

DELAY-DEPENDENT ROBUST STABILITY CRITERIA FOR UNCERTAIN NEUTRAL SYSTEMS WITH MIXED TIME-VARYING DISCRETE AND NEUTRAL DELAYS

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ABSTRACT

In this paper, a new class of augmented quasi full size Lyapunov- Krasovskii functional is introduced for the robust stability of uncertain neutral systems with mixed time-varying discrete and neutral delays. The nonlinear parameter perturbations and norm-bounded uncertainties are taken into consideration separately. Delay-dependent robust stability criteria are derived in the form of linear matrix inequalities. Numerical examples are presented to illustrate the significant improvement on the conservativeness of the delay bound over some reported results in the literature.

KeyWords: Neutral systems, nonlinear parameter perturbations, norm-bounded uncertainty, delay-dependent stability, augmented quasi full size Lyapunov functional.

I. INTRODUCTION

The stability issue of neutral delay-differential systems which involve time-delay simultaneously in both state and state derivative has received a considerable amount of attention from researchers, for example [1-6] and the references therein. A delay-dependent linear matrix inequality (LMI) is proposed in Han [7] for neutral systems with norm-bounded uncertainty using the model transformation technique. The delay-dependent stability methods under various model transformations are compared in Fridman and Shaked [8], such that the advantages of the descriptor type model transformation are underlined. The parametrized model transformation method is combined in Wu *et al.* [9] with another approach that takes the relationships between terms in the Leibnitz-Newton formula into account for delay-dependent robust stability of neutral systems. The problem of the robust stability of neutral systems with nonlinear parameter perturbations is addressed in Han and Yu [10] for the case of a time-varying neutral delay. An augmented Lyapunov functional involving the delayed state is proposed in He *et al.* [11] in order to derive improved delay-dependent stability criteria for neutral systems using the free weighting matrix method. The problem of robust stability analysis for neutral

systems with norm- bounded time-varying uncertainties is investigated in Xu *et al.* [12]. On the basis of a model transformation method, the original system with a discrete delay is transformed in Cao and Lam [13] into a system with a distributed delay and using Lyapunov-Krasovskii functional approach, delay-dependent stability criteria are obtained. The descriptor model transformation technique [8] and decomposition of the delayed state matrix approach are employed in Han [14] to investigate the robust stability of systems with a single time-varying discrete delay. The delay- dependent robust stability for time-delay systems is investigated in Kim [15], where bounded inequalities are used. The exponential stability of uncertain systems with time- varying delays is studied in Niculescu *et al.* [16]. The problem of delay-dependent robust stability of neutral systems with mixed discrete and neutral delays is investigated in He *et al.* [17]. Robust stability of uncertain systems is studied in Parlakci [18] by employing a new class of Lyapunov- Krasovskii functional combined with the descriptor model transformation. The delay-dependent robust stability and robust state-feedback stabilization of uncertain time-varying state-delayed systems are considered in Parlakci [19].

In this paper, the stability problem of uncertain neutral systems with mixed time-varying discrete and neutral delays is investigated. A novel form of augmented quasi full size Lyapunov-Krasovskii functional is introduced to take into account a new state integral term $\int_{t-\tau}^t [s-t+d(t)] \cdot x(s)ds$ having a time-derivative coupled with the current

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state of the system and the state integral term, which is also considered in the augmented Lyapunov functional proposed

in He *et al.* [11], i.e. $x(t)$ and $\int_{t-d(t)}^t x(s)ds$. Moreover, the

well-known descriptor type model transformation [8] is utilized and an extended free weighting matrix approach is employed in order to achieve some kind of relaxation. Both the cases of nonlinear parameter perturbations and norm-bounded uncertainties are taken into consideration. Delay-dependent robust stability criteria are formulated in the form of linear matrix inequalities. Three numerical examples, which were previously considered in the literature, are presented to demonstrate the effective application of the proposed stability method. The results show that the stability method developed in this paper yields less conservative delay bounds than the existing ones.

II. PROBLEM STATEMENT

Let us consider a class of uncertain neutral system with mixed time-varying discrete and neutral delays and nonlinear parameter perturbations:

$$\begin{aligned} \dot{x}(t) = & Ax(t) + A_d x(t-d(t)) + C\dot{x}(t-\tau(t)) \\ & + f(x(t), t) + g(x(t-d(t)), t) + h(\dot{x}(t-\tau(t)), t) \end{aligned} \quad (1)$$

$$x(\theta) = \Phi(\theta), \dot{x}(\theta) = \dot{\Phi}(\theta),$$

$$\forall \theta \in [-\max\{\bar{d}, \bar{\tau}\}, 0], \bar{d}, \bar{\tau} > 0 \quad (2)$$

where $x(t) \in \mathfrak{R}^n$ is the state, $A, A_d, C \in \mathfrak{R}^{n \times n}$ are constant system matrices, $d(t)$ and $\tau(t)$ denote time-varying discrete and neutral delays, respectively, $f(x(t), t), g(x(t-d(t)), t), h(\dot{x}(t-\tau(t)), t)$ are time-varying vector-valued functions which are unknown and represent the nonlinear parameter perturbations, and $\Phi(t)$ is a continuous vector-valued initial function of $t \in [-\max\{\bar{d}, \bar{\tau}\}, 0]$ which represents an initial condition. One shall make the following assumptions for the time-delays $d(t)$ and $\tau(t)$ (Niculescu *et al.*) [16].

Assumption 1. The time-varying delays $d(t)$ and $\tau(t)$ are positive continuously-differentiable functions satisfying:

$$0 \leq d(t) \leq \bar{d}, \quad |\dot{d}(t)| \leq \mu \quad (3)$$

$$0 \leq \tau(t) \leq \bar{\tau}, \quad |\dot{\tau}(t)| \leq \delta \quad (4)$$

where $\bar{d} > 0, \mu > 0, \bar{\tau} > 0, \delta > 0$ are given real numbers.

The admissible uncertainties f, g , and h are assumed to satisfy the following boundedness conditions.

Assumption 2. The nonlinear parameter perturbations can be structured or unstructured and satisfy:

$$f(0, t) = 0, \quad g(0, t) = 0, \quad h(0, t) = 0 \quad (5)$$

Moreover, there exist nonnegative numbers α, β, γ such that for all $x(t) \in \mathfrak{R}^n$ and for all t :

$$\begin{aligned} \|f(x(t), t)\| &\leq \alpha \|x(t)\|, \\ \|g(x(t-d(t)), t)\| &\leq \beta \|x(t-d(t))\|, \\ \|h(\dot{x}(t-\tau(t)), t)\| &\leq \gamma \|\dot{x}(t-\tau(t))\| \end{aligned} \quad (6)$$

One can define a difference operator $\nabla: \mathfrak{S}([- \max\{\bar{d}, \bar{\tau}\}, 0], \mathfrak{R}^n) \rightarrow \mathfrak{R}^n$ such that:

$$\nabla(x_t) = x(t) - Cx(t-\tau(t)) \quad (7)$$

Definition 1. [1,11] The difference operator ∇ is said to be stable if the zero solution of the homogeneous difference equation $\nabla x_t = 0, t \geq 0, x_0 = \Psi \in \{\Phi \in C([- \bar{\tau}, 0]): \nabla \Phi = 0\}$ is uniformly asymptotically stable.

