Journal of Combinatorial Theory, Series B 102 (2012) 715–725

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Minimum H-decompositions of graphs: Edge-critical case ☆

Lale Özkahya [a], Yury Person [b]

[a] İstanbul Bilgi Üniversitesi, Matematik Bölümü, Kurtuluş Deresi Cad. 47, 34435 Dolapdere Beyoğlu İstanbul, Turkey
[b] Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

ARTICLE INFO

Article history:
Received 4 May 2010
Available online 19 October 2011

Keywords:
Graph decomposition
Edge-critical
Turán graph
Stability approach

ABSTRACT

For a given graph H let $\phi_H(n)$ be the maximum number of parts that are needed to partition the edge set of any graph on n vertices such that every member of the partition is either a single edge or it is isomorphic to H. Pikhurko and Sousa conjectured that $\phi_H(n) = \text{ex}(n, H)$ for $\chi(H) \geqslant 3$ and all sufficiently large n, where $\text{ex}(n, H)$ denotes the maximum size of a graph on n vertices not containing H as a subgraph. In this article, their conjecture is verified for all edge-critical graphs. Furthermore, it is shown that the graphs maximizing $\phi_H(n)$ are $(\chi(H)-1)$-partite Turán graphs.

1. Introduction and results

For two graphs G and H, an *H-decomposition* of G is a partition of the edges of G into $G_1, \ldots, G_t$ such that every G_i is either a single edge or is isomorphic to H. An H-decomposition of G with smallest possible t is called *minimum* and $\phi_H(G) = t$ denotes its cardinality. It is not difficult to see that $\phi_H(G) = e(G) - (e(H) - 1)N_H(G)$, where $N_H(G)$ denotes the maximum number of edge-disjoint copies of H in G.

In this paper, we study the function

$$\phi_H(n) := \max_{G \in \mathcal{G}_n} \phi_H(G),$$

where $\mathcal{G}_n$ denotes the family of all graphs on n vertices.

This function was studied first by Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They showed that $\phi_{K_3}(n) = \text{ex}(n, K_3)$, where $\text{ex}(n, F)$ denotes the maximum size of a graph on n vertices, that does not contain H as a subgraph. Moreover,

☆ Part of this work was done while both authors were visiting the Institute for Pure and Applied Mathematics at UCLA. The second author was supported by GIF grant No. I-889-182.6/2005.

E-mail addresses: ozkahya@illinoisalumni.org (L. Özkahya), person@math.fu-berlin.de (Y. Person).

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2011.10.004

they proved that the only graph that maximizes this function is the complete balanced bipartite graph. Consequently, they conjectured that $\phi_{K_r}(n) = \text{ex}(n, K_r)$ and the only optimal graph is the Turán graph $T_{r-1}(n)$, the complete balanced $(r-1)$-partite graph on n vertices, where the sizes of the partite sets differ from each other by at most one. Clearly, $\text{ex}(n, K_r)$ is a lower bound, as $T_{r-1}(n)$ does not contain any copy of K_r and therefore in the optimal K_r-decomposition, every part consists of a single edge. Later, Bollobás [1] proved that $\phi_{K_r}(n) = \text{ex}(n, K_r)$ for all $n \geqslant r \geqslant 3$.

Recently, Pikhurko and Sousa [6] studied $\phi_H(n)$ for arbitrary graphs H. Their result is the following.

Theorem 1. *(See Theorem 1.1 from [6].) Let H be any fixed graph with chromatic number $r \geqslant 3$. Then,*

$$\phi_H(n) = \text{ex}(n, H) + o\left(n^2\right).$$

The same authors also made the following conjecture.

Conjecture 2. *For any graph H with chromatic number at least 3, there is an $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$ for all $n \geqslant n_0$.*

This conjecture has been verified by Sousa for clique extensions of order $r \geqslant 4$ $(n \geqslant r)$ [10], the cycles of length 5 $(n \geqslant 6)$ and 7 $(n \geqslant 10)$ [11,9].

We say a graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$. Cliques and odd cycles are examples of edge-critical graphs. We verify Conjecture 2 for all edge-critical graphs.

Theorem 3. *For any edge-critical graph H with chromatic number at least 3, there is an $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$ for all $n \geqslant n_0$. Moreover, the only graph attaining $\phi_H(n)$ is the Turán graph $T_{\chi(H)-1}(n)$.*

Note that $\text{ex}(n, H) = \text{ex}(n, K_{\chi(H)})$ for all large n and all edge-critical graphs H, and this is a result of Simonovits [8], where he also shows that the unique extremal graph is $T_{\chi(H)-1}(n)$.

To prove Theorem 3, we first show in Lemma 4 an approximate structural result about the function $\phi_H(n)$. Namely, graphs G with $\phi_H(G) \geqslant \text{ex}(n, H) - o(n^2)$ look almost as Turán graphs. Then we exploit small imperfections by finding too many edge-disjoint copies of H in G which would give us a contradiction to our assumptions about $\phi_H(G)$. Such approach (stability method) has been extensively used to study problems in extremal (hyper)graph theory.

