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Spherically symmetric Finsler metrics with constant Ricci and flag curvature

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Abstract. Spherically symmetric metrics form a rich and important class of metrics. Many well-known Finsler metrics of constant flag curvature can be locally expressed as a spherically symmetric metric on R^n. In this paper, we study spherically symmetric metrics with constant Ricci curvature (tensor) and constant flag curvature.

1. Introduction

It is one of important problems in Finsler geometry to study and characterize Finsler metrics with constant flag curvature or constant Ricci curvature (tensor). Let $R_j{}^i{}_{kl}$ denote the Riemann curvature tensor of the Berwald connection and $R^i_k := R^i_{j\ kl} y^j y^l$. A Finsler metric F is said to be of *constant flag curvature* if

$$R^i{}_k = K\{F^2 \delta^i_k - g_{kl} y^l y^i\}.$$

Many Finsler metrics of constant flag curvatures can be locally expressed on a ball $B^n(\rho) \subset R^n$ in the following form

$$F = |y|\phi(r,s), \quad r = |x|,\ s = \frac{\langle x, y\rangle}{|y|}, \quad y \in T_x B(\rho) \equiv R^n,$$

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where $\phi = \phi(r,s)$ is a positive smooth function defined on $[0,\rho)\times(-\rho,\rho)$. Finsler metrics in this form are called *spherically symmetric metrics* which are first studied by L. Zhou [9]. For example, the well-known Funk metric on $B^n(1)\subset R^n$ is projectively flat with constant flag curvature $K=-1/4$. $\phi=\phi(r,s)$ is given by

$$\phi = \frac{\sqrt{1-(r^2-s^2)}+s}{1-r^2}. \tag{1.1}$$

Using the above Funk metric, one can construct another projectively flat metric on $B^n(1)$ with zero flag curvature $K=0$ (due to L. Berwald). $\phi=\phi(r,s)$ is given by

$$\phi = \frac{(\sqrt{1-(r^2-s^2)}+s)^2}{(1-r^2)^2\sqrt{1-(r^2-s^2)}}. \tag{1.2}$$

One can also construct a projectively flat metric with constant flag curvature $K=-1$ ([6]). $\phi=\phi(r,s)$ is given by

$$\phi = \frac{1}{2}\left\{\frac{\sqrt{1-(r^2-s^2)}+s}{1-r^2} - \epsilon\frac{\sqrt{1-\epsilon^2(r^2-s^2)}+\epsilon s}{1-\epsilon^2 r^2}\right\}, \tag{1.3}$$

where $-1\le\epsilon<1$ is a constant. They are all spherically symmetric metrics with constant flag curvature.

Recently, Mo–Zhou [5] and Mo–Zhou–Zhu [4] find three equations that characterize spherically symmetric metrics of constant curvature and find some new locally projectively flat metrics of constant flag curvature. We shall show that these three equations can be reduced to two equations (Theorems 1.1 and 1.2 below).

To state our results, we introduce the following notations. For a positive smooth function $\phi=\phi(r,s)$ on $[0,\rho)\times(-\rho,\rho)$, let

$$R_1 := P^2 - \frac{1}{r}(sP_r + rP_s) + 2Q[1+sP+(r^2-s^2)P_s]$$

$$R_2 := \frac{1}{r}(2Q_r - sQ_{rs} - rQ_{ss}) + 2Q(2Q - sQ_s) + (r^2-s^2)(2QQ_{ss} - [Q_s]^2),$$

$$R_3 := \frac{1}{r}(P_r - sP_{rs} - rP_{ss}) + 2Q[1+sP+(r^2-s^2)P_s]_s,$$

where

$$P = \frac{1}{2r\phi}(s\phi_r + r\phi_s) - \frac{Q}{\phi}\{s\phi + (r^2-s^2)\phi_s\}$$

$$Q = \frac{1}{2r}\frac{s\phi_{rs} + r\phi_{ss} - \phi_r}{\phi - s\phi_s + (r^2-s^2)\phi_{ss}}.$$

