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Application Of Stochastic Optimal Control For The Analysis Of Turkish Foreign Debt

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Rastgele Süreçlerde Eniyileme Yönteminin Türkiye Dış Borcu Analizi için Uygulanması

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Abstract

In this thesis, we apply a method in order to provide a short analysis of the Turkish economy from the period that starts 1980 and ends in 2013. For the analysis, we used the model of Fleming and Stein (2004) that introduced a controlled stochastic net worth process and applied the framework of Merton (1971) in order to obtain an early warning signal for a possible debt crisis. This work, on the other hand, uses the approach of Yener et al. (2014) under the setting of Fleming and Stein (2004) to have a view on the performance of the Turkish economy within the last three decades. Our results show that the Turkish economy is non-resilient since 2005 and this persistent non-resilience is coupled with increasing realized leverage. Therefore, Turkish economy is currently on fragile grounds and policy makers need to take actions that will alleviate the effects of internal and external events to save the economy from a possible debt crisis.

Özet

Tez olarak sunulan bu çalışmada, Türkiye ekonomisinin 1980-2013 aralığını kapsayan kısa bir geçmişinin analizi, belli bir yöntem izlenerek yapılmıştır. Bu analiz çalışması için, Fleming ve Stein (2004) ' ın çalışmasındaki yönteme paralel olarak, net öz sermaye Merton (1971) ' ın çalışması çerçevesinde bir etkin kontrol süreci olarak ele alınmış ve borç krizi için bir erken uyarı sinyali olarak kullanılmaya çalışılmıştır. Öte yandan bu çalışmada, Yener ve arkadaşlarının 2014 yılında yayımladıkları çalışmalarında kullanılan Fleming ve Steinın yöntemi izlenerek, Türkiye ekonomisinin son on yıllık dilimi incelenmektedir. Elde edilen sonuçlar, Türkiye ekonomisinin 2005 yılına kadar borç krizine karşı dirençlilik gösterdiğini, buna karşın, bu tarih sonrasından itibaren borçluluk oranını artırarak dirençsiz bir ekonomi haline dönüştüğünü göstermektedir. Bu nedenle, Türkiye ekonomisi günümüzde kırılgan bir temeldedir ve politikacıların ülkeyi bir borçluluk krizinden kurtarma adına önlemler alarak ekonominin sırtındaki bu yükü hafifletecek önlemler alması gerekmektedir.

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Contents

Chapter 1

Introduction

There is a multitude of economic indicators that are developed in order to analyse the economic situation of countries or companies. Though the list exhaustive, among many, a notable study by Reinhart and Rogoff (2010) shows the weak relationship between growth and debt for those countries with debt to growth domestic product (GDP) ratio below 90%. Such ratio is sensible in view of the recent Eurozone debt crisis; Greece, Italy, Ireland, Portugal and Spain lead their debt/GDP ratio become over 130%. Concordantly, the effect of the leverage of debt became the key concern for the financial authorities and central banks.

In this thesis, rather than concentrating on the debt-to-GDP ratio, we consider a framework that takes capital and debt stock of a sovereign country into consideration. To this end, we analysed the case of one of the major emerging countries, Turkey, by employing the approach of Yener et al. (2014). The approach is developed upon the model of Fleming and Stein (2004) who structured a problem by using the techniques of stochastic optimal control theory. The model accounts the net worth of an economy as a controlled stochastic process which is defined as the difference of capital and debt stocks[1]. In this way, Fleming and Stein (2004) devises a method to see when overly leveraged economics may fall into financial difficulties; if an economy is leverage beyond a maximal amount over the optimal leverage, then, that economy is prone to a possible debt crisis.

Stochastic optimal control was first introduced to economics and finance literature by Merton (1971) who studied the optimal portfolio selection problem in continuous time. In this framework, Merton modeled the portfolio as a controlled stochastic process and found the optimal investment strategy that maximizes a given objective. Thanks to the similarity of Fleming and Stein (2004) model to Merton's stochastically traded portfolio, we then consider his framework in order to solve the problem undertaken in this thesis. As in Merton, we do not consider any market frictions[2].

Our main approach is due to Yener et al. (2014) that also studied the optimal economy as in Fleming and Stein (2004). However, rather than providing a direct approach for finding the growth optimal leverage strategy, we use hyperbolic abolute risk aversion (HARA) utility function as Fleming and Stein (2004) did. Within such consideration, we derive the optimal leverage and consumption strategies, then, show how growth optimal leverage strategy can be derived from the one derived under HARA type utility function. We show explicitly how the derivation may be done by referring to Björk (2004), then apply the optimal results to provide an analysis of Turkish economy. The latter is therefore the major

[1]The stochasticity of the model comes from two uncertainities: the rate of return on capital and the net effective rate of return on net foreign assets.