The stability of the difference operator ∇ is necessary for the stability of neutral system (1).

Assumption 3. It follows from Theorem 1.6 in [2] that for the robust asymptotic stability of neutral system (1), the following sufficient condition needs to be satisfied:

$$\|C\| + \gamma < 1 \quad (8)$$

Remark 1. It can be seen that Assumptions 1, 2, and 3 are sufficient conditions for the existence and uniqueness of a solution to the functional differential Eq. (1).

III. MAIN RESULTS

In this section, a delay-dependent robust stability result is presented for the robust stability of system (1) with nonlinear parameter perturbations satisfying (5), (6). Employing descriptor type model transformation [8], one can represent system (1) in descriptor form as follows:

$$\begin{aligned} \dot{x}(t) = & y(t), \quad y(t) = Ax(t) + A_d x(t-d(t)) + Cy(t-\tau(t)) \\ & + f(x(t), t) + g(x(t-d(t)), t) + h(y(t-\tau(t)), t) \end{aligned} \quad (9)$$

where $y(t)$ is the “fast varying” descriptor variable.

Theorem 1. Given nonnegative scalars \bar{d}, μ, δ , the neutral system (1) with nonlinear parameter perturbations satisfying (5), (6) is robustly asymptotically stable for any time-delay satisfying (3), (4) if (8) holds and there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R, S, Z$; and matrices $P_{12}, P_{13}, P_{23}, Q_{12}, M_i$ ($i = 1, \dots, 9$) and nonnegative scalars ε_j ($j = 1, 2, 3$) satisfying:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix} \geq 0, \quad \text{with } P_{11} > 0 \quad (10)$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \geq 0 \tag{11}$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} & \bar{d}P_{13} & \mu P_{12} & \mu P_{13} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} & \Omega_{27} & \Omega_{28} & \Omega_{29} & 0 & 0 & 0 \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} & \Omega_{38} & \Omega_{39} & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & -P_{33} & M_4^T & M_4^T & M_4^T C & M_4^T & \bar{d}P_{23} & \mu P_{22} & \mu P_{23} \\ * & * & * & * & -3Z & M_5^T & M_5^T & M_5^T C & M_5^T & \bar{d}P_{33} & \mu P_{23}^T & \mu P_{33} \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & \Omega_{68} & \Omega_{69} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77} & \Omega_{78} & \Omega_{79} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88} & \Omega_{89} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Omega_{99} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\bar{d}T & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\mu U & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\mu Y \end{bmatrix} < 0 \tag{12}$$

where

$$\Omega_{11} = P_{12}^T + P_{12} + \bar{d}T + \bar{d}^2 Q_{11} - Q_{22} + R + \bar{d}^4 Z + \varepsilon_1 \alpha^2 I + A^T M_1 + M_1^T A,$$

$$\Omega_{12} = P_{11} + \bar{d}^2 Q_{12} - M_1^T + A^T M_2,$$

$$\Omega_{13} = -P_{12} + Q_{22} + M_1^T A_d + A^T M_3,$$

$$\Omega_{14} = -P_{13} + P_{22} - Q_{12}^T + A^T M_4, \quad \Omega_{15} = P_{23} + A^T M_5,$$

$$\Omega_{16} = M_1^T + A^T M_6, \quad \Omega_{17} = M_1^T + A^T M_7,$$

$$\Omega_{18} = M_1^T C + A^T M_8, \quad \Omega_{19} = M_1^T + A^T M_9,$$

$$\Omega_{22} = \bar{d}^2 Q_{22} + S - M_2^T - M_2, \quad \Omega_{23} = M_2^T A_d - M_3,$$

$$\Omega_{24} = P_{12} - M_4, \quad \Omega_{25} = P_{13} - M_5, \quad \Omega_{26} = M_2^T - M_6,$$

$$\Omega_{27} = M_2^T - M_7, \quad \Omega_{28} = M_2^T C - M_8, \quad \Omega_{29} = M_2^T - M_9,$$

$$\Omega_{33} = \mu U - Q_{22} - (1 - \mu)R + \varepsilon_2 \beta^2 I + A_d^T M_3 + M_3^T A_d,$$

$$\Omega_{34} = -P_{22} + Q_{12}^T + A_d^T M_4, \quad \Omega_{35} = -P_{23} + A_d^T M_5,$$

$$\Omega_{36} = M_3^T + A_d^T M_6, \quad \Omega_{37} = M_3^T + A_d^T M_7,$$

$$\Omega_{38} = M_3^T C + A_d^T M_8, \quad \Omega_{39} = M_3^T + A_d^T M_9,$$

$$\Omega_{44} = -P_{23}^T - P_{23} + \mu Y - Q_{11}, \quad \Omega_{66} = -\varepsilon_1 I + M_6^T + M_6,$$

$$\Omega_{67} = M_6^T + M_7, \quad \Omega_{68} = M_6^T C + M_8,$$

$$\Omega_{69} = M_7^T + M_9, \quad \Omega_{77} = -\varepsilon_2 I + M_7^T + M_7,$$

$$\Omega_{78} = M_7^T C + M_8, \quad \Omega_{79} = M_7^T + M_9,$$

$$\Omega_{88} = -(1 - \delta)S + \varepsilon_3 \gamma^2 I + C^T M_8 + M_8^T C,$$

$\Omega_{89} = M_8^T + C^T M_9, \quad \Omega_{99} = -\varepsilon_3 I + M_9^T + M_9$, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. Let us choose a new class of augmented quasi full size Lyapunov-Krasovskii functional as follows:

$$V(x(t), t) = \sum_{i=1}^5 V_i \tag{13}$$

where

$$V_1 = \eta^T(t) P \eta(t), \quad V_2 = \bar{d} \int_{t-\bar{d}}^t (s-t+\bar{d}) \xi^T(s) Q \xi(s) ds,$$

$$V_3 = \int_{t-d(t)}^t x^T(s) R x(s) ds, \quad V_4 = \int_{t-\tau(t)}^t y^T(s) S y(s) ds,$$

$$V_5 = \bar{d} \int_{t-\bar{d}}^t (s-t+\bar{d})^3 x^T(s) Z x(s) ds,$$

$$\eta(t) = \begin{bmatrix} x^T(t) & \left(\int_{t-d(t)}^t x(s) ds \right)^T & \left(\int_{t-d(t)}^t [s-t+d(t)] x(s) ds \right)^T \end{bmatrix}^T,$$

$\xi(s) = [x^T(s) \quad y^T(s)]^T$. The time derivative of $V(x(t), t)$ along the state trajectory of system (1), (9) is given by