Throughout the sections we will sometimes omit floors and ceilings as they will not affect our calculations. We use standard notations from graph theory. Thus, for $t \in \mathbb{N}$ we denote by $[t]$ the set $\{1, \ldots, t\}$. For a given graph $G = (V, E)$ and for a subset $U \subseteq V$ we denote by $E_G(U) = E \cap \binom{U}{2}$ and $G[U] = G(U, E_G(U))$. We set $e_G(U) = |E_G(U)|$, and for a vertex $v \in V$ we write $\deg_{G,U}(v) = |\{u \in U: \{v, u\} \in E(G)\}|$, i.e., we are only counting the neighbors of v in U. Similarly, for two disjoint subsets $U, W \subseteq V$ we set $E_G(U, W) = \{\{u, w\} \in E(G): u \in U, w \in W\}$, $G[U, W] = G(U \cup W, E_G(U, W))$ and $e_G(U, W) = |E_G(U, W)|$. We will sometimes omit G when there is no danger of confusion, and we write $\deg_U(v)$, $e(U)$, $E(U, W)$, $e(U, W)$.

2. A stability result

In this section, we prove the following approximate result about graphs $G \in \mathcal{G}_n$ with $\phi_H(G) \geqslant \text{ex}(n, H) - o(n^2)$.

Lemma 4. *For every H with $\chi(H) = r \geqslant 3$, $H \neq K_r$, and for every $\gamma > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every graph G on $n \geqslant n_0$ vertices the following is true. If*

$$\phi_H(G) \geqslant \text{ex}(n, H) - \varepsilon n^2$$

then there exists a partition of $V(G) = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ with $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

In a sense, Lemma 4 is a corollary from the result of Pikhurko and Sousa [6, Theorem 1.1], where they show $\phi_H(n) = \mathrm{ex}(n, H) + o(n^2)$. Roughly speaking, one follows the proof of Theorem 1 (see [6, Theorem 1.1]), and at the end, an application of the following stability result of Erdős [3] and Simonovits [8] along with some computation is required.

Theorem 5 *(Stability theorem). For every H with $\chi(H) = r \geqslant 2$, and every $\gamma > 0$ there exist a $\delta > 0$ and an n_0 such that the following holds. If G is a graph on $n \geqslant n_0$ vertices with $e(G) \geqslant \mathrm{ex}(n, H) - \delta n^2$ and if it does not contain H as a subgraph, then there exists a partition of $V(G) = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.*

Note that by the theorem of Erdős and Stone [5], one has $\mathrm{ex}(n, H) = \mathrm{ex}(n, K_{\chi(H)}) + o(n^2)$, and though the error term is small, it is not necessarily zero.

One can extract the following consequence of Theorem 1.1 from [6].

Corollary 6. *For every $H \ncong K_r$ with $\chi(H) = r \geqslant 3$ and for every $c > 0$ there exist $k, n_0 \in \mathbb{N}$ such that the following holds. For every graph G with $|V(G)| = n \geqslant n_0$ vertices there exists $\alpha \geqslant 0$ which satisfies the following:*

(i) $N_H(G) \geqslant (1 - c)\frac{\alpha}{e(H)}\binom{r}{2}(\frac{n}{k})^2$, *and*
(ii) $e(G) \leqslant (\binom{r}{2} - 1)\alpha(\frac{n}{k})^2 + \mathrm{ex}(n, K_r) + cn^2$.

In the following we briefly sketch the proof of Corollary 6 by giving the argument from [6]. Yet we refrain from introducing the tools needed for the proof, but instead we only mention them.

Sketch of the proof of Corollary 6. In the proof of Theorem 1 from [6, Theorem 1.1], the following hierarchy of constants (chosen exactly in the same order) has been used:

$$c_0 := c \gg c_1 \gg c_2 \gg c_3 \gg c_4 \gg c_5 > 0, \tag{1}$$

where c_0 corresponds to the error term in the claim of Theorem 1. The sketch is given in four steps.

In the first step, one applies the regularity lemma of Szemerédi [12] to G and obtains a $c_4/2$-regular partition $V(G) = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ such that $1/c_3 \leqslant k < 1/c_5$ and $|V_1| = |V_2| = \cdots = |V_k| \geqslant (1 - c_4/2)n/k$. Then, we update G by removing all edges in $G[V_i]$'s, in $c_4/2$-irregular pairs and in pairs of density less than c_1. Due to (1), we removed at most $c_1 n^2$ ($\ll c_0 n^2$) edges.