[2]In the portfolio optimization literature see for example Davis and Norman (1990), Zariphopoulou (1992) and Shreve and Soner (1994) who extended Merton's framework by considering different utility functions and market frictions.

contribution of this thesis.

The approach for the analysis on the other hand involves, as explained in Yener et al. (2014), defining the growth optimal leverage strategy as a limiting case. This, in turn, enables us to show that if actual leverage strategy becomes higher than the optimal one, which is denoted by the growth optimal strategy, then the economy is taking much risk by giving up growth. In other words, if the difference of the actual leverage strategy and the optimal strategy is positive, this means that this economy takes a high level of risk according to our model. It is possible to take excessive risk for an economy if it has a sustainable structure that gauges the strength of the economy. To measure the sustainability, we borrow from the analysis in Browne (1999). Browne showed that, when the growth optimal strategy is followed, the strategy minimizes the expected time if the trend is greater than zero. Otherwise, the optimal solution can only postpone a possible default. Yener et al. (2014) utilize this conclusion to differentiate the health of the economy.

Given the underlying aim of the thesis, we first provide in the next chapter, a brief background of the stochastic optimal control theory. In the third chapter, we solve the main problem and provide a summary of the underlying idea for our analysis. Fourth chapter includes the application to Turkish case with some remarks on the state of this economy. In the final chapter we conclude by providing a summary of our results.

Chapter 2

Stochastic Optimal Control

In this chapter, we will introduce the techniques of stochastic optimal control that is employed in this thesis. The technique is used in many fields such as engineering, physics and mathematics in various forms. Merton (1971) introduced this method to economics and finance literature by using partial differential equations known as Hamilton-Jacobi-Bellman equations to obtain more precise solutions in continuous time portfolio optimization. Here, we will present the technique in its most general form. To this end, we assume N stochastic processes driven by N risk factors and steered by k dimensional control process.

In what follows, we will set up the stochastic optimal control problem and present the solution with theorem and proof by following the outline in Saß (2006) and Björk (2004).

2.1 Setup

We start by fixing a finite horizon via setting $T < \infty$ as the terminal time. We then consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t < \infty}, \mathbb{P})$, where the market filtration is spanned by a N-dimensional standard Brownian motion $W(t) = (W_1(t), \ldots, W_N(t))^{'}, 0 \leq t < \infty$, where $'$ denotes the transpose. Here, $\mathbb{P}$ is the real measure and $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ is the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W := \sigma\{W(s) \mid s \leq t\}$. Furthermore, Ω represents the set of all possible random events while the filtration represents their evolution within the finite time. More on this will follow in the next chapter.

Next, we let $\{X_t^{\mathbf{u}}, 0 \leq t \leq T\}$ be the N-dimensional controlled stochastic process and $\mathbf{u}(t, x) \in \mathcal{U}$ be the k-dimensional control vector, where $\mathcal{U}$ denotes the set of all strategies. The dynamics of the N-dimensional controlled stochastic process are given by the stochastic differential equation

$$\mathrm{d}X_t^{\mathbf{u}} := \mu(t, X_t^{\mathbf{u}}, \mathbf{u})\mathrm{d}t + \sigma(t, X_t^{\mathbf{u}}, \mathbf{u})\mathrm{d}W_t, \tag{2.1}$$

where μ and σ are measurable and represent respectively the drift and deviation functions of $\{X_t, 0 \leq t \leq T\}$ with

$$\mu : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n;$$
$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times d}.$$

We will assume that the above functions are regular enough so that a solution of the above stochastic differential equation exists.

Associated to the above mentioned controlled vector of stochastic processes, the performance criterion considered is given by

$$J(t,x,\mathbf{u}) = \mathbb{E}_{t,x}\Big[\int_0^T F(t,X_t^{\mathbf{u}},\mathbf{u}(t,x))\mathrm{d}t + \Phi(X_T)\Big], \tag{2.2}$$

where $E_{t,x}[\cdot] = E[\cdot|X_t = x]$ for a fixed (t,x), F is the infinitesimal cost function whose value depends on time, the control process and the level of the controlled stochastic process $\{X_t^{\mathbf{u}}, t \geq 0\}$. Moreover, Φ is the legacy function and its value depends on the terminal value of X. Then, it follows that

$$F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R};$$
$$\Phi : \mathbb{R}^n \to \mathbb{R}.$$