$\dot{V}(x(t), t) = \sum_{i=1}^5 \dot{V}_i$. Then, one can obtain:

$$\begin{aligned} \dot{V}_1 &= 2\eta^T(t) P \frac{d}{dt} \eta(t) \\ &= 2\eta^T(t) P \begin{bmatrix} y(t) \\ x(t) - x(t-d(t)) + \dot{d}(t)x(t-d(t)) \\ - \int_{t-d(t)}^t x(s) ds + d(t)x(t) + \dot{d}(t) \int_{t-d(t)}^t x(s) ds \end{bmatrix} \\ &= 2\eta^T(t) P [\eta_1(t) + \eta_2(t) + \eta_3(t) + \eta_4(t)] \end{aligned} \tag{14}$$

where

$$\eta_1(t) = \begin{bmatrix} y^T(t) & x^T(t) - x^T(t-d(t)) & \left(- \int_{t-d(t)}^t x(s) ds \right)^T \end{bmatrix}^T,$$

$$\eta_2(t) = d(t)[0 \ 0 \ I]^T x(t),$$

$$\eta_3(t) = \dot{d}(t)[0 \ I \ 0]^T x(t-d(t)),$$

$$\eta_4(t) = \dot{d}(t)[0 \ 0 \ I]^T \int_{t-d(t)}^t x(s) ds.$$

Let us define the following expressions:

$$\eta(t) = \Gamma \chi(t), \quad \eta_1(t) = \Gamma_1 \chi(t) \tag{15}$$

where

$$\chi(t) = \begin{bmatrix} x^T(t) & y^T(t) & x^T(t-d(t)) \\ \left(\int_{t-d(t)}^t x(s) ds \right)^T & \left(\int_{t-d(t)}^t [s-t+d(t)] x(s) ds \right)^T \\ f^T(x(t), t) & g^T(x(t-d(t)), t) \\ y^T(t-\tau(t)) & h^T(y(t-\tau(t)), t) \end{bmatrix}^T,$$

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Rewriting $2\eta^T(t)P\eta_i(t)$, $i = 2, 3, 4$ gives

$$2\eta^T(t)P\eta_2(t) = 2d(t)\chi^T(t)\Gamma^T P[0 \ 0 \ I]^T x(t) = 2d(t)\chi^T(t)\Gamma_2^T \Gamma_3 \chi(t) \tag{16a}$$

$$2\eta^T(t)P\eta_3(t) = 2\dot{d}(t)\chi^T(t)\Gamma^T P[0 \ I \ 0]^T x(t-d(t)) = 2\dot{d}(t)\chi^T(t)\Gamma_4^T \Gamma_5 \chi(t) \tag{16b}$$

$$2\eta^T(t)P\eta_4(t) = 2\dot{d}(t)\chi^T(t)\Gamma^T P[0 \ 0 \ I]^T \int_{t-d(t)}^t x(s) ds = 2\dot{d}(t)\chi^T(t)\Gamma_2^T \Gamma_6 \chi(t) \tag{16c}$$

where

$$\Gamma_2 = [P_{13}^T \ 0 \ 0 \ P_{23}^T \ P_{33} \ 0 \ 0 \ 0 \ 0],$$

$$\Gamma_3 = [I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0],$$

$$\Gamma_4 = [P_{12}^T \ 0 \ 0 \ P_{22} \ P_{23} \ 0 \ 0 \ 0 \ 0],$$

$$\Gamma_5 = [0 \ 0 \ I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0],$$

$\Gamma_6 = [0 \ 0 \ 0 \ I \ 0 \ 0 \ 0 \ 0 \ 0]$. For some symmetric and positive definite matrices T, U , and Y , one can utilize the bounding inequalities in (16) to obtain:

$$2\eta^T(t)P\eta_2(t) \leq \bar{d}\chi^T(t)\Gamma_2^T T^{-1} \Gamma_2 \chi(t) + \bar{d}\chi^T(t)\Gamma_3^T T \Gamma_3 \chi(t) \tag{17a}$$

$$2\eta^T(t)P\eta_3(t) \leq \mu\chi^T(t)\Gamma_4^T U^{-1} \Gamma_4 \chi(t) + \mu\chi^T(t)\Gamma_5^T U \Gamma_5 \chi(t) \tag{17b}$$

$$2\eta^T(t)P\eta_4(t) \leq \mu\chi^T(t)\Gamma_2^T Y^{-1} \Gamma_2 \chi(t) + \mu\chi^T(t)\Gamma_6^T Y \Gamma_6 \chi(t) \tag{17c}$$

Substituting (15), (17) into (14) gives:

$$\dot{V}_1 \leq \chi^T(t)(\Omega_1 + \bar{d}\Gamma_2^T T^{-1} \Gamma_2 + \mu\Gamma_4^T U^{-1} \Gamma_4 + \mu\Gamma_2^T Y^{-1} \Gamma_2) \chi(t) \tag{18}$$

where

$$\Omega_1 = \Gamma^T P \Gamma + \Gamma_1^T P \Gamma + \bar{d}\Gamma_3^T T \Gamma_3 + \mu(\Gamma_5^T U \Gamma_5 + \Gamma_6^T Y \Gamma_6).$$

The time derivative of V_2 can be computed as

$$\dot{V}_2 = \bar{d}^2 \xi^T(t) Q \xi(t) - \bar{d} \int_{t-\bar{d}}^t \xi^T(s) Q \xi(s) ds \leq \bar{d}^2 \xi^T(t) Q \xi(t) - d(t) \int_{t-d(t)}^t \xi^T(s) Q \xi(s) ds \tag{19}$$

Applying Jensen's integral inequality (Gu *et al.*) [3] to (19) yields:

$$\dot{V}_2 \leq \bar{d}^2 \xi^T(t) Q \xi(t) - \left(\int_{t-d(t)}^t \xi(s) ds \right)^T Q \int_{t-d(t)}^t \xi(s) ds \tag{20}$$

Let us rewrite $\xi(t), \int_{t-d(t)}^t \xi(s) ds$ in the following form:

$$\xi(t) = \Gamma_7 \chi(t), \quad \int_{t-d(t)}^t \xi(s) ds = \begin{bmatrix} \left(\int_{t-d(t)}^t x(s) ds \right)^T \\ x^T(t) - x^T(t-d(t)) \end{bmatrix}^T = \Gamma_8 \chi(t) \tag{21}$$

where

$$\Gamma_7 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_8 = \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Substituting (21) into (20) gives:

$$\dot{V}_2 \leq \chi^T(t) \Omega_2 \chi(t) \tag{22}$$

where $\Omega_2 = \bar{d}^2 \Gamma_7^T Q \Gamma_7 - \Gamma_8^T Q \Gamma_8$. The time-derivative of V_3 is calculated as:

$$\dot{V}_3 = x^T(t) R x(t) - [1 - \dot{d}(t)] x^T(t-d(t)) R x(t-d(t)) \leq x^T(t) R x(t) - (1 - \mu) x^T(t-d(t)) R x(t-d(t)) = \chi^T(t) \Omega_3 \chi(t) \tag{23}$$

where $\Omega_3(1, 1) = R$, $\Omega_3(3, 3) = -(1-\mu)R$, $\Omega_3(i, j) = 0$ for $i, j = 1, \dots, 9$. One can compute the time-derivative of V_4 in a similar manner as follows:

$$\begin{aligned} \dot{V}_4 &= y^T(t)Sy(t) - [1 - \dot{\tau}(t)]y^T(t - \tau(t))Sy(t - \tau(t)) \\ &\leq y^T(t)Sy(t) - (1 - \delta)y^T(t - \tau(t))Sy(t - \tau(t)) \\ &= \chi^T(t)\Omega_4\chi(t) \end{aligned} \tag{24}$$

where $\Omega_4(2, 2) = S$, $\Omega_4(8, 8) = -(1-\delta)S$, $\Omega_4(i, j) = 0$, $i, j = 1, \dots, 9$. The time-derivative of V_5 is calculated as:

$$\begin{aligned} \dot{V}_5 &= \frac{d}{dt} \int_{t-\bar{d}}^t [s^3 - 3s^2(t-\bar{d}) + 3s(t-\bar{d})^2 - (t-\bar{d})^3] x^T(s)Zx(s) ds \end{aligned} \tag{25}$$