We have the following

Theorem 1.1. *Let $F = |y|\phi(r, s)$ be a spherically symmetric Finsler metric on an open ball $B^n(\rho) \subset R^n$ $(n \geq 3)$. Then F is of constant flag curvature K if and only if*

$$R_1 = K\phi^2, \quad R_2 = 0, \tag{1.4}$$

In fact, the terms R_1, R_2 and R_3 are related mysteriously. We can show that $R_2 = 0$ and $R_3 = 0$ are equivalent to that $R_1 = K\phi^2$ and $R_2 = 0$.

Theorem 1.2. *Let $F = |y|\phi(r, s)$ be a spherically symmetric Finsler metric on an open ball $B^n(\rho) \subset R^n$ $(n \geq 3)$. Then F is of constant flag curvature if and only if*

$$R_2 = 0, \quad R_3 = 0.$$

The Ricci curvature Ric is defined as $\mathrm{Ric} = R^m{}_m$. A Finsler metric F is said to be of *constant Ricci curvature* if

$$\mathrm{Ric} = (n-1)KF^2.$$

In [5], MO–ZHOU study spherically symmetric metrics with constant Ricci curvature. They find one equation that characterizes spherically symmetric metrics of constant Ricci curvature (Theorem 1.3 below).

Theorem 1.3 ([5])**.** *Let $F = |y|\phi(r, s)$ be a spherically symmetric Finsler metric where $r := |x|$ and $s := \frac{\langle x,y\rangle}{|y|}$. Then* $\mathrm{Ric} = (n-1)KF^2$ $(K = \text{constant})$ *if and only if ϕ satisfies the PDE below:*

$$(n-1)K\phi^2 = (n-1)R_1 + (r^2 - s^2)R_2. \tag{1.5}$$

There is a notion of Ricci curvature tensor Ric_{ij} introduced in [2].

$$\mathrm{Ric}_{ij} := \frac{1}{2}\Big\{R_i{}^m{}_{mj} + R_j{}^m{}_{mi}\Big\}, \tag{1.6}$$

Note that

$$\mathrm{Ric} = \mathrm{Ric}_{ij}\, y^i y^j. \tag{1.7}$$

By (1.7), one can easily see that $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$ implies that $\mathrm{Ric} = (n-1)KF^2$. It is an interesting problem to see the difference between $\mathrm{Ric} = (n-1)KF^2$ and $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$. We shall discuss this problem via spherically symmetric Finsler metrics on R^n.

We have the following

Theorem 1.4. *Let $F = |y|\phi(r, s)$ be a spherically symmetric Finsler metric where $r := |x|$ and $s := \frac{\langle x,y\rangle}{|y|}$. Then* $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$ $(K = \text{constant})$ *if and only if ϕ satisfies*

$$(n-1)K\phi^2 = (n-1)R_1 + (r^2 - s^2)R_2, \quad (n+1)R_3 + (r^2 - s^2)[R_2]_s = 0. \tag{1.8}$$

Note that for the above three functions, $\phi = \phi(r,s)$, in (1.1), (1.2) and (1.3), $Q = 0$, i.e., $s\phi_{rs} + r\phi_{ss} - \phi_r = 0$. In this case, $R_2 = 0$, R_1 and R_3 can be simplified further. For a spherically symmetric metric with $R_2 = 0$, $\mathrm{Ric} = (n-1)K$ if and only if $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$ if and only if the flag curvature is a constant K.