Note that, for the problem considered in the thesis, we may interpret the first part of the above statement as the utility accumulated in interval $[t,T]$ and the second part as the utility derived from the terminal value of X. Our aim is to find a control process $\mathbf{u}^*$ that maximizes[1] the performance criterion,

$$V(t,x) = \sup_{\mathbf{u}\in\tilde{\mathcal{U}}(x)} J(t,x,\mathbf{u}) \tag{2.3}$$

where $\tilde{\mathcal{U}}$ is the set of admissible strategies which is defined as

$$\tilde{\mathcal{U}}(t,x) := \Big\{\mathbf{u} \in \mathcal{U} \,\Big|\, |J(t,x,\mathbf{u})| < \infty\Big\}. \tag{2.4}$$

[1]Depending on the problem, the aim might as well be the minimization of the performance criterion.

Hence, our aim is to find $\mathbf{u}^*$ for which $V(t,x)$ is attained which can be expressed mathematically as

$$\mathbf{u}^* := \arg \sup_{\mathbf{u}\in\hat{\mathcal{U}}(t,x)} J(t,x,\mathbf{u}). \tag{2.5}$$

In sum, the aim of the stochastic control problem is to find an optimal control strategy $\mathbf{u}^*$ which controls X and eventually enables us to reach optimal value function, $V(t,x)$. In the following section we will present the solution to the problem defined in (2.3) and show the conditions in which a unique solution exists.

2.2 Hamilton-Jacobi-Bellman Equation

To obtain a solution to the specified problem in the previous section, we will utilize a special type of equation, known as the Hamilton-Jacobi-Bellman (HJB, hereafter) Equation. This equation was proposed by Bellman (1954) and is widely applied in various scientific fields. For the problem we consider in the thesis, we refer the readers to Merton (1971) on the application in the continuous time portfolio optimization literature and to Fleming and Stein (2004) on the application to a debt crisis prediction model. As we mentioned previously, we refer to Björk (2004) and Saß (2006) in addition to the aforementioned references.

In sequel, we will use $\mathbf{u}$ instead of $\mathbf{u}(t,x)$ when necessary and provide the definition of the following operators that will ease the expressions given throughout this section.

Definition 2.1. For any fixed vector $\mathbf{u}$ at point (t',x'), the second order partial differential operator $\mathcal{L}^{\mathbf{u}}$ is defined by,

$$\mathcal{L}^{\mathbf{u}} = \sum_{i=1}^{n} \mu_i^{\mathbf{u}}(t,x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C_{i,j}^{\mathbf{u}}(t,x) \frac{\partial^2}{\partial x_i x_j} \tag{2.6}$$

where the functions μ^u, σ^u and C^u are

$$\begin{aligned} \mu^{\mathbf{u}}(t,x) &= \mu(t,x,\mathbf{u}), \\ \sigma^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}), \\ C^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u})\sigma(t,x,\mathbf{u})', \end{aligned}$$

and $'$ on the right-hand side of the final equation above denotes the transpose.

Before proceeding to the main theorem of this section, we first provide the list of assumptions so that the HJB equation can have a solution.

Assumption 2.1. The standing assumptions are:

- There exists an optimal control law $\mathbf{u}^*$;
- The optimal value function V is regular in the sense that $V \in C^{1,2}$.

Theorem 2.1. *Under assumption 2.1, V satisfies the HJB equation*

$$\frac{\partial}{\partial t} V(t,x) + \sup_{\boldsymbol{u} \in \mathfrak{U}} \{F(t,x,\boldsymbol{u}) + \mathcal{L}^{\boldsymbol{u}} V(t,x)\} = 0, \quad \forall (t,x) \in (0,T) \times R^N,$$

with the boundary condition

$$\Phi(x) = V(T,x), \quad \forall x \in R^N.$$

Moreover, for each $(t,x) \in [0,T] \times R^N$ *the supremum above is attained by* $\boldsymbol{u} = \boldsymbol{u}^*(t,x)$.

For the proof, we emphasize the idea that the optimal control strategy relies on the dynamic programming principle which aims to find a control strategy responded by a feedback for each (t,x). Hence, if $\mathbf{u}^*$ is an optimal strategy on $[t,T]$, this implies that it is also the optimal strategy on every sub-interval for $t \leq s \leq T$.