Therefore, elaborating the expression in (25) gives:

$$\begin{aligned} \dot{V}_5 &= \bar{d}^4 x^T(t)Zx(t) - 3\bar{d} \int_{t-\bar{d}}^t (s-t+\bar{d})^2 x^T(s)Zx(s)ds \leq \bar{d}^4 x^T(t)Zx(t) \\ &\quad - 3d(t) \int_{t-d(t)}^t [s-t+d(t)]^2 x^T(s)Zx(s) ds \end{aligned} \tag{26}$$

Employing Jensen’s integral inequality (Gu *et al.*) [3] in (26), one can obtain:

$$\begin{aligned} \dot{V}_5 &\leq \bar{d}^4 x^T(t)Zx(t) - 3 \left(\int_{t-d(t)}^t [s-t+d(t)]x(s)ds \right)^T Z \int_{t-d(t)}^t [s-t+d(t)]x(s) ds \\ &= \chi^T(t)\Omega_5\chi(t) \end{aligned} \tag{27}$$

where $\Omega_5(1, 1) = \bar{d}^4 Z$, $\Omega_5(5, 5) = -3Z$, $\Omega_5(i, j) = 0$ for $i, j = 1, \dots, 9$. In order to accommodate the nonlinear parameter perturbations, one can consider the inequalities in (6) and build the following combined inequality:

$$\begin{aligned} &\varepsilon_1 \left[\alpha^2 x^T(t)x(t) - f^T(x(t), t)f(x(t), t) \right] \\ &+ \varepsilon_2 \left[\beta^2 x^T(t-d(t))x(t-d(t)) - g^T(x(t-d(t)), t)g(x(t-d(t)), t) \right] \\ &+ \varepsilon_3 \left[\gamma^2 y^T(t-\tau(t))y(t-\tau(t)) - h^T(y(t-\tau(t)), t)h(y(t-\tau(t)), t) \right] \\ &= \chi^T(t)\Omega_6\chi(t) \geq 0 \end{aligned} \tag{28}$$

where $\Omega_6(1, 1) = \varepsilon_1\alpha^2 I$, $\Omega_6(3, 3) = \varepsilon_2\beta^2 I$, $\Omega_6(6, 6) = -\varepsilon_1 I$, $\Omega_6(7, 7) = -\varepsilon_2 I$, $\Omega_6(8, 8) = \varepsilon_3\gamma^2 I$, $\Omega_6(9, 9) = -\varepsilon_3 I$, $\Omega_6(i, j) = 0$ for $i, j = 1, \dots, 9$. Let us reconsider the descriptor form representation (9) as follows:

$$0 = -y(t) + Ax(t) + A_d x(t-d(t)) + Cy(t-\tau(t)) + f(x(t), t) + g(x(t-d(t)), t) + h(y(t-\tau(t)), t) = \Gamma_9 \chi(t) \tag{29}$$

where $\Gamma_9 = [A \ -I \ A_d \ 0 \ 0 \ I \ I \ C \ I]$. One can get a null expression for a relaxation as:

$$2\chi^T(t)M^T\Gamma_9\chi(t) = 0 \tag{30}$$

where $M = [M_1 \ M_2 \ M_3 \ M_4 \ M_5 \ M_6 \ M_7 \ M_8 \ M_9]$. The null Eq. (30) can be rewritten as:

$$\chi^T(t)\Omega_7\chi(t) = 0 \tag{31}$$

where $\Omega_7 = M^T\Gamma_9 + \Gamma_9^T M$. Finally, substituting (18), (22)-(24), (27) into $\dot{V}(x(t), t)$ and adding (28), (31), to the resulting expression yields:

$$\dot{V}(x(t), t) \leq \chi^T(t)\Omega_0\chi(t) \tag{32}$$

where $\Omega_0 = \sum_{i=1}^7 \Omega_i + \bar{d}\Gamma_2^T T^{-1}\Gamma_2 + \mu\Gamma_4^T U^{-1}\Gamma_4 + \mu\Gamma_2^T Y^{-1}\Gamma_2$. In order to guarantee the robust stability of system (1), one is required to satisfy $\Omega_0 < 0$. Applying Schur’s complement to $\Omega_0 < 0$, one gets (12). Therefore, if the LMI’s (10)-(12) are satisfied, then $\dot{V}(x(t), t) < 0$ is assured and system (1) is guaranteed to be robustly globally asymptotically stable. This completes the proof.

If both the neutral and discrete delays are time-invariant, *i.e.* $d(t) \equiv d$, $\tau(t) \equiv \tau$ with $\dot{d}(t) = 0$, $\dot{\tau}(t) = 0$, then one has the following result.

Corollary 1. Given the nonnegative scalar d , the neutral system (1) with time-invariant neutral and discrete delays and nonlinear parameter perturbations satisfying (5), (6) is robustly asymptotically stable if (8) holds and there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R, S, Z$; matrices $P_{12}, P_{13}, P_{23}, Q_{12}, M_i, (i = 1, \dots, 9)$ and nonnegative scalars $\varepsilon_j (j = 1, 2, 3)$ satisfying (10), (11); and

$$\begin{aligned} \Phi &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Omega_{13} & \Phi_{14} & \Phi_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\ * & \Phi_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} & \Omega_{27} & \Omega_{28} & \Omega_{29} \\ * & * & \Omega_{33}|_{\mu=0} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} & \Omega_{38} & \Omega_{39} \\ * & * & * & \Omega_{44}|_{\mu=0} & -P_{33} & M_4^T & M_4^T & M_4^T C & M_4^T \\ * & * & * & * & -3Z & M_5^T & M_5^T & M_5^T C & M_5^T \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & \Omega_{68} & \Omega_{69} \\ * & * & * & * & * & * & \Omega_{77} & \Omega_{78} & \Omega_{79} \\ * & * & * & * & * & * & * & \Omega_{88}|_{\delta=0} & \Omega_{89} \\ * & * & * & * & * & * & * & * & \Omega_{99} \end{bmatrix} \\ &< 0 \end{aligned} \tag{33}$$

where $\Phi_{11} = P_{12}^T + P_{12} + d(P_{13}^T + P_{13}) + d^2 Q_{11} - Q_{22} + R + d^4 Z + \varepsilon_1 \alpha^2 I + A^T M_1 + M_1^T A$, $\Phi_{12} = P_{11} + d^2 Q_{12} - M_1^T + A^T M_2$, $\Phi_{14} = -P_{13} + P_{22} + dP_{23}^T - Q_{12}^T + A^T M_4$, $\Phi_{15} = P_{23} + dP_{33} + A^T M_5$, $\Phi_{22} = d^2 Q_{22} + S - M_2^T - M_2$, Ω_{ij} 's $i, j \in \{1, \dots, 9\}$ are defined in Theorem 1, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. Similar to the proof of Theorem 1, thus it is omitted. ■

If the neutral and discrete delays are also coincident, i.e. $d \equiv \tau$, then the following result can be given.