2. Preliminaries

Let $F = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on R^n, where $r := |x|$ and $s := \frac{\langle x,y\rangle}{|y|}$. According to Mo–Zhou–Zhu ([4]), the Riemann curvature tensor $R^i{}_j$ is given by

$$R^i{}_j = R_1(|y|^2\delta^i_j - y^iy^j) + |y|R_2(|y|x^j - sy^j)x^i + R_4(|y|x^j - sy^j)y^i, \qquad (2.1)$$

where R_1, R_2, R_3 are given in the introduction above and R_4 is given by

$$R_4 := \frac{1}{2}\{3R_3 - [R_1]_s\}. \qquad (2.2)$$

Note that R_3 here is not the R_3 in [4] and R_1, R_2, R_4 are the same terms as in [4]. Using the identity $g_{ik}R^i{}_j = g_{ij}R^i{}_k$, we immediately obtain

$$R_4 + sR_2 = -\frac{\phi_s}{\phi}\big\{(r^2 - s^2)R_2 + R_1\big\}. \qquad (2.3)$$

Recall that F is of scalar flag curvature if and only if $R^i{}_j = R\delta^i_j - \tau_j y^i$ with $\tau_j y^j = R$. Thus it is easy to see from (2.1) that $F = |y|\phi(r,s)$ is of scalar flag curvature if and only if $R_2 = 0$.

Lemma 2.1 ([1])**.** *Let $F = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on a ball $B^n(\rho) \subset R^n$ $(n \geq 3)$. Then F is of scalar flag curvature if and only if $R_2 = 0$.*

By the above formula (2.1), Mo–Zhou–Zhu [4] prove the following

Theorem 2.2 ([4])**.** *Let $F = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on a ball $B^n(\rho) \subset R^n$ $(n \geq 3)$. Then F is of constant flag curvature K if and only if*

$$R_1 = K\phi^2, \qquad (2.4)$$

$$R_2 = 0, \qquad (2.5)$$

$$R_3 = 0, \qquad (2.6)$$

where R_1, R_2 and R_3 are given as above.

In fact, two equations (2.4) and (2.5) in Theorem 2.2 will be sufficient (Theorem 1.1 above).

There is an important non-Riemannian quantity, $\chi = \chi_i dx^i$, defined by the S-curvature [7].

$$\chi_i := \frac{1}{2}\big\{S_{\cdot i|m}y^m - S_{|i}\big\},$$

where S denotes the S-curvature of F with respect to the Busemann–Hausdorff volume. It can be also expressed in terms of the Riemann curvature $R^i{}_k = R_j{}^i{}_{kl}y^jy^l$ by

$$\chi_i = -\frac{1}{6}\big\{2R^m{}_{i\cdot m} + R^m{}_{m\cdot i}\big\}, \tag{2.7}$$

where " $\cdot$ " denotes the vertical covariant derivative. The importance of this χ-curvature lies in the following:

Lemma 2.3 ([7])**.** *For a Finsler metric of scalar flag curvature on an n-dimensional manifold, $\chi_i = 0$ if and only if the flag curvature is isotropic (constant if $n \geq 3$).*

Let $F = |y|\phi(r,s)$ be a spherically symmetric metric on $B^n(\rho) \subset R^n$. By differentiating (2.1) and using (2.7), we can easily obtain a formula for χ_i:

$$\chi_i = -\frac{1}{2}\big\{(n+1)R_3 + (r^2 - s^2)[R_2]_s\big\}(|y|x^i - sy^i). \tag{2.8}$$

We have the following

Lemma 2.4. *For a spherically symmetric metric on R^n, $\chi_i = 0$ if and only if*

$$(n+1)R_3 + (r^2 - s^2)[R_2]_s = 0. \tag{2.9}$$

There is another important non-Riemannian quantity, the H-curvature $H = H_{ij}dx^i \otimes dx^j$, defined by $H_{ij} := E_{ij|m}y^m$, where $E_{ij} := \frac{1}{2}S_{\cdot i \cdot j}$ denotes the mean Berwald curvature. Here S is the S-curvature. H can be also expressed in terms of χ_i by

$$H_{ij} = \frac{1}{2}\big\{\chi_{i\cdot j} + \chi_{j\cdot i}\big\}. \tag{2.10}$$