2.2.1 Proof of Hamilton Jacobi Bellman Equation

For the outline of the proof, we will follow Björk (2004). To obtain the HJB equation, we will compare success of two associated strategies. The first one is the optimal control law, $\mathbf{u}^*$, which is assumed to be obtained, and the latter is a composition of the form

$$\mathbf{u}(v,l) = \begin{cases} \hat{\mathbf{u}}(v,l), & \text{for } (v,l) \in [t,t+p] \times R^N; \\ \mathbf{u}^*(v,l), & \text{for } (v,l) \in (t+p,T] \times R^N, \end{cases}$$

where $\hat{\mathbf{u}}$ is a control strategy whose value is given by the arbitrary control strategy $\mathbf{u}$ within the time interval at or before time $t+p$. On the other hand, for the time period after $t+p$, the control strategy becomes the optimal control strategy $\mathbf{u}^*$ until the terminal time T.

To obtain the HJB equality, we will compare expected utilities for both strategies. To this end, we will use the fact that $\hat{\mathbf{u}}$ is as successful as $\mathbf{u}^*$ and obtain an equality when $p \downarrow 0$.

With the **first strategy**, we have

$$J(t, x, \mathbf{u}^*) = V(t, x)$$

by definition. For the **second strategy**, we will examine each cases of $\hat{\mathbf{u}}$ separately. So, the conditional expected utility of $\hat{\mathbf{u}}$ for $[t, t+p]$ is,

$$E_{t,x}\left[\int_t^{t+p} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)\mathrm{d}s\right].$$

Note that there is no need for the legacy function $\Phi(\cdot)$ since the time interval is limited with $[t, t+p]$ for the first case. For the second case (when $\hat{\mathbf{u}} = \mathbf{u}^*$), it is trivial that conditional expected utility is $V(t+p, X_{t+p}^{\mathbf{u}^*})$. Then, the total conditional expected utility is

$$E_{t,x}\left[\int_t^{t+p} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)\mathrm{d}s + V(t+p, X_{t+p}^{\mathbf{u}})\right]. \tag{2.7}$$

Now, since the first strategy is at least as successful as the second, we have the following inequality,

$$V(t, x) \geq E_{t,x}\left[\int_t^{t+p} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)\mathrm{d}s + V\left(t+p, X_{t+p}^{\mathbf{u}}\right)\right]. \tag{2.8}$$

Here, the inequality comes from the optimality of the first strategy and the arbitrariness of the $\mathbf{u}$ in the second, so it will cease if $\mathbf{u} = \mathbf{u}^*$ is chosen. To proceed, we write by the application of Itô's formula

$$V(t+p, X^{\mathbf{u}}_{t+p}) = V(t,x) + \int_t^{t+p} \left\{ \frac{\partial V}{\partial t}(s, X^{\mathbf{u}}_s) + \mathcal{L}^{\mathbf{u}} V(s, X^{\mathbf{u}}_s) \right\} \mathrm{d}s$$
$$+ \int_t^{t+p} \frac{\partial}{\partial x} V(s, X^{\mathbf{u}}_s) \sigma^{\mathbf{u}} \mathrm{d}W_s. \tag{2.9}$$

By replacing the above into (2.8), we obtain,

$$E_{t,x}\left[\int_t^{t+p} \left[F(s, X^{\mathbf{u}}_s, \mathbf{u}_s) + \frac{\partial V}{\partial t}(s, X^{\mathbf{u}}_s) + \mathcal{L}^{\mathbf{u}} V(s, X^{\mathbf{u}}_s) \right] \mathrm{d}s \right] \leq 0. \tag{2.10}$$

Note that the conditional expectation of the stochastic integral in (2.9) vanishes if it's a martingale. That is,

$$E_{t,x}\left[\int_t^{t+p} \left\| \frac{\partial}{\partial x} V(s, X^{\mathbf{u}}_s) \sigma^{\mathbf{u}} \right\|^2 \mathrm{d}s \right] < \infty$$

Then, it follows by taking the limit $p \downarrow 0$ that

$$F(t,x,u) + \frac{\partial V}{\partial t}(t,x) + \mathcal{L}^{\mathbf{u}} V(t,x) \leq 0, \tag{2.11}$$

where x and u are fixed vectors at t so that $x = X_t$ and $u = \mathbf{u}(t,x)$. As we mentioned above, the equality is applicable if u is optimal. Finally, we obtain HJB partial differential equation,

$$\frac{\partial V}{\partial t}(t,x) + \sup_{\mathbf{u} \in \mathcal{U}} \{ F(t,x,u) + \mathcal{L}^u V(t,x) \} = 0. \tag{2.12}$$

As mentioned in Björk (2004), the above holds $\forall (t,x) \in (0,T) \times R^N$ since the point, (t,x) was fixed arbitrarily. So we need a boundary condition which obviously is

$V(T, x) = \Phi(x)$. The condition with the equation yields us the HJB equation.