Corollary 2. Given the nonnegative scalar d , the neutral system (1) with time-invariant coincident neutral and discrete delays and nonlinear parameter perturbations satisfying (5), (6) is robustly asymptotically stable if (8) holds and there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R_{11}, R_{22}, S$; and matrices $P_{12}, P_{13}, P_{23}, Q_{12}, R_{12}, M_i (i = 1, \dots, 9)$ and nonnegative scalars $\varepsilon_j (j = 1, 2, 3)$ satisfying (10), (11); and

$$R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} \geq 0 \tag{34}$$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Omega_{13} & \Phi_{14} & \Phi_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\ * & \Pi_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} & \Omega_{27} & \Omega_{28} & \Omega_{29} \\ * & * & \Pi_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} & \Pi_{38} & \Omega_{39} \\ * & * & * & \Omega_{44}|_{\mu=0} & -P_{33} & M_4^T & M_4^T & M_4^T C & M_4^T \\ * & * & * & * & -3Z & M_5^T & M_5^T & M_5^T C & M_5^T \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & \Omega_{68} & \Omega_{69} \\ * & * & * & * & * & * & \Omega_{77} & \Omega_{78} & \Omega_{79} \\ * & * & * & * & * & * & * & \Pi_{88} & \Omega_{89} \\ * & * & * & * & * & * & * & * & \Omega_{99} \end{bmatrix} < 0 \tag{35}$$

where $\Pi_{11} = P_{12}^T + P_{12} + d(P_{13}^T + P_{13}) + d^2 Q_{11} - Q_{22} + R_{11} + d^4 Z + \varepsilon_1 \alpha^2 I + A^T M_1 + M_1^T A$, $\Pi_{12} = \Phi_{12} + R_{12}$, $\Pi_{22} = \Phi_{22} + R_{22}$, $\Pi_{33} = -Q_{22} - R_{11} + \varepsilon_2 \beta^2 I + A_d^T M_3 + M_3^T A_d$, $\Pi_{38} = \Omega_{38} - R_{12}$, $\Pi_{88} = -R_{22} + \varepsilon_3 \gamma^2 I + C^T M_8 + M_8^T C$, Ω_{ij} 's, Φ_{ij} 's, $i, j \in \{1, \dots, 9\}$ are defined in Theorem 1, Corollary 1, respectively, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. As the neutral and discrete delays are coincident then the Lyapunov-Krasovskii functional terms V_3, V_4 in (13) are replaced by $\int_{t-d}^t \xi^T(s) R \xi(s) ds$ where R is defined in (34). The rest of the proof follows in a similar manner as in the proof of Theorem 1. ■

Remark 2. Note that the Lyapunov-Krasovskii functional (10) is an augmented form of the one introduced in Wu *et al.* [9]. Moreover, although an augmented Lyapunov functional is also proposed in He *et al.* [11], the approach of this work and that considered in He *et al.* [11] are quite different. The delayed state, $x(t - \tau)$ is used in [11] to con-

struct the augmented Lyapunov functional. However, in this paper, a new integral term of $\int_{t-d(t)}^t [s - t + d(t)] \cdot$

$x(s) ds$ is introduced into the Lyapunov-Krasovskii functional for the first time in order to augment the conventional Lyapunov functional. Another point is that the augmented Lyapunov functional introduced in [11] is useful only for neutral systems because the delayed state, $x(t - \tau)$ is utilized as the augmenting element for the Lyapunov functional. However, the time-derivative of $x(t - \tau)$ is involved only in neutral systems. When a retarded time-delay system is considered, the time-derivative of the delayed state, $x(t - \tau)$ is just decoupled from the system equation and the augmented Lyapunov functional given in [11], thus remains not so beneficial. However, the augmented Lyapunov functional proposed in this paper is useful both for retarded and neutral time-delay systems as the time-derivative of the augmenting integral element

$\int_{t-d(t)}^t [s - t + d(t)] x(s) ds$ involves terms like $x(t), x(t - d(t))$

and $\int_{t-d(t)}^t x(s) ds$ which directly or inherently exist in the

system equation and, thus, is coupled with the system dynamics.

Remark 3. Note that the augmented Lyapunov-Krasovskii functional approach is applied for uncertain neutral systems with mixed time-varying discrete and neutral delays. However, the augmented Lyapunov functional approach developed in He *et al.* [9] and the stability methods introduced in [5], [12] are applicable only to uncertain neutral systems with coincident time-invariant discrete and neutral delays. Therefore, from this point of view, the present paper's method gives an insight into a more generalized approach in the field of stability analysis for uncertain neutral systems with mixed time-varying discrete and neutral delays.

Remark 4. It can be clearly seen that free weighting matrices are embedded into the Lyapunov functional derivative via utilization of the descriptor form model transformation which is shown in Fridman and Shaked [8] as the most efficient model transformation. The motivation for this approach arises from the fact that the descriptor form based Lyapunov functional inherently involves free weighting matrices. Therefore, in this paper, this idea is extensively used by choosing a separate free weighting matrix for each state element that appear in the extended state vector.

IV. NORM-BOUNDED UNCERTAINTY

In this section, a well-known issue (Boyd *et al.*) [20] in robust control of uncertain systems is considered, that is: when $f(x(t), t), g(x(t - d(t)), t)$ and $h(\dot{x}(t - \tau(t)), t)$ are

defined in the form of norm-bounded uncertainty. System (1) is thus rewritten as follows:

$$\dot{x}(t) = [A + DF(t)E_a]x(t) + [A_d + DF(t)E_d]x(t-d(t)) + [C + DF(t)E_c]\dot{x}(t-\tau(t)) \quad (36)$$

where $F(t) \in \mathfrak{R}^{n \times q}$ is an unknown real time-varying matrix with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \leq I \quad (37)$$

and D, E_a, E_d, E_c are known real constant matrices that determine how the uncertainty appears in the nominal matrices A, A_d, C . Utilizing the approach of Han and Yu [10] combined with descriptor form representation allows one to rewrite system (36) as:

$$\begin{aligned} \dot{x}(t) &= y(t), \quad y(t) = Ax(t) + A_d x(t-d(t)) + Cy(t-\tau(t)) + Du \\ z &= E_a x(t) + E_d x(t-d(t)) + E_c y(t-\tau(t)) \end{aligned} \quad (38)$$

with the constraint $u = F(t)z$. Then, it follows from (37) that:

$$u^T u \leq [E_a x(t) + E_d x(t-d(t)) + E_c y(t-\tau(t))]^T [E_a x(t) + E_d x(t-d(t)) + E_c y(t-\tau(t))] \quad (39)$$

In the following sequel, the delay-dependent robust stability criteria are presented.

