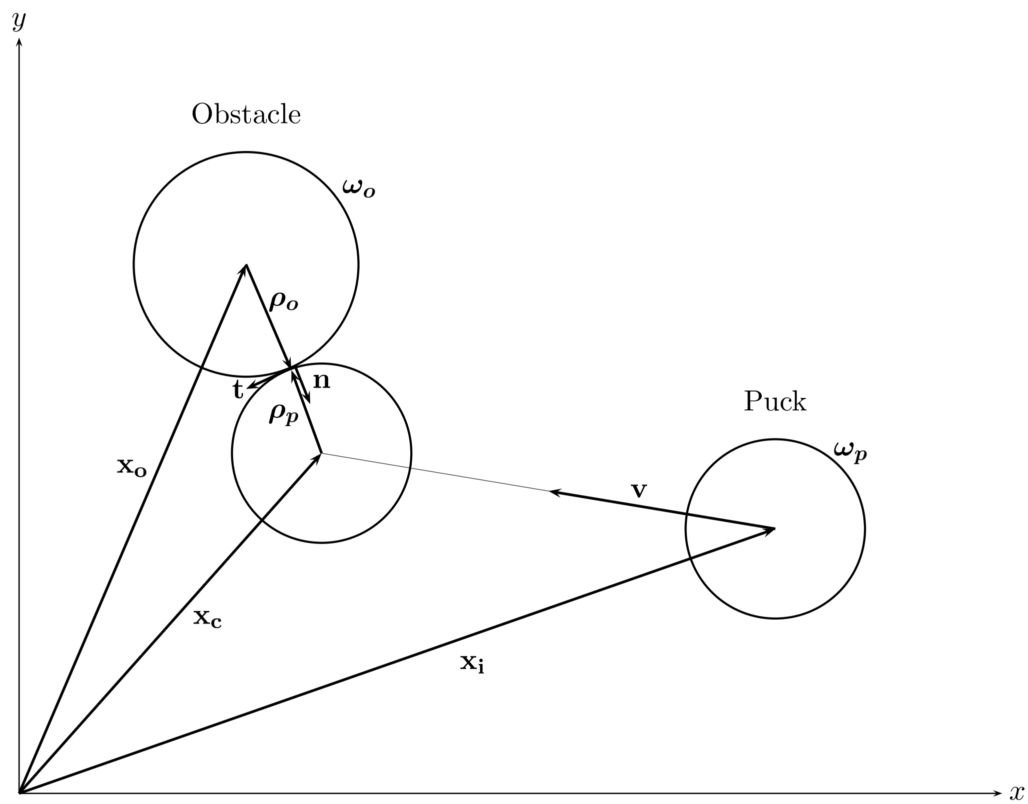

Figure 6. *A puck (smaller disk) starts at* $\mathbf{x_i}$ *and travels to* $\mathbf{x_c}$*, where it collides with an obstacle situated at* $\mathbf{x_o}$*.*

impact time,

$$|\mathbf{x_c} - \mathbf{x_o}| = R_p + R_o \tag{4}$$

will hold, where R_p and R_o are the radii of the puck and obstacle, respectively. Substituting (3) into (4) and defining $\mathbf{z} = \mathbf{x_i} - \mathbf{x_o}$ leads to

$$|\mathbf{z} + \mathbf{v}\, t_c| = R_p + R_o.$$

After squaring both sides, we get a quadratic equation for the collision time t_c of the form $a\, t_c^2 + b\, t_c + c\, t_c = 0$, where $a = v^2$, $b = 2\,\mathbf{z}\cdot\mathbf{v}$, and $c = z^2 - (R_p + R_o)^2$.