2.3 Verification Theorem

The theorem above shows that under given conditions $V(t, x)$ satisfies the HJB equation and the supremum is attained with the maximizing strategy, $\mathbf{u}^*$. Here, we will verify that the HJB equation is sufficient to obtain the optimal solution. In particular, we will show that if the maximizing strategy is admissible, then $V(t, x)$ is the optimal solution and $\mathbf{u}^*$ is the optimal control strategy. The following theorem details this feature.

Theorem 2.2. *Suppose $G(t, x) \in C^{1,2}$ with $G' > 0$, $G'' < 0$, besides $G(t, x)$ is dominated by $k \int_0^t (1 + (X_s^{\boldsymbol{u}})^p) \mathrm{d}s$ for $k > 0, p \geq 2$. That $G(t, x)$ satisfies the HJB equation with its boundary condition, i.e. $\Phi(T, x) = G(T, x)$ and*

$$\frac{\partial G}{\partial t}(t, x) + \sup_{\boldsymbol{u} \in \mathcal{U}} \{F(t, x, u) + \mathcal{L}^{\boldsymbol{u}} G(t, x)\} = 0,$$

and the supremum in the above expression is attained by $u = w(t, x)$. Then,

(i) $G(t, x) = V(t, x)$ is the optimal value function,

(ii) $\boldsymbol{u}^* = w(t, x)$ is the optimal control law.

2.3.1 Proof of Verification Theorem

Fix $(t, x) \in [0, T] \times \mathbb{R}^n$ arbitrarily. We first introduce,

$$\tau_n = T \wedge \inf\{s > t : \|X_s - X_t\| \geq n\}, \quad n \in \mathbb{N}.$$

For a fixed (t, x) we then write by the application of Itô's Formula, for an admissible $\mathbf{u}$,

$$G(\tau_n, X_{\tau_n}) = G(t, x) + \int_t^{\tau_n} \left\{ \frac{\partial G}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{L}^{\mathbf{u}} G(s, X_s^{\mathbf{u}}) \right\} \mathrm{d}s + \int_t^{\tau_n} \frac{\partial}{\partial x} G(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) \mathrm{d}W_s. \tag{2.13}$$

Now consider the function,

$$E_{t,x} \left[\int_t^{\tau_n} F(s, X_s^{\mathbf{u}}, u_s) \mathrm{d}s + G(\tau_n, X_{\tau_n}^{\mathbf{u}}) \right].$$

Since u is admissible, $G(t, x)$ is continuous, X is bounded on $[t, \tau_n]$, we have from the stochastic integral above

$$E_{t,x} \left[\int_t^{\tau_n} \left\| \frac{\partial}{\partial x} G(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) \right\|^2 \mathrm{d}s \right] < \infty.$$

Therefore, the conditional expectation of the stochastic integral will yield zero. Then, we can rewrite the above expression by substituting $G(\tau_n, X_{\tau_n}^{\mathbf{u}})$ with (2.13) as

$$G(t, X_t) + E_{t,x} \left[\int_t^{\tau_n} \left(F(s, X_s^{\mathbf{u}}, u_s) + \frac{\partial}{\partial t} G(s, X_s^{\mathbf{u}}) + \mathcal{L}^{\mathbf{u}} G(s, X_s^{\mathbf{u}}) \right) \mathrm{d}s \right].$$

Furthermore, for every $\mathbf{u} \in \mathbb{R}^k$ and for each s, the inequality

$$F(s, X_s^{\mathbf{u}}, u_s) + \frac{\partial}{\partial t} G(s, X_s^{\mathbf{u}}) + \mathcal{L}^{\mathbf{u}} G(s, X_s^{\mathbf{u}}) \leq 0 \tag{2.14}$$

holds $\mathbb{P}$-almost surely. As a result, we obtain

$$G(t,x) \geq E_{t,x}\left[\int_t^{\tau_n} F(s,X_s,u_s)\mathrm{d}s + G(\tau_n, X_{\tau_n})\right].$$

As $n \to \infty, \tau_n \to T$, we have from the condition provided the theorem, thus by the dominated convergence theorem,

$$E_{t,x}\left[\int_t^{\tau_n} F(s,X_s,u_s)\mathrm{d}s + G(\tau_n, X_{\tau_n})\right] \to J(t,x,u).$$

By combining together, we have $G(t,x) \geq J(t,x,u)\ \forall u \in \mathfrak{U}$. Hence,

$$G(t,x) \geq V(t,x). \tag{2.15}$$

On the other hand, with the optimal control law $\mathbf{u}^*$ we can write, $G(t,x) = J(t,x,\mathbf{u}^*)$ and since $V(t,x) = \sup_{\mathbf{u}\in\mathfrak{U}} J(t,x,\mathbf{u})$,

$$V(t,x) \geq J(t,x,\mathbf{w}) = G(t,x). \tag{2.16}$$

Hence, from Equations 2.15 and 2.16 we can conclude that $V(t,x) = G(t,x)$. Then, it follows that $\mathbf{u}^*(t, X_t^{\mathbf{u}^*})$ is the optimal control strategy and $X_t^{\mathbf{u}^*}$ is the optimal portfolio process.