Theorem 2. Given the nonnegative scalars \bar{d}, μ, δ , the neutral system (36), (38) with norm-bounded uncertainties satisfying (37), (39) is robustly asymptotically stable if there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R, S, Z$ and matrices $P_{12}, P_{13}, P_{23}, Q_{12}, M_i, (i = 1, \dots, 7)$; and nonnegative scalars $\rho, \sigma, \varepsilon$ satisfying (10), (11) and the following LMIs:

$$\begin{bmatrix} C^T C - I + \sigma E_c^T E_c & C^T D \\ * & -(\rho I - D^T D) \end{bmatrix} < 0 \quad (40)$$

$\Sigma =$

$$\begin{bmatrix} \Sigma_{11} & \Omega_{12} & \Sigma_{13} & \Omega_{14} & \Omega_{15} & \Sigma_{16} & \Sigma_{17} & \bar{d}P_{13} & \mu P_{12} & \mu P_{13} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Sigma_{26} & \Sigma_{27} & 0 & 0 & 0 \\ * & * & \Sigma_{33} & \Omega_{34} & \Omega_{35} & \Sigma_{36} & \Sigma_{37} & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & -P_{33} & M_4^T D & M_4^T C & \bar{d}P_{23} & \mu P_{22} & \mu P_{23} \\ * & * & * & * & -3Z & M_5^T D & M_5^T C & \bar{d}P_{33} & \mu P_{23}^T & \mu P_{33} \\ * & * & * & * & * & \Sigma_{66} & \Sigma_{67} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{d}T & 0 & 0 \\ * & * & * & * & * & * & * & * & -\mu U & 0 \\ * & * & * & * & * & * & * & * & * & -\mu Y \end{bmatrix} < 0 \quad (41)$$

where $\Sigma_{11} = P_{12}^T + P_{12} + \bar{d}T + \bar{d}^2 Q_{11} - Q_{22} + R + \bar{d}^4 Z$
 $+ \varepsilon E_a^T E_a + A^T M_1 + M_1^T A, \Sigma_{13} = -P_{12} + Q_{22} + \varepsilon E_a^T E_d$
 $+ M_1^T A_d + A^T M_3, \Sigma_{16} = M_1^T D + A^T M_6, \Sigma_{17} = \varepsilon E_a^T E_c$
 $+ M_1^T C + A^T M_7, \Sigma_{26} = M_2^T D - M_6, \Sigma_{27} = M_2^T C - M_7,$
 $\Sigma_{33} = \mu U - Q_{22} - (1-\mu)R + \varepsilon E_d^T E_d + A_d^T M_3 + M_3^T A_d,$
 $\Sigma_{36} = M_3^T D + A_d^T M_6, \Sigma_{37} = \varepsilon E_d^T E_c + M_3^T C + A_d^T M_7,$
 $\Sigma_{66} = -\varepsilon I + D^T M_6 + M_6^T D, \Sigma_{67} = M_6^T C + D^T M_7,$
 $\Sigma_{77} = -(1-\delta)S + \varepsilon E_c^T E_c + C^T M_7 + M_7^T C, \Omega_{ij}, i, j \in \{1, \dots, 9\}$ are defined in Theorem 1, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. The stability of the difference operator $\nabla(x_t)$ is guaranteed if and only if $\|C + DF(t)E_c\| < 1$ which implies that $[C + DF(t)E_c]^T [C + DF(t)E_c] < I$. Applying Lemma 2 given in [10] yields the condition (40). The Lyapunov-Krasovskii functional candidate given in the proof of Theorem 1 is reconsidered in order to derive the condition (41). The time derivative of $V(x(t), t)$ along the state trajectory of (36), (38) is computed in a similar manner as in the proof of Theorem 1. However, the extended state vector, $\chi(t)$ is modified as:

$$\zeta(t) = \begin{bmatrix} x^T(t) & y^T(t) & x^T(t-d(t)) & \left(\int_{t-d(t)}^t x(s) ds \right)^T \\ \left(\int_{t-d(t)}^t [s-t+d(t)]x(s) ds \right)^T & u^T & y^T(t-\tau(t)) \end{bmatrix}$$

Applying steps (14)-(17) allows one to rewrite \dot{V}_1 as follows:

$$\dot{V}_1 \leq \zeta^T(t) (\Sigma_1 + \bar{d}\Gamma_2^T T^{-1} \Gamma_2 + \mu\Gamma_4^T U^{-1} \Gamma_4 + \mu\Gamma_2^T Y^{-1} \Gamma_2) \zeta(t) \quad (42)$$

where

$$\Sigma_1 = \Gamma^T P \Gamma_1 + \Gamma_1^T P \Gamma + \bar{d}\Gamma_3^T T \Gamma_3 + \mu\Gamma_5^T U \Gamma_5 + \mu\Gamma_6^T Y \Gamma_6$$

with

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_2 = [P_{13}^T \ 0 \ 0 \ P_{23}^T \ P_{33} \ 0 \ 0], \Gamma_3 = [I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$\Gamma_4 = [P_{12}^T \ 0 \ 0 \ P_{22} \ P_{23} \ 0 \ 0], \Gamma_5 = [0 \ 0 \ I \ 0 \ 0 \ 0 \ 0],$$

$$\Gamma_6 = [0 \ 0 \ 0 \ I \ 0 \ 0 \ 0].$$
 Following the similar steps in

(19)-(21) allows to obtain the time-derivative of V_2 as

$$\dot{V}_2 \leq \zeta^T(t) \Sigma_2 \zeta(t) \tag{43}$$

where $\Sigma_2 = \bar{d}^2 \Gamma_7^T Q \Gamma_7 - \Gamma_8^T Q \Gamma_8$ with

$$\Gamma_7 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \Gamma_8 = \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix},$$

In a similar manner as in (23)-(27), the time-derivative of $V_i, i = 3, 4, 5$ can be obtained as follows:

$$\dot{V}_i \leq \zeta^T(t) \Sigma_i \zeta(t) \tag{44}$$

where $\Sigma_3(1, 1) = R, \Sigma_3(3, 3) = -(1-\mu)R, \Sigma_4(2, 2) = R, \Sigma_4(7, 7) = -(1-\delta)S, \Sigma_4(i, j) = 0, \Sigma_5(1, 1) = \bar{d}^4 Z, \Sigma_5(5, 5) = -3Z$ for $i, j = 1, \dots, 7$. It follows from (39) that one can obtain the following inequality:

$$\begin{aligned} & \left\{ -u^T u + [E_a x(t) + E_d x(t-d(t)) + E_c y(t-\tau(t))]^T \right. \\ & \left. \cdot [E_a x(t) + E_d x(t-d(t)) + E_c y(t-\tau(t))] \right\} \geq 0 \end{aligned} \tag{45}$$