In the second step, one defines a weighted graph K on the vertex set $[k]$, by setting the weight $w(i, j)$ for each $\{i, j\} \subset \binom{[k]}{2}$ to be the density of the bipartite graph $G[V_i, V_j]$. Another result from [6] (Lemma 2.4) states that one can find a family $\{(A_h, \alpha_h)\colon h \in [t']\}$ such that

- each $A_h \subset [k]$ with $|A_h| = 2$ or $|A_h| = r$,
- each $\alpha_h > 0$, where α_h is called the *weight* of A_h,
- for any distinct $i, j \in [k]$ one has $w(i, j) = \sum_{h\colon \{i,j\} \subset A_h} \alpha_h$,
- for every A_h with $|A_h| = r$, we have $\alpha_h \geqslant c_2$, and
- $$\sum_{h=1}^{t'} \alpha_h \leqslant \mathrm{ex}(k, K_r) + 2c_1 k^2, \tag{2}$$

  where this sum is called the *total weight of* $\{(A_h, \alpha_h)\colon h \in [t']\}$.

This family is called *weighted K_r-decomposition* which in some sense suggests how the $c_4/2$-regular pairs of G should be splitted into regular ones with smaller density. Without loss of generality, we assume that $|A_h| = r$ for all $h \in [t]$ and $|A_h| = 2$ for all $t < h \leqslant t'$ for some $t \in [t'] \dot\cup \{0\}$.

In the third step, we partition every pair $G[V_i, V_j]$ into bipartite subgraphs $B_{ij,1}, \ldots, B_{ij,t}$ with vertex sets $V_i \dot\cup V_j$, where each edge of $G[V_i, V_j]$ is included into $B_{ij,\ell}$ with probability $\alpha_\ell / w(i, j)$

(if $i, j \in A_\ell$ and otherwise 0) independent of the other edges. Thus, for $1 \leqslant \ell \leqslant t$, the expected density of $B_{ij,\ell}$ is α_ℓ if $i, j \in A_\ell$ and 0 otherwise. Because

$$t \leqslant \frac{\binom{k}{2}}{c_2\binom{r}{2}},$$

Chernoff's [2] inequality shows that, with high probability, every $B_{ij,\ell}$ is c_4-regular with density approximately α_ℓ.

In the fourth step, we concentrate on $F_\ell := \dot{\bigcup}_{i<j,\, i,j\in A_\ell} B_{ij,\ell}$ which form balanced r-partite graphs for every $\ell \in [t]$. Moreover, for fixed ℓ, the density between any two classes is approximately α_ℓ. For each $\ell \in [t]$, one defines an $e(H)$-uniform hypergraph with vertex set $E(F_\ell) = \dot{\bigcup}_{i<j,\, i,j\in A_\ell} E(B_{ij,\ell})$ and edge set being the family of copies of H in F_ℓ. By the theorem of Pippenger and Spencer [7], $E(F_\ell)$ can be almost perfectly decomposed into edge-disjoint copies of H. More precisely, Pikhurko and Sousa compute that each F_ℓ contains at least

$$(1-2c_2)\frac{\alpha_\ell}{e(H)}\binom{r}{2}\left(\frac{n}{k}\right)^2 \tag{3}$$

edge-disjoint copies of H.

We set $\alpha := \sum_{\ell=1}^{t}\alpha_\ell$, and due to (3) there are at least

$$(1-2c_2)\frac{\alpha}{e(H)}\binom{r}{2}\left(\frac{n}{k}\right)^2 \geqslant (1-c)\frac{\alpha}{e(H)}\binom{r}{2}\left(\frac{n}{k}\right)^2 \tag{4}$$

edge-disjoint copies of H in G. On the other hand, the number of edges in G can be bounded above in terms of α as follows:

$$\begin{aligned}
e(G) &\leqslant \left(\sum_{i=1}^{t'}\alpha_i + \left(\binom{r}{2}-1\right)\alpha\right)\left(\frac{n}{k}\right)^2 + c_1n^2 \\
&\overset{(2)}{\leqslant} \left(\binom{r}{2}-1\right)\alpha\left(\frac{n}{k}\right)^2 + \left(\mathrm{ex}(k,K_r)+2c_1k^2\right)\left(\frac{n}{k}\right)^2 + c_1n^2 \\
&\leqslant \left(\binom{r}{2}-1\right)\alpha\left(\frac{n}{k}\right)^2 + \mathrm{ex}(n,K_r) + 4c_1n^2 \\
&\leqslant \left(\binom{r}{2}-1\right)\alpha\left(\frac{n}{k}\right)^2 + \mathrm{ex}(n,K_r) + cn^2,
\end{aligned} \tag{5}$$

where we use $\mathrm{ex}(k,K_r)(\frac{n}{k})^2 \leqslant \mathrm{ex}(n,K_r) + c_1n^2$.

Thus, (4) and (5) prove the assertion of Corollary 6. □

Now we show how Corollary 6 together with Theorem 5 implies Lemma 4.