Chapter 3

Analysis of Debt Capital Structure of a Virtual Country

In this chapter, we will define and introduce the model of Fleming and Stein (2004). As we will see in what follows, the model is a controlled stochastic process that resembles to the process that were introduced in the previous chapter. Furthermore, the model is used to capture the evolution of the net worth of an economy and the aim is to find the optimal consumption and leverage that fulfil a certain objective given as the maximization of a utility function.

As evident from the aforementioned, we designate the leverage strategy as the control law and net worth $X(t)$ as the controlled process. To obtain the solutions, same procedure as in the previous chapter is followed. The utility function is on the other hand is Hyperbolic Absolute Risk Aversion (HARA) type utility. In particular, the objective is to find a unique consumption and leverage strategies that maximize the utility from consumption and terminal net worth.

3.1 The Economy

We consider infinite horizon economy that is modelled under a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbb{P})$ with infinite time horizon. The uncertainty is modelled by two dimensional Brownian Motion $W(t) := (W_1(t), W_2(t))'$, where $W_1(\cdot)$ and $W_2(\cdot)$ are two correlated Brownian motions. We denote the correlated case by

$$\mathrm{d}W_1(t) \cdot \mathrm{d}W_2(t) = \rho \mathrm{d}t \quad \text{where } |\rho| < 1 \text{ is constant.}$$

In the above notation for the probability space, Ω states all possible events in the economy and $\mathbb{P}$ is the measure of these events in the probability space. Furthermore, we denote by $\{\mathcal{F}_{0\leq t<\infty}\}$ the filtration which is the $\mathbb{P}$-augmentation of the natural filtration and it is defined as $\mathcal{F}_t^w := \sigma\{W(u)|u \leq t\}$. The filtration is used to model the evolution of the uncertainty in the economy, while $\mathcal{F}_t$, which is the σ-algebra[1], gives the status of the economy.

Given the model of uncertainty for the economy, we then use the Brownian motions to model the uncertainties arising from the return on debt and return on capital in an economy. To this end, we write

$$\begin{aligned} r(t)\mathrm{d}t &= r_t \mathrm{d}t + \sigma_1 \mathrm{d}W_1(t); \\ b(t)\mathrm{d}t &= b_t \mathrm{d}t + \sigma_2 \mathrm{d}W_2(t), \end{aligned} \tag{3.1}$$

where r_t is the deterministic mean rate of return on debt and $b_t > 0$ is the deterministic mean rate of return on capital. Additionally, we assume that the deviations

[1]The collection of the subsets of all possible events.

both processes are non-negative and constant scalars, $\sigma_1 > 0$ and $\sigma_2 > 0$.

In our economy, the capital stock $K(.)$ is assumed to be the product of quantity and price, which are respectively denoted by $N(\cdot)$ and $P(\cdot)$. That is, the equation for capital stock at time t is given by

$$K(t) = N(t) \cdot P(t).$$

By the product rule, the change in capital stock is then given by

$$\begin{aligned} \mathrm{d}K(t) &= N(t)\mathrm{d}P(t) + P(t)\mathrm{d}N(t) \\ &= K(t)\frac{\mathrm{d}P(t)}{P(t)} + I(t)\mathrm{d}t, \quad P(t) > 0, \end{aligned} \tag{3.2}$$

where $I(t)$ states the investment rate and $\{\mathrm{d}P(t)/P(t), t \geq 0\}$ is the stochastic capital gain. Moreover, GDP, denoted by $Y(t)$, is considered to be the product of productivity rate and capital stock

$$Y(t)\mathrm{d}t = \alpha(t)K(t)\mathrm{d}t, \tag{3.3}$$

where $\{\alpha(t), t > 0\}$ is the stochastic productivity rate. Then, we specify the debt value as $L(t)$ and the consumption as $C(t) > 0$. With these, the debt balance for this economic model is written as pointed out in **?**,

$$\mathrm{d}L(t) = [r(t)L(t) + I(t) + C(t) - Y(t)]\,\mathrm{d}t, \tag{3.4}$$

showing that the debt balance of an economy can be described as the net foreign assets. Mainly, $\mathrm{d}L(t)$ can be described as the current account deficit (or surplus) of an economy. Here, $r(t)$ denotes the time t net effective rate of return which includes dividends and interest. Therefore, the product $r(t)L(t)$ is the net transfer payments. Then, the sum $I(t) + C(t) - Y(t)$ is the trade balance. Also note that, when $L(t) > 0$ an economy borrows from the rest of the world and when $L(t) < 0$ it lends to the rest of the world. Furthermore, $C(t) = c_t X(t)$ denotes the level of consumption at time t and $\{c_t, t \geq 0\}$ is the proportional consumption rate.