One can rewrite (45) in quadratic form as:

$$\zeta^T(t) \Sigma_6 \zeta(t) \geq 0 \tag{46}$$

where

$\Sigma_6(1, 1) = \varepsilon E_a^T E_a, \Sigma_6(1, 3) = \varepsilon E_a^T E_d, \Sigma_6(1, 7) = \varepsilon E_a^T E_c, \Sigma_6(3, 3) = \varepsilon E_d^T E_d, \Sigma_6(3, 7) = \varepsilon E_d^T E_c, \Sigma_6(6, 6) = -\varepsilon I, \Sigma_6(7, 7) = \varepsilon E_c^T E_c$ for $i, j = 1, \dots, 7$. Let us reconsider the descriptor form representation (38) and obtain:

$$\begin{aligned} 0 = & -y(t) + Ax(t) + A_d x(t-d(t)) \\ & + Cy(t-\tau(t)) + Du = \Gamma_9 \zeta(t) \end{aligned} \tag{47}$$

where $\Gamma_9 = [A \ -I \ A_d \ 0 \ 0 \ D \ C]$. In order to obtain a relaxation, one can consider a null expression of:

$$2\zeta^T(t) M^T \Gamma_9 \zeta(t) = \zeta^T(t) \Sigma_7 \zeta(t) = 0 \tag{48}$$

where $M = [M_1 \ M_2 \ M_3 \ M_4 \ M_5 \ M_6 \ M_7], \Sigma_7 = M^T \Gamma_9 + \Gamma_9^T M$. Finally, substituting (42)-(44) into $\dot{V}(x(t), t)$ and adding (46), (48) to the resulting expression yields:

$$\dot{V}(x(t), t) \leq \zeta^T(t) \Sigma_0 \zeta(t) \tag{49}$$

where $\Sigma_0 = \sum_{i=1}^7 \Sigma_i + \bar{d} \Gamma_2^T T^{-1} \Gamma_2 + \mu \Gamma_4^T U^{-1} \Gamma_4 + \mu \Gamma_2^T Y^{-1} \Gamma_2$.

The robust stability of system (36) is guaranteed if $\Sigma_0 < 0$ can be satisfied. Applying Schur's complement to $\Sigma_0 < 0$ gives LMI condition (41). Hence, if LMIs (10), (11), (40), (41) are satisfied, then one gets $\dot{V}(x(t), t) < 0$ which assures that system (36) is robustly globally asymptotically stable. This completes the proof. ■

For the case when the neutral and discrete delays are time-invariant, i.e. $d(t) \equiv d, \tau(t) \equiv \tau$ with $\dot{d}(t) = 0, \dot{\tau}(t) = 0$, the following result is given.

Corollary 3. Given a nonnegative scalar d , the neutral system (36), (38) with time-invariant neutral and discrete delays and norm-bounded uncertainties satisfying (37), (39) is robustly asymptotically stable if there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R, S, Z$ and matrices $P_{12}, P_{13}, P_{23}, Q_{12}, M_i (i = 1, \dots, 7)$; and nonnegative scalars $\rho, \sigma, \varepsilon$ satisfying (10), (11), (40) and

$$\Psi = \begin{bmatrix} \Psi_{11} & \Phi_{12} & \Sigma_{13} & \Phi_{14} & \Phi_{15} & \Sigma_{16} & \Sigma_{17} \\ * & \Phi_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Sigma_{26} & \Sigma_{27} \\ * & * & \Sigma_{33}|_{\mu=0} & \Omega_{34} & \Omega_{35} & \Sigma_{36} & \Sigma_{37} \\ * & * & * & \Omega_{44}|_{\mu=0} & -P_{33} & M_4^T D & M_4^T C \\ * & * & * & * & -3Z & M_5^T D & M_5^T C \\ * & * & * & * & * & \Sigma_{66} & \Sigma_{67} \\ * & * & * & * & * & * & \Sigma_{77}|_{\delta=0} \end{bmatrix} < 0 \tag{50}$$

where $\Psi_{11} = P_{12}^T + P_{12} + d(P_{13}^T + P_{13}) + d^2 Q_{11} - Q_{22} + R + d^4 Z + \varepsilon E_a^T E_a + A^T M_1 + M_1^T A$, and Ω_{ij} 's, Φ_{ij} 's, Σ_{ij} 's, $i, j \in \{1, \dots, 7\}$ are defined in Theorem 1, Corollary 1 and Theorem 2, respectively, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. Similar to the proof of Theorem 2, hence it is omitted.

If neutral and discrete delays are time-invariant and coincident, i.e. $d \equiv \tau$, then the following result is presented.

Corollary 4. Given the nonnegative scalar d , the neutral system (36), (38) with coincident time-invariant neutral and discrete delays, and norm-bounded uncertainties satisfying (37), (39) is robustly asymptotically stable if there exist a symmetric positive definite matrix P_{11} ; symmetric positive semi-definite matrices $P_{22}, P_{33}, Q_{11}, Q_{22}, R_{11}, R_{22}, S$ and matrices $P_{12}, P_{13}, P_{23}, Q_{12}, R_{12}, M_i (i = 1, \dots, 7)$; and nonnegative scalars $\rho, \sigma, \varepsilon$ satisfying (10), (11), (34), (40) and the following LMI:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Pi_{12} & \Sigma_{13} & \Phi_{14} & \Phi_{15} & \Sigma_{16} & \Sigma_{17} \\ * & \Pi_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Sigma_{26} & \Sigma_{27} \\ * & * & \Theta_{33} & \Omega_{34} & \Omega_{35} & \Sigma_{36} & \Theta_{37} \\ * & * & * & \Omega_{44}|_{\mu=0} & -P_{33} & M_4^T D & M_4^T C \\ * & * & * & * & -3Z & M_5^T D & M_5^T C \\ * & * & * & * & * & \Sigma_{66} & \Sigma_{67} \\ * & * & * & * & * & * & \Theta_{77} \end{bmatrix} < 0 \tag{51}$$

where

$$\Theta_{11} = P_{12}^T + P_{12} + d(P_{13}^T + P_{13}) + d^2 Q_{11} - Q_{22} + R_{11} + d^4 Z + \varepsilon E_a^T E_a + A^T M_1 + M_1^T A,$$

$$\Theta_{33} = -Q_{22} - R_{11} + \varepsilon E_d^T E_d + A_d^T M_3 + M_3^T A_d,$$

$$\Theta_{37} = \Sigma_{37} - R_{12}, \quad \Theta_{77} = -R_{22} + \varepsilon E_c^T E_c + C^T M_7 + M_7^T C$$

and Ω_{ij} 's, Φ_{ij} 's, Π_{ij} 's, Σ_{ij} 's, $i, j \in \{1, \dots, 7\}$ are defined in Theorem 1, Corollary 1, Corollary 2 and Theorem 2, respectively, and the notation (*) denotes the symmetric terms in a symmetric matrix.