Proof of Lemma 4. Let γ and H be given. Since $\chi(H) = r$ and H is not the complete graph K_r it follows $\binom{r}{2} < e(H)$, and thus we define $\beta > 0$ such that

$$1-\beta = \binom{r}{2}/e(H). \tag{6}$$

Theorem 5 asserts the existence of $\delta = \delta(\gamma/2) > 0$ for $\gamma > 0$. Further we set

$$c := \frac{\beta}{4r^4}\cdot\min\{\gamma,\delta\} \quad \text{and} \quad \varepsilon := c. \tag{7}$$

Next we choose n_0 sufficiently large. Let G be any graph of order $n \geqslant n_0$ with $\phi_H(G) \geqslant \mathrm{ex}(n,H) - \varepsilon n^2$.

Applying Corollary 6 to G with c and H we obtain k and α such that the assertions of Corollary 6 hold.

With the upper bound on $e(G)$ and the lower bound on $N_H(G)$, asserted to us by Corollary 6, we bound $\phi_H(G)$ from above by

$$\begin{aligned}\phi_H(G) &= e(G) - \big(e(H)-1\big)N_H(G)\\ &\leqslant \left(\binom{r}{2}-1\right)\alpha\left(\frac{n}{k}\right)^2 + \mathrm{ex}(n,K_r) + cn^2 - \big(e(H)-1\big)(1-c)\frac{\alpha}{e(H)}\binom{r}{2}\left(\frac{n}{k}\right)^2\\ &\overset{(6)}{\leqslant} \mathrm{ex}(n,K_r) + cn^2 + c\binom{r}{2}n^2 - \alpha\beta\left(\frac{n}{k}\right)^2 \leqslant \mathrm{ex}(n,H) + 2cr^2n^2 - \alpha\beta\left(\frac{n}{k}\right)^2,\end{aligned} \tag{8}$$

since $\mathrm{ex}(n,K_r) \leqslant \mathrm{ex}(n,H)$ for $\chi(H) = r$. We deduce from the assumption of our lemma that

$$\mathrm{ex}(n,H) - \varepsilon n^2 \leqslant \phi_H(G) \leqslant \mathrm{ex}(n,H) + 2cr^2n^2 - \alpha\beta\left(\frac{n}{k}\right)^2$$

which implies

$$\alpha \leqslant \frac{\varepsilon + 2cr^2}{\beta}k^2 \overset{(7)}{=} \frac{c(2r^2+1)}{\beta}k^2. \tag{9}$$

Intuitively, this means that α is indeed very small in comparison to k^2 (if c was chosen small). Therefore the number of edge-disjoint copies of H in G is only $o(n^2)$. To verify this precisely, we consider the following inequality:

$$\mathrm{ex}(n,H) - \varepsilon n^2 \leqslant \phi_H(G) = e(G) - \big(e(H)-1\big)N_H(G).$$

Again, the upper bound on $e(G)$ implies that

$$\big(e(H)-1\big)N_H(G) \leqslant \left(\binom{r}{2}-1\right)\alpha\left(\frac{n}{k}\right)^2 + \mathrm{ex}(n,K_r) + cn^2 - \mathrm{ex}(n,H) + \varepsilon n^2,$$

and thus

$$N_H(G) \leqslant \frac{(\binom{r}{2}-1)\frac{\alpha}{k^2} + 2c}{e(H)-1}n^2 \overset{(9)}{\leqslant} \frac{cr^4}{\beta(e(H)-1)}n^2.$$

We also know that $e(G) \geqslant \phi_H(G) \geqslant \mathrm{ex}(n,H) - \varepsilon n^2 \overset{(7)}{=} \mathrm{ex}(n,H) - cn^2$. Since one can obtain an H-free subgraph G' by deleting at most $e(H)\cdot N_H(G)$ edges from G, we deduce that

$$e\big(G'\big) \geqslant \mathrm{ex}(n,H) - cn^2 - \frac{cr^4}{\beta(e(H)-1)}e(H)n^2 \overset{(7)}{\geqslant} \mathrm{ex}(n,H) - \delta n^2.$$

Therefore, the stability theorem, Theorem 5, is applicable and there exists a partition of $V(G')$ (and of $V(G)$) as $V(G) = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ such that

$$\sum_{i=1}^{r-1} e(V_i) < (\gamma/2)n^2 + \frac{cr^4}{\beta(e(H)-1)}e(H)n^2 \overset{(7)}{<} \gamma n^2,$$

which finishes the proof. □

3. Proof of Theorem 3

In the following proof we will use auxiliary Claims 7, 8 and 9, which will be shown in Section 4.