The discriminant $\Delta = b^2 - 4ac$ leads to several regimes of qualitatively different physical behavior: $\Delta < 0$ implies that the puck will miss the obstacle entirely, whereas $\Delta = 0$ means that the puck will pass tangentially to the obstacle. No interaction is assumed between the two bodies in either case. A value of $\Delta > 0$, on the other hand, means that there will be two values of t, t_+ and t_-, for which (4) is satisfied (one for each side of the obstacle). If these are positive, we pick the smaller of the two values. Of course $t_+, t_- < 0$ correspond to meaningless collisions for negative times. In short,

$$t_c = \min\{t_+, t_-\} \quad (t_\pm > 0), \tag{5}$$

and substituting this back into (3) yields the collision coordinate $\mathbf{x_c}$. For the remainder of this discussion assume that a given set of initial conditions will lead to a collision satisfying (5) above.

The second and final step is to determine what happens in a collision. Define unit vectors in the normal and tangential directions at the point of contact as

$$\mathbf{n} = (\mathbf{x_c} - \mathbf{x_o})/|\mathbf{x_c} - \mathbf{x_o}|,$$
$$\mathbf{t} = \mathbf{n} \times \mathbf{k},$$

where $\mathbf{k}$ is the unit vector in the z direction (we assume a right-handed coordinate system) and where a "$\times$" denotes the usual vector cross product. We have defined $\mathbf{n}$ to be pointing away from the obstacle. Position vectors from the centers of the puck and obstacle that extend to the point of contact are then given by $\boldsymbol{\rho_p} = -R_p\,\mathbf{n}$ and $\boldsymbol{\rho_o} = R_o\,\mathbf{n}$. These in turn allow us to write the tangential velocities of the rims of the puck and obstacle at the point of contact:

$$\mathbf{V_p} = (\mathbf{v}\cdot\mathbf{t})\,\mathbf{t} + \boldsymbol{\omega_p}\times\boldsymbol{\rho_p}, \tag{6}$$

$$\mathbf{V_o} = \boldsymbol{\omega_o}\times\boldsymbol{\rho_o}. \tag{7}$$

We now turn our attention from kinematics to dynamics. The collision is instantaneous but can be considered as the limit of a more physically realistic, very brief collision. All these approximations have the same impulse J. Throughout the collision, the obstacle will exert a normal force $N(t)$, where t lies in the small interval during which the collision takes place. The time integral of this over that small interval is the impulse $J \equiv \int N(t)dt$, which acts in the direction $\mathbf{n}$. We do not calculate $N(t)$ itself; instead, we work with J not only for motion in the normal direction but also for the tangential and rotational degrees of freedom. To begin, recall from mechanics that if $\mathbf{p} = \mathbf{v}$ is the initial momentum of the puck having mass $m = 1$ and $\mathbf{p_n} = (\mathbf{p}\cdot\mathbf{n})\mathbf{n}$ is the component of this momentum in the normal direction, then the new value of $\mathbf{p_n}$ is given by $\mathbf{p'_n} = \mathbf{p_n} + J\,\mathbf{n}$. On the other hand, the coefficient of restitution [3, 4, 5], $e \in (0,1]$, satisfies $\mathbf{p'_n} = -e\,\mathbf{p_n}$. Therefore,

$$J = -(1+e)\mathbf{v}\cdot\mathbf{n}, \tag{8}$$

which is always positive since $\mathbf{v}\cdot\mathbf{n}$ is negative.

The impulse imparted to the puck in the tangential direction involves the force of friction, $F(t)$. If the puck and obstacle are sliding with respect to each other during the entire duration of the impact (the "sliding" regime), this force will be assumed to have the form $F(t) = \mu_s N(t)$, where μ_s is the coefficient of sliding friction. The impulse generated by $F(t)$ will then be $\mu_s J$. If, on the other hand, at some point during the collision the tangential velocities $\mathbf{V_p}$ and $\mathbf{V_o}$ are equalized due to friction, $F(t)$ has to be set to zero for the remainder of the impact (we neglect rolling friction here), and the impulse in this direction will then be correspondingly less than $\mu_s J$. Physically, this corresponds to the puck and obstacle rolling about each other (the "rolling" regime, even though the puck may have been initially sliding).