(See [7]). The Ricci curvature tensor Ric_{ij} in (1.6) is related to the Ricci curvature $\mathrm{Ric} = R^m{}_m$ by the following identity:

$$\mathrm{Ric}_{ij} = \frac{1}{2}[\mathrm{Ric}]_{y^iy^j} + H_{ij}. \tag{2.11}$$

For spherically symmetric metrics on R^n, by differentiating χ_i and using (2.8) we obtain

$$H_{ij} = M_s|y|^{-2}(|y|x^i - sy^i)(|y|x^j - sy^j) - sM|y|^{-2}(|y|^2\delta_{ij} - y^iy^j), \tag{2.12}$$

where $M := -\frac{1}{2}\{(n+1)R_3 + (r^2 - s^2)[R_2]_s\}$.

We have the following:

Lemma 2.5. *For spherically symmetric metrics on R^n, $\chi_i = 0$ if and only if $H_{ij} = 0$.*

Proof. Assume that $H_{ij} = 0$. Contracting (2.12) with x^i and x^j yields

$$H_{ij}x^i x^j = \{(r^2 - s^2)M_s - sM\}(r^2 - s^2) = 0.$$

Thus $(r^2 - s^2)M_s = sM$. Plugging it into (2.12) gives

$$H_{ij} = M_s|y|^{-2}\{(|y|x^i - sy^i)(|y|x^j - sy^j) - (r^2 - s^2)(|y|^2\delta_{ij} - y^i y^j)\} = 0.$$

Clearly we have $M_s = 0$, hence $M = 0$. Then $\chi_i = M(|y|x^i - sy^i) = 0$. This proves the lemma. □

3. Proof of main theorems

With the above preparation, the proofs of the main results become quite simple.

Proof of Theorem 1.1. We only prove the sufficiency. Assume that (1.4) holds. Since $R_2 = 0$, we see that F is of scalar flag curvature by Lemma 2.1. We take the trace of the formula (2.1). The trace is the Ricci curvature given by

$$\mathrm{Ric} = (n-1)|y|^2 R_1 + (r^2 - s^2)|y|^2 R_2. \tag{3.1}$$

Thus

$$\mathrm{Ric} = (n-1)R_1|y|^2 + (r^2 - s^2)R_2|y|^2 = (n-1)K\phi^2|y|^2 = (n-1)KF^2.$$

Namely, F is of constant Ricci curvature K. Then F must be of constant flag curvature K since F is of scalar flag curvature ($R_2 = 0$). This completes the proof. □

Proof of Theorem 1.4. We know that for any Finsler metric, $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$ if and only if $\mathrm{Ric} = (n-1)KF^2$ and $H_{ij} = 0$. By Lemma 2.5, for any spherically symmetric metric, $H_{ij} = 0$ if and only if $\chi_i = 0$. Thus for a spherically symmetric metric $F = |y|\phi(r, s)$ on R^n, $\mathrm{Ric}_{ij} = (n-1)Kg_{ij}$ if and only if $\mathrm{Ric} = (n-1)KF^2$ and $\chi_i = 0$. By (2.8) and (3.1), we prove the theorem. □

PROOF OF THEOREM 1.2. Assume that F is of constant flag curvature. Then it follows from Theorems 1.1 and 1.4 that $R_2 = 0$ and $R_3 = 0$. Conversely, assume that $R_2 = 0$ and $R_3 = 0$. First by (2.8), we see that $\chi_i = 0$. By Lemma 2.1, we see that F is of scalar flag curvature. Then the theorem follows from Lemma 2.3. We can also prove this using (2.2) and (2.3). Under the assumption that $R_2 = 0$ and $R_3 = 0$, we get from (2.2) and (2.3) that

$$-\frac{1}{2}[R_1]_s = R_4 = -\frac{\phi_s}{\phi} R_1.$$

Thus

$$\left[\frac{R_1}{\phi^2}\right]_s = 0.$$

This gives

$$R_1 = K\phi^2,$$

where $K = K(r)$ is independent of s. Then F is of isotropic flag curvature by Theorem 1.1. K must be a constant by the Schur Lemma. □