Proof. The similar approach introduced in the proof of Corollary 2 is considered and the rest of the proof follows from the proof of Theorem 2.

V. NUMERICAL EXAMPLES

In this section, three numerical examples which have already been reported in the literature are taken into consideration in order to demonstrate the application of theorems and corollaries which yield less conservative results.

Example 1. Let us consider system (1) with

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

where $0 \leq |c| \leq 1$ and $\alpha = 0.1, \beta = 0.1, \gamma \geq 0$ are given nonnegative scalars.

Case I. Assume that $c = 0$ and $h(\dot{x}(t - \tau(t)), t) = 0$, i.e. $\gamma = 0$, and choose $\mu = 0.5$, then the system under consideration reduces to system studied in Han and Yu [10], Cao and Lam [13], Han [14].

It can be clearly seen from Table 1 that the results of this note are better than those obtained in Han and Yu (2004) [10], Cao and Lam (2000) [13], Han (2004) [14].

Case II. Let us consider the case when $c = 0.1, \mu = 0.5$, and $\delta = 0.5$ and see the effect of the uncertainty bound, γ on the

maximum delay bound, \bar{d} . Note that both the neutral time-delay and discrete time-delay are time-varying and Table 2 shows that the proposed method of this paper gives quite less conservative delay bounds than those obtained in Han and Yu [10].

Example 2. Let us consider an uncertain linear neutral system with time-varying discrete and neutral delays

$$\dot{x}(t) = \begin{bmatrix} -2 + \delta_1 & 0 \\ 0 & -1 + \delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1 + \delta_3 & 0 \\ -1 & -1 + \delta_4 \end{bmatrix} x(t - d(t)) + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \dot{x}(t - \tau(t)) \tag{52}$$

where $0 \leq |c| \leq 1$ and $\delta_1, \delta_2, \delta_3, \delta_4$ are unknown parameters

Table 1. Delay bound \bar{d} .

Cao and Lam [13]	Han [14]	Han and Yu [10]	This work
0.4950	0.5716	0.9952	1.0097

Table 2. Delay bound \bar{d} .

γ	0.0	0.1	0.2	0.3
Han and Yu [10]	0.7437	0.5131	0.3112	0.1398
This work	0.7749	0.5658	0.3859	0.2357

satisfying $|\delta_1| \leq 1.6, |\delta_2| \leq 0.05, |\delta_3| \leq 0.1, |\delta_4| \leq 0.3$. Note that the numerical system (52) can be represented in the form of system (40) with

$$D = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad E_a = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad E_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}$$

Case I. For $c = 0$ and $\mu = 0$, the maximum allowable sizes of the time-delay which are reported in the literature are listed in Table 3 along with those obtained with the method proposed in this paper.

It is clearly seen that the result obtained in this paper is less conservative than the existing methods.

Case II. For $\mu = 0.1$ and $\delta = 0$, Table 4 presents the maximum value of \bar{d} along with those obtained in Han and Yu [10] for various values of the parameter, c . Table 4 shows that the proposed method in this paper yields less conservative result than that given in Han and Yu [10].

Example 3. Let us consider the following time-delay system with norm-bounded uncertainty as in [7], [9]

$$\dot{x}(t) = [A + DF(t)E_a]x(t) + [A + DF(t)E_d]x(t-d) + C\dot{x}(t-\tau) \tag{53}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, D = I,$$

$E_a = E_d = \alpha I$. When $\alpha = 0.2$, the delay bounds are obtained for each of the different values of c . The results are listed in Table 5 along with those given in Han [7], Wu *et al.* [9], and those achieved using the method of Xu *et al.* [12], and He *et al.* [17].

Table 5 shows that the proposed methodology of this paper gives better results than those introduced in Han [7], Wu *et al.* [9], Xu *et al.* [12], and He *et al.* [17].

Finally, considering the nominal form of the neutral system (53), *i.e.* $\alpha = 0$, and assuming that $\tau = d$, Table 6 presents the allowable maximum values of the time-delay, d for different values of c along with the results obtained in Fridman and Shaked [8], Wu *et al.* [9], He *et al.* [11], and those estimated by the method given in Xu *et al.* [12], and He *et al.* [17]. It is clearly seen from Table 6 that the proposed method gives better namely less conservative delay bounds when compared with those given in Han [8], Wu *et al.* [9], He *et al.* [11], Xu *et al.* [12], He *et al.* [17].

VI. CONCLUSIONS

In this paper, a new quasi full size augmented Lyapunov-Krasovskii functional is proposed in order to investigate the delay-dependent stability problem for uncertain neutral systems with mixed time-varying discrete and neutral delays. Both the nonlinear parameter perturbations and the norm-bounded uncertainty cases are studied. The stability criteria are formulated in terms of LMIs and shown to be less conservative than the existing stability methods. Three numerical examples are utilized to illustrate the lower conservativeness of the method developed in this paper.

Table 3. Upper bound of the time-delay, $d(t)$.

Kim [15]	Fridman and Shaked [8]	Han and Yu [10]	He <i>et al.</i> [17]	Parlakci [18], [19]	This work
0.2412	1.0000	1.0345	1.1490	1.1623	1.1721

Table 4. Upper bound of the time-delay, $d(t)$.

$ c $	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Han and Yu [10]	0.97	0.78	0.60	0.45	0.31	0.19	0.10	0.02
This work	1.11	0.92	0.75	0.60	0.46	0.34	0.23	0.11

Table 5. Upper bound of the time-delay, d .

c	0.0	0.05	0.1	0.15	0.2	0.25	0.30
Han [7], $\tau = d$	1.77	1.63	1.48	1.33	1.16	0.98	0.79
He <i>et al.</i> [17], $\tau = d$	2.39	2.05	1.75	1.49	1.27	1.08	0.91
Xu <i>et al.</i> [12], $\tau = d$	2.39	2.15	1.93	1.74	1.57	1.41	1.26
He <i>et al.</i> [17], $\tau = 0.1$	2.39	2.25	2.11	1.96	1.81	1.66	1.50
Wu <i>et al.</i> [9], $\tau = d$	2.43	2.33	2.24	2.14	2.03	1.91	1.78
This work, $0 \leq \tau < \infty$	2.47	2.19	1.95	1.72	1.52	1.35	1.19
This work, $\tau = d$	2.47	2.38	2.28	2.16	2.04	1.91	1.79

Table 6. Upper bound of the time delay, d .

$ c $	0.0	0.1	0.3	0.5	0.7	0.9
Fridman and Shaked [8]	4.47	3.49	2.06	1.14	0.54	0.13
Xu <i>et al.</i> [12]	4.47	3.58	2.30	1.46	0.86	0.32
He <i>et al.</i> [17]	4.47	4.04	3.23	2.51	1.83	1.02
Wu <i>et al.</i> [9]	4.47	4.35	4.13	3.67	2.87	1.41
He <i>et al.</i> [11]	4.47	4.42	4.17	3.69	2.87	1.41
This work	4.63	4.57	4.29	3.75	2.88	1.41

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