Proof of Theorem 3. We may assume that H is not a clique, as otherwise this is a result of Bollobás [1] mentioned in the introduction. We will apply Lemma 4 in a slightly weaker form, i.e., when $\epsilon = 0$, and we choose γ sufficiently small, for definiteness it is enough to set

$$\gamma := \frac{1}{3600((r-1)e(H))^4}. \tag{10}$$

We also choose $n_0(H) \geqslant n_0 + \binom{n_0}{2}$, where n_0 is given to us by Lemma 4. Suppose that there exists a graph G on $n \geqslant n_0(H)$ vertices with $\phi_H(G) \geqslant \mathrm{ex}(n, H)$. Further assume that G is not isomorphic to the Turán graph $T_{r-1}(n)$, where $r = \chi(H)$. We will derive a contradiction, by finding too many edge-disjoint copies of H in G and thus showing that

$$\phi_H(G) = e(G) - \bigl(e(H) - 1\bigr)N_H(G) < \mathrm{ex}(n, H). \tag{11}$$

We may assume without loss of generality that

$$\delta(G) \geqslant \delta\bigl(T_{r-1}(n)\bigr) \geqslant (r-2)\left\lfloor \frac{n}{r-1} \right\rfloor, \tag{12}$$

since otherwise we apply the following claim and further proceed with G' instead of G.

Claim 7. *Let $m \geqslant 0$ and $\phi_H(G) = \mathrm{ex}(n, H) + m$. Then there is a graph G' on $n' = n - i$ vertices that is obtained by removing i vertices from G, $i \in [\binom{n_0}{2}] \cup \{0\}$, such that*

$$\delta\bigl(G'\bigr) \geqslant \delta\bigl(T_{r-1}\bigl(n'\bigr)\bigr)$$

and $\phi_H(G') \geqslant \mathrm{ex}(n', H) + m + i$.

Let $\mathcal{P} = \{V_1, \ldots, V_{r-1}\}$ be a partition of $V(G)$ such that

$$\sum_{1 \leqslant i < j \leqslant r-1} e_G(V_i, V_j) \tag{13}$$

is maximized. Note that $\sum_{i=1}^{r-1} e_G(V_i) > 0$ because we assumed that G is not isomorphic to $T_{r-1}(n)$.

Because of the maximality of the partition $\mathcal{P}$, for every $v \in V_i$, $i \in [r-1]$, we have $\deg_{G,V_i}(v) \leqslant \deg_{G,V_j}(v)$ for $j \in [r-1] \setminus \{i\}$, otherwise, moving v to V_j would increase the size of the multicut (13). Let m_1 denote the number of missing edges in $\mathcal{P}$, i.e.,

$$m_1 := \sum_{1 \leqslant i < j \leqslant r-1} \bigl(|V_i| \cdot |V_j| - e_G(V_i, V_j)\bigr) \geqslant 0, \tag{14}$$

while

$$m_2 := \sum_{i=1}^{r-1} e(V_i) > 0 \tag{15}$$

denotes the number of edges within the partition classes. Also note that

$$\sum_{1 \leqslant i < j \leqslant r-1} |V_i| \cdot |V_j| - m_1 + m_2 \geqslant \mathrm{ex}(n, H) = e\bigl(T_{r-1}(n)\bigr).$$

We also have

$$m_2 < \gamma n^2 \quad \text{and} \quad e(G) \leqslant \mathrm{ex}(n, H) + m_2 \tag{16}$$

which is asserted to us by Lemma 4.

The following claim establishes bounds on the sizes of the partition classes V_i.

Claim 8.

$$\forall i \in [r-1]: \quad |V_i| \geqslant \frac{n}{r-1} - 2\sqrt{\gamma} n \quad \textit{and} \quad |V_i| \leqslant \frac{n}{r-1} + 2(r-2)\sqrt{\gamma} n. \tag{17}$$

Because of the edge-criticality of H, there exists $e = \{x, y\} \in E(H)$ such that $\chi(H - e) = r - 1$, where x and y are connected to every other class in any coloring of $H - e$. We let $s_x = \deg_H(x) - 1$ and $s_y = \deg_H(y) - 1$, so

$$e(H) - s_x - s_y - 1 \geqslant \binom{r-2}{2} \geqslant 0.$$

We also assume without loss of generality that $s_x \leqslant s_y$. Below we show that $N_H(G) > \frac{m_2}{e(H)-1}$ which is a contradiction to $\phi_H(G) \geqslant \mathrm{ex}(n, H)$. For this purpose we will describe a procedure to find that many edge-disjoint copies of H in G such that each copy uses exactly one edge within some class V_i. Before doing that, we need the following terminology.

For $v \in V_i$, $i \in [r-1]$, we call v a *bad* vertex if

$$\deg_{V_i}(v) > \frac{n}{12(r-1)e(H)}, \tag{18}$$

otherwise we call v *good*. Note that there are at most

$$\frac{2\gamma n^2}{(\frac{n}{12(r-1)e(H)})} \leqslant 24(r-1)e(H)\gamma n \tag{19}$$

bad vertices in G. Another observation is that we can give a sufficiently good (for our purposes) lower bound on $\deg_{V_j}(v)$ for a bad vertex $v \in V_i$, $i \neq j$. Namely,

$$\deg_{V_j}(v) \geqslant \max\left\{\deg_{V_i}(v), \frac{\delta(T_{r-1}(n)) - \sum_{\ell \in [r-1], \ell \neq i, j} |V_\ell|}{2}\right\} \geqslant \frac{n}{2(r-1)} - 2\sqrt{\gamma} n \tag{20}$$

which follows from (12), (17) and the maximality of the vertex partition $\mathcal{P}$.