Given the capital stock and debt equations we define the net worth process as

$$X(t) = K(t) - L(t), \quad X(t) > 0. \tag{3.5}$$

Furthermore, for the existence of further derivations, we should assume that b_t, r_t and c_t satisfy

$$\int_0^t (|r_s| + |b_s| + |c_s|)\mathrm{d}s < \infty, \text{ for } t < \infty. \tag{3.6}$$

Now, to obtain the rate of change in $X(t)$, we combine (3.2) with (3.4) and write the change in net worth as,

$$\begin{aligned} \mathrm{d}X(t) &= \mathrm{d}K(t) - \mathrm{d}L(t) \\ &= K(t)\left[\alpha(t) + \frac{\mathrm{d}P(t)}{P(t)}\right] - r_t L(t)\mathrm{d}t - \sigma_1 L(t)\mathrm{d}W_1(t) - C(t)\mathrm{d}t. \end{aligned} \tag{3.7}$$

From above, we observe that the drift of the net worth process is in positive relationship with increasing increasing productivity and capital gain, while decreasing with the net transfer payments and consumption. That is, the model does capture what is beneficial to an economy while penalizes those that cause cash outflow. In addition, we observe that the rate of investment $I(\cdot)$ is not present in the above expression; both capital stock and debt stock is increasing in $I(\cdot)$ (see equations (3.2) and (3.4)), and therefore the net effect is an offsetting one as increase in capital is nullified with the increase in the debt stock.

Finally, we see in (3.7) that the change in $X(t)$ over $K(t)$ is determined by $\{\alpha(t) + \mathrm{d}P(t)/P(t), t \geq 0\}$ and is actually the effective rate of return on capital. Therefore, we use $b(t)$ that was previously defined in (3.1) to account for that change. Thus, by substituting $b(t)$ and $K(t)$ with (3.1) and (3.5) respectively, we can rewrite (3.7) as,

$$\begin{aligned} \mathrm{d}X(t) = &[b_t X(t) + (b_t - r_t)L(t) - C(t)]\mathrm{d}t + [X(t) + L(t)]\sigma_2 \mathrm{d}W_2(t) \\ &- \sigma_1 L(t)\mathrm{d}W_1(t), \end{aligned} \tag{3.8}$$

a more suitable expression for the change in the net worth of an economy.

The evolution of the net worth process will be controlled via a leverage strategy that is defined by $f_t := L(t)/X(t)$[2]. Note that from $k_t := K(t)/X(t)$, we obtain $k_t = 1 + f_t$ showing that controlling the net worth process via leverage is equivalent to controlling it via capital investment strategy. Note that leverage must be done within reasonable limits and its value must depend on the observed state of the net worth. Therefore, we say that a leverage strategy is admissible for an initial net worth amount x, that is, $f_t \in \mathcal{A}(x)$, if f_t is $\{\mathcal{F}_t\}$-progressively measurable and satisfies $\int_0^t f_s^2 \mathrm{d}s < \infty$ for $t < \infty$. Then, the net worth process associated with an

[2]We assume no frictions. That is, the change in leverage is done costlessly.

admissible leverage strategy satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= \left[(b_t - c_t) + (b_t - r_t)f_t\right] X(t)\mathrm{d}t - f_t X(t)\sigma_1 \mathrm{d}W_1(t) \\ &\quad + (1+f_t)X(t)\sigma_2 \mathrm{d}W_2(t). \end{aligned} \tag{3.9}$$

Since we have $K(t) > 0$ and $X(t) > 0$ $\forall t < \infty$, then $f_t > -1$ provided that $k_t = 1 + f_t$[3]. That is, the capital stock of an economy must never be allowed to be sold short as that economy must always have a certain level of capital in order to operate. Moreover, we have have $X(\cdot) > 0$ as borrowing must not be allowed to be financed with further borrowing. In this way, a Ponzi scheme will be avoided.