4. Special solutions

We now look at the special case when $Q = 0$, i.e.,

$$\phi_r - s\phi_{rs} - r\phi_{ss} = 0. \tag{4.1}$$

In this case, $F = |y|\phi(r, s)$ must be projectively flat and

$$R_1 = \psi^2 - \frac{1}{r}(s\psi_r + r\psi_s) \tag{4.2}$$

$$R_2 = 0, \tag{4.3}$$

$$R_3 = \frac{1}{r}\{\psi_r - s\psi_{rs} - r\psi_{ss}\}, \tag{4.4}$$

$$R_4 = \frac{1}{r}(2\psi_r - r\psi\psi_s - s\psi_{rs} - r\psi_{ss}), \tag{4.5}$$

where

$$\psi := \frac{1}{2r\phi}(s\phi_r + r\phi_s).$$

By Theorems 1.1 and 1.2, F is of constant flag curvature K if ϕ satisfies one of the following equations:

$$K\phi^2 = \psi^2 - \frac{1}{r}(r\psi_s + s\psi_r), \tag{4.6}$$

$$\psi_r - s\psi_{rs} - r\psi_{ss} = 0. \tag{4.7}$$

Note that (4.1) and (4.7) are similar.

(4.1) and (4.6) are solvable (see Shen–Yu [8]), hence we obtain special solutions of (1.4). The key idea is to use the following special substitution

$$u := r^2 - s^2, \quad v := s.$$

Then

$$\phi_r = 2r\phi_u, \quad \phi_s = -2s\phi_u + \phi_v.$$

This gives

$$\psi = \frac{\phi_v}{2\phi} = \Big(\ln\sqrt{\phi}\Big)_v.$$

Similarly, we have

$$\frac{s\psi_r + r\psi_s}{r} = \psi_v.$$

Thus (4.6) can be written as

$$K\phi^2 = \psi^2 - \psi_v = \Big(\frac{\phi_v}{2\phi}\Big)^2 - \Big(\frac{\phi_v}{2\phi}\Big)_v. \tag{4.8}$$

This is just an ODE in v. After solving this ODE, then plugging it into (4.1), one obtains all solutions to (4.1) and (4.6). The corresponding spherical symmetric metrics must be of constant curvature.

Proposition 4.1 ([8]). *The non-constant solutions of equations* (4.1) *and* (4.6) *are given by*

$$\phi(r,s) = \frac{1}{2\sqrt{-K}}\frac{1}{\sqrt{C - r^2 + s^2} + s} \tag{4.9}$$

or

$$\phi(r,s) = \frac{q}{q^2(Dq+v)^2 + K} \quad \text{where} \tag{4.10}$$

$$q \neq 0 \quad \text{is determined by the following equation}$$

$$0 = D^2q^4 + (u - C)q^2 - K. \tag{4.11}$$

where $u = r^2 - s^2$, $v = s$, C and D given above are both constant numbers.

Three interesting solutions are given as follows

(a) $D \neq 0$ and $K = 0$, ϕ is given by

$$\phi(r,s) = \frac{D}{\sqrt{C - r^2 + s^2}(\sqrt{C - r^2 + s^2} - s)^2} \tag{4.12}$$

In this case, the corresponding spherically symmetric Finsler metrics are Berwald metrics.

(b) $D \neq 0$ and $K = -1$, ϕ is given by

$$\phi(r,s) = \frac{1}{2}\left\{\frac{1}{\sqrt{C+2D-r^2+s^2}-s} - \frac{1}{\sqrt{C-2D-r^2+s^2}-s}\right\}$$

In this case, the corresponding spherically symmetric Finsler metrics are first given by Z. Shen in [6].

(c) $D \neq 0$, $K = 1$, and q is real, ϕ is given by

$$\phi(r,s) = Re\left(\frac{1}{\sqrt{C+2iD-r^2+s^2}-is}\right)$$

In this case, the corresponding spherically symmetric Finsler metrics are Bryant's metrics.

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