For each bad vertex v in some $V_i, i \in [r-1]$, we choose $\deg_{V_i}(v)/(2s_x) + 1$ edges in $G[V_i]$ that connect v to good vertices which is always possible because of the bound on the number of bad vertices (19). We keep these edges and delete the remaining edges incident to v from $G[V_i]$. After repeating this for each bad vertex in G, we call the final graph $\tilde{G}$. Note that

$$\sum_{i=1}^{r-1} e_{\tilde{G}}(V_i) > \frac{m_2}{2s_x} \geqslant \frac{m_2}{e(H)-1} \tag{21}$$

which is used in the following claim.

Claim 9. *There are at least* $\lfloor \frac{m_2}{e(H)-1} \rfloor + 1$ *edge-disjoint copies of* H *in* $\tilde{G}$.

As stated earlier, $\phi_H(G) = e(G) - (e(H) - 1)N_H(G)$. With Claim 9 and (16) we obtain that $\phi_H(G) < \mathrm{ex}(n, H)$ which is a contradiction to the assumption of Theorem 3. □

4. Proofs of Claims 7, 8 and 9

Proof of Claim 7. If $\delta(G) \geqslant \delta(T_{r-1}(n))$, then $i = 0$. Otherwise, let v be a vertex of G with $\deg_G(v) < \delta(T_{r-1}(n))$. Then we delete v from G obtaining $G_1 := G - v$ with

$$\phi_H(G_1) \geqslant \phi_H(G) - \deg_G(v) \geqslant \mathrm{ex}(n, H) + m - \delta\big(T_{r-1}(n)\big) + 1 = \mathrm{ex}(n-1, H) + m + 1, \tag{22}$$

where we used the fact

$$\mathrm{ex}(n, H) - \delta\big(T_{r-1}(n)\big) = e\big(T_{r-1}(n)\big) - \delta\big(T_{r-1}(n)\big) = e\big(T_{r-1}(n-1)\big) = \mathrm{ex}(n-1, H)$$

for edge-critical H and sufficiently large n.

If G_1 does not satisfy condition on the minimum degree, then we iterate this procedure, until we arrive at a graph G' that satisfies (12), or we stop when $n' = n_0$. In the latter case, G' has n_0 vertices and $\phi_H(G') > \binom{n_0}{2}$ which is a contradiction. In the case when (12) holds, we know that G' is not isomorphic to the Turán graph, since $\phi_H(G) > \mathrm{ex}(n, H)$.

In general, if G' is obtained after removing i vertices from the original graph, then (22) implies

$$\phi_H\big(G'\big) \geqslant \mathrm{ex}\big(\big|V\big(G'\big)\big|, H\big) + m + i, \tag{23}$$

where $i, m \geqslant 0$. □

Proof of Claim 8. Suppose without loss of generality that

$$|V_{r-1}| = n/(r-1) - a,$$

where $a > 0$. Then:

$$\sum_{1 \leqslant i < j \leqslant r-1} |V_i||V_j| + \gamma n^2 \geqslant e(G), \tag{24}$$

while on the other side

$$e(G) \geqslant \mathrm{ex}(n, H) = \mathrm{ex}\big(n, T_{r-1}(n)\big) \geqslant \binom{r-1}{2}\left(\frac{n}{r-1} - 1\right)^2. \tag{25}$$

We also further estimate $\sum_{1 \leqslant i < j \leqslant r-1} |V_i||V_j|$ from above by

$$\binom{r-2}{2}\left(\frac{n}{r-1} + \frac{a}{r-2}\right)^2 + \left(\frac{(r-2)n}{r-1} + a\right)\left(\frac{n}{r-1} - a\right) \geqslant \sum_{1 \leqslant i < j \leqslant r-1} |V_i||V_j|, \tag{26}$$

as Turán graphs maximize the number of edges in complete partite graphs. Thus, by (24) and (26), we have

$$\binom{r-1}{2}\left(\frac{n}{r-1}\right)^2 + \frac{(r-3)a^2}{2(r-2)} - a^2 + \gamma n^2 \geqslant e(G).$$

With (25) we obtain:

$$\binom{r-1}{2}\left(\frac{n}{r-1}\right)^2 + \frac{(r-3)a^2}{2(r-2)} - a^2 + \gamma n^2 \geqslant \binom{r-1}{2}\left(\frac{n}{r-1} - 1\right)^2$$

which implies

$$2\gamma n^2 \geqslant (r-2)n + \gamma n^2 \geqslant a^2/2$$

and yields an upper bound that $a \leqslant 2\sqrt{\gamma}n$. Thus, $\forall i \in [r-1]$: $|V_i| \geqslant n/(r-1) - 2\sqrt{\gamma}n$ implying that

$$\forall i \in [r-1]: \quad |V_i| \leqslant \frac{n}{r-1} + 2(r-2)\sqrt{\gamma}n. \qquad \square$$