Before we discuss the details of the two regimes, we note that (6) and (7) allow us to define a unit vector in the direction of friction as $\mathbf{f} = -(\mathbf{V_p} - \mathbf{V_o})/|\mathbf{V_p} - \mathbf{V_o}|$. This simply states that friction acts to oppose the motion of the puck relative to the obstacle at the point of contact. The force of friction is then written as $\mathbf{F}(t) = F(t)\mathbf{f}$. Since $\mathbf{f}$ and $\mathbf{t}$ have the same sense, $\mathbf{f} \cdot \mathbf{t} = \pm 1$.

The sliding regime. The velocity of the puck's center of mass after the collision is

$$\mathbf{v}' = \mathbf{v} + J\,\mathbf{n} + \mu_s J\,\mathbf{f}. \tag{9}$$

In addition to this, the presence of friction will create a torque $\boldsymbol{\rho_p} \times \mathbf{F}(t)$ on the puck, which will cause a change in its angular momentum $\mathbf{L} = I\boldsymbol{\omega_p}$, where I is the moment of inertia of the puck. The final angular velocity $\boldsymbol{\omega'_p}$ is then found to be

$$\boldsymbol{\omega'_p} = \boldsymbol{\omega_p} + (1/I)(\boldsymbol{\rho_p} \times \mathbf{f})\,\mu_s J. \tag{10}$$

Together with (8), equations (9) and (10) express the final state of the puck in the sliding regime.

The rolling regime. This case differs from the sliding friction case in that the changes in the angular and tangent velocities are smaller than those predicted by (9) and (10). As mentioned above, if rolling takes hold during a collision, the force due to friction will drop discontinuously to zero. As a result, the values of J for the puck in the tangential and rotational directions in (9) and (10) will take a new value J^L, where the superscript denotes the "locking" of the bodies. To calculate J^L we recall that the onset of rolling happens when $\mathbf{V'_p} = \mathbf{V_o}$. More explicitly, using (6) and (7),

$$(\mathbf{v}' \cdot \mathbf{t})\,\mathbf{t} + \boldsymbol{\omega'_p} \times \boldsymbol{\rho_p} = \boldsymbol{\omega_o} \times \boldsymbol{\rho_o}.$$

Substituting the previous solutions (9) and (10) into this and using the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}),$$

we can solve for J^L,

$$J^L = \frac{m}{\mu_s}\frac{\alpha}{1+\alpha}|\mathbf{V_p} - \mathbf{V_o}|,$$

which is also positive and where $\alpha = I/(mR_p^2)$ ($\alpha = \frac{1}{2}$ for a solid disk). The final state for the case of rolling is then

$$\mathbf{v}' = \mathbf{v} + (1/m)J\,\mathbf{n} + (1/m)\mu_s J^L\,\mathbf{f}, \tag{11}$$

$$\boldsymbol{\omega'_p} = \boldsymbol{\omega_p} + (1/I)(\boldsymbol{\rho_p} \times \mathbf{f})\,\mu_s J^L, \tag{12}$$

where J is still given by (8).

Deciding between the regimes. The question of whether a set of initial conditions leads to sliding or rolling is easily addressed by considering the relative velocities $\mathbf{V_p} - \mathbf{V_o}$ (at collision) and $\mathbf{V'_p} - \mathbf{V_o}$ (after the collision, assuming the sliding case). If the relative motion of the puck at the rim has changed direction, i.e., if $(\mathbf{V_p} - \mathbf{V_o}) \cdot (\mathbf{V'_p} - \mathbf{V_o}) < 0$, then rolling must take place, and we use (11) and (12) instead of (9) and (10) for the final state of the puck. It is also possible that $\mathbf{V_p} - \mathbf{V_o} = 0$; in this case, the bodies start rolling immediately after impact, and we take the rolling case equations and set $J^L = 0$.

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