Proof of Claim 9. We will find a family of edge-disjoint copies of H such that each edge from $\dot{\bigcup}_{i=1}^{r-1} E_{\tilde{G}}(V_i)$ will belong to exactly one of the copies of H that we will find. We also introduce a threshold t as

$$t := \frac{n}{r-1} - 2r^2\sqrt{\gamma}n - \frac{n}{12(r-1)e(H)} - 24(r-1)e(H)\gamma n - \frac{n}{60(r-1)} \tag{27}$$

such that if for a good vertex $v \in V_i$ it happens that there is a $j \neq i$ such that

$$\deg_{\tilde{G},V_j}(v) < t,$$

then we call such a good vertex *inactive*. Initially, every good vertex of $\tilde{G}$ is *active*. Indeed, by using (12), (17) and (18), we can bound the initial number of neighbors of every good vertex $v \in V_i$ in a set V_j, $j \neq i$ as follows:

$$\begin{aligned}
\deg_{\tilde{G},V_j}(v) &\geqslant (r-2)\left\lfloor \frac{n}{r-1} \right\rfloor - (r-3)\left(\frac{n}{r-1} + 2(r-2)\sqrt{\gamma}n\right) - \frac{n}{12(r-1)e(H)} \\
&\geqslant \frac{n}{r-1} - 2r^2\sqrt{\gamma}n - \frac{n}{12(r-1)e(H)} \\
&\overset{(27)}{=} t + 24(r-1)e(H)\gamma n + \frac{n}{60(r-1)}.
\end{aligned} \tag{28}$$

In fact during the process of finding edge-disjoint copies of H, only small amount of good vertices becomes inactive. The rough idea will be to find one copy of H after another in $\tilde{G}$. By doing so, we try to decrease the degrees of good vertices rather "uniformly" while embedding more H's into $\tilde{G}$. In this way, it can be ensured that we create no sparse bipartite subgraphs and it will help in our analysis later.

We proceed as follows. We take an edge $e = \{v, u\}$ from $\dot{\bigcup}_{i=1}^{r-1} E_{\tilde{G}}(V_i)$. Note that among v and u there is at most one bad vertex. First suppose that exactly one vertex from $\{v, u\}$ is bad, say v. We find an embedding φ of H into $\tilde{G}$ such that $\varphi(x) = v$, $\varphi(y) = u$. Moreover,

- $\forall z \in V(H) \setminus \{x, y\}$, $\varphi(z)$ is an active and good vertex of some $V_i, i \in [r-1]$,
- for every $e' = \{z_1, z_2\} \neq e$, $\{\varphi(z_1), \varphi(z_2)\} \in E_{\tilde{G}}(V_i, V_j)$ for some $i, j \in [r-1]$, $i \neq j$.

If such an embedding exists, then we delete from $\tilde{G}$ the edges of the corresponding copy of H and repeat until either $\dot{\bigcup}_{i=1}^{r-1} E_{\tilde{G}}(V_i) = \emptyset$ or we cannot find any more copies of H. After any edge deletion we still denote the remaining graph by $\tilde{G}$ and only update the status of good vertices depending on whether they become inactive or not after that step. We need the threshold t mainly for the sake of simpler analysis as we do not impose any order in which we take edges from $\dot{\bigcup}_{i=1}^{r-1} E_{\tilde{G}}(V_i)$.

In the following, we argue that our procedure succeeds at every iteration step. So, let $e = \{v, u\}$ be a current edge chosen at some point of the iteration. Assume without loss of generality that $e \in \tilde{G}[V_i]$ and suppose that v is a bad vertex. Note that whenever an edge incident to v is used, $\deg_{\tilde{G},V_j}(v)(j \neq i)$ reduces by at most s_x and this step happens at most $\deg_{G,V_i}(v)/(2s_x) + 1$ times. Thus, it follows from (20) that

$$\deg_{\tilde{G},V_j}(v) \geqslant \frac{n}{4(r-1)} - \sqrt{\gamma}n - s_x. \tag{29}$$

For a good vertex u, we write a simple lower bound

$$\deg_{\tilde{G},V_j}(u) \geqslant t - s_y \cdot \frac{n}{12(r-1)e(H)} \geqslant t - \frac{n}{12(r-1)}, \tag{30}$$

because after u becomes inactive, we use at most $\deg_{\tilde{G},V_i}(u)$ copies of H that use the vertex u (u was assumed to be from V_i). We define for each $j \in [r-1]$ a vertex set $L_j(u,v)$ as the good and active vertices in V_j that are connected to both v and u. Thus, we have with (17) (the upper bound on $|V_j|$), (29) and (30):

$$\begin{aligned} \left|L_j(u,v)\right| &\geqslant \deg_{\tilde{G},V_j}(u) + \deg_{\tilde{G},V_j}(v) - \left(\frac{n}{r-1} + 2(r-2)\sqrt{\gamma}n\right) \\ &\overset{(30)}{\geqslant} t - \frac{n}{12(r-1)} - \frac{n}{r-1} - 2(r-1)\sqrt{\gamma}n + \frac{n}{4(r-1)} - \sqrt{\gamma}n - s_x \\ &\overset{(27)}{\geqslant} \frac{n}{r-1}\left(\frac{1}{4} - \frac{1}{12} - \frac{1}{60} - \frac{1}{12e(H)}\right) - \left(24(r-1)e(H)\gamma + 4r^2\sqrt{\gamma}\right)n \\ &\geqslant \frac{n}{9(r-1)}. \end{aligned} \tag{31}$$

Moreover, let L_i be the set of current active good vertices in V_i, except u (and v, if v is good). Note that it follows from (28) that initially every good vertex is adjacent to at least

$$t + \frac{n}{60(r-1)} \tag{32}$$

good vertices in any other class. Due to (32), the definition of our threshold t in (27) and in view of the fact that there are at most $\gamma n^2/(e(H)-1)$ steps, the total number of good vertices that become inactive is bounded above by

$$\frac{2\gamma n^2}{n/(60(r-1))} \leqslant 120\gamma(r-1)n. \tag{33}$$

From (33) and (19), we know

$$|L_i| \geqslant |V_i| - 24\gamma(r-1)e(H)n - 120\gamma(r-1)n \overset{(10)}{\geqslant} \frac{n}{2(r-1)}.$$

And therefore we can consider subsets $L_i' \subseteq L_i$, $L_j' \subseteq L_j(u,v)$ for each $j \neq i$ such that

$$\left|L_i'\right|, \left|L_j'\right| \geqslant \frac{n}{10(r-1)}.$$

We let G' denote a graph on the vertex set $L_1' \dot{\cup} L_2' \dot{\cup} \cdots \dot{\cup} L_{r-1}'$, whose edge set is $\dot{\bigcup}_{i \neq j} E_{\tilde{G}}(L_i', L_j')$. Then, G' is an $(r-1)$-partite graph on $n/10$ vertices and it contains at least

$$\binom{r-1}{2}\frac{n^2}{100(r-1)^2} - e(H)\gamma n^2 > \operatorname{ex}(n/10, K_{r-1}) + \frac{n^2}{240(r-1)^2}$$

edges. But then, the theorem of Erdős and Stone [5] implies that G' contains a complete $(r-1)$-partite graph with each part of size $|V(H)|-2$ and therefore there is a copy of H in the subgraph of $\tilde{G}$ induced by the vertices $V(G') \cup \{u,v\}$. This finishes the analysis of an arbitrary iteration step, when the vertex v is bad. The case, where both u and v are good, is analyzed exactly in the same manner except that we do not need (29).

Thus, we have shown that we succeed at finding a copy of H in each iteration. By our choice of $\tilde{G}$, there are at least (cf. (21))

$$\left\lfloor \frac{m_2}{e(H)-1} \right\rfloor + 1$$

edges in $\tilde{G}$ that lie within the classes $V_1, \ldots, V_{r-1}$, and therefore our procedure generates this many edge-disjoint copies of H in $\tilde{G}$. $\square$

Acknowledgments

The authors would like to thank the organizers of the program "Combinatorics: Methods and Applications in Mathematics and Computer Science" held at IPAM. The authors thank Oleg Pikhurko for useful suggestions. They also thank the referees for reading the paper carefully and for helpful comments.

References

[1] B. Bollobás, On complete subgraphs of different orders, Math. Proc. Cambridge Philos. Soc. 79 (1976) 19–24.
[2] H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, Ann. Math. Statistics 23 (1952) 493–507.
[3] P. Erdős, On some new inequalities concerning extremal properties of graphs, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 77–81.
[4] P. Erdős, A.W. Goodman, L. Pósa, The representation of a graph by set intersections, Canad. J. Math. 18 (1966) 106–112.
[5] P. Erdös, A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946) 1087–1091.
[6] O. Pikhurko, T. Sousa, Minimum H-decompositions of graphs, J. Combin. Theory Ser. B 97 (2007) 1041–1055.
[7] N. Pippenger, J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, J. Combin. Theory Ser. A 51 (1) (1989) 24–42.
[8] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 279–319.
[9] T. Sousa, Decompositions of graphs into 5-cycles and other small graphs, Electron. J. Combin. 12 (R49) (2005) 1.
[10] T. Sousa, Decompositions of graphs into a given clique-extension, Ars Combin. C (2011) 465–472.
[11] T. Sousa, Decomposition of graphs into cycles of length seven and single edges, Ars Combin., in press.
[12] E. Szemerédi, Regular partitions of graphs, in: Problèmes combinatoires et théorie des graphes, Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976, in: Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401.