In fact, from the closed form solution of (3.9), we see that

$$\begin{aligned} X(t) &= x \exp\left\{ \int_0^t \left[(b_s - c_s) + f_s(b_s - r_s) - \frac{1}{2}\left(f_s\sigma_1^2 + (1+f_s)^2\sigma_2^2\right) \right.\right. \\ &\quad \left.\left. - f_s(1+f_s)\rho\sigma_1\sigma_2 \right] \mathrm{d}s - \int_0^t f_s\sigma_1 \mathrm{d}W_1(s) + \int_0^t (1+f_s)\sigma_2 \mathrm{d}W_2(s) \right\} \end{aligned} \tag{3.10}$$

is always lager than zero for $t < \infty$. In this way, a Ponzi scheme is avoided. The next goal is then to solve a maximization problem subject to such a process.

3.2 The Problem

A method to find the optimal consumption and leverage of an economy entails solving a maximization problem under dynamic programming principle (see **?**).

[3]To account for this constraint we select a lower leverage level $\zeta \in (-1, 0)$ and set $f_t = \zeta$ if its unconstrained value is ever below -1.

Our aim is to obtain the optimal control variables via maximizing a utility function $U : (0, \infty) \to (-\infty, \infty)$ which is assume to be concave, non-decreasing, and continuously differentiable, and satisfies

$$\begin{aligned} U^{'}(0) &:= \lim_{x \downarrow 0} U^{'}(x) = \infty; \\ U^{'}(\infty) &:= \lim_{x \to \infty} U^{'}(x) = 0. \end{aligned} \tag{3.11}$$

For the problem, as in **?**, we consider Hyperbolic Absolute Risk Aversion (HARA, hereafter) type utility function defined as

$$U\big(C(t)\big) = \frac{1}{\gamma} C^{\gamma}(t). \tag{3.12}$$

Our objective is then given by

$$V(X) = \sup_{f,c \in \mathcal{A}} \mathbb{E}_x \left[\int_0^{\infty} \frac{1}{\gamma} C^{\gamma}(t) e^{-\delta t} dt \right], \tag{3.13}$$

where $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot \mid X(0) = x]$, the risk aversion parameter γ satisfies $\gamma < 1$, $\gamma \neq 0$ and $\delta > 0$ is the subjective discount factor. The associated HJB equation that the value function satisfies is given by

$$\begin{aligned} \sup_{f,c} \Big\{ & - V_t \frac{1}{\gamma} C^{\gamma}(t) + [(b-c) + (b-r)f] x V_x \\ & + \frac{1}{2} [f^2 \sigma_1^2 + (1+f)^2 \sigma_2^2 - 2f(1+f)\rho \sigma_1 \sigma_2] x^2 V_x x \Big\} = 0. \end{aligned} \tag{3.14}$$

Thus, by taking the derivative with respect to f and c, maximizing consumption and leverage strategies are obtained as,

$$\begin{aligned} c^*(x) &= x^{-1}V_x^{1/(1-\gamma)}; && (3.15)\\ f^*(x) &= -\frac{b-r}{\sigma}\frac{V_x}{xV_{xx}} + \alpha(\rho\kappa - 1), && (3.16) \end{aligned}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, $\alpha = \sigma_2^2/\sigma^2$, and $\kappa = \sigma_1/\sigma_2$.

Next, by substituting c^* and f^* into (3.14), we obtain,

$$\Bigg\{ -\delta V + \frac{1}{\gamma}V_x^{-\gamma/(1-\gamma)} + bxV_x + (b-r)\alpha(\rho\kappa-1)xV_x - \frac{1}{2}\frac{(b-r)^2}{\sigma^2}\frac{V_x^2}{V_{xx}} - \frac{1}{2}[\sigma^2(\alpha(\rho\kappa-1)^2 - \sigma_b^2]x^2V_{xx} \Bigg\} = 0. \quad (3.17)$$

Furthermore, to find the explicit values of f^* and c^*, we will guess a solution for V in the form $V(x) = M\frac{X^\gamma}{\gamma}$, where M is a constant. Thus, we have

$$\begin{aligned} V_x &= Mx^{\gamma-1},\\ V_{xx} &= M(\gamma-1)x^{\gamma-2}. && (3.18) \end{aligned}$$

After substituting V_x and V_{xx} into (3.17), we obtain,

$$\Bigg\{ -\delta M\frac{x^\gamma}{\gamma} + \frac{1}{\gamma}(Mx^{\gamma-1})^{-\gamma/(1-\gamma)} + bx(Mx^{\gamma-1}) + (b-r)\alpha(\rho\kappa-1)x(Mx^{\gamma-1}) - \frac{1}{2}\frac{(b-r)^2}{\sigma^2}\frac{(Mx^{\gamma-1})^2}{(M(\gamma-1)x^{\gamma-2})} - \frac{1}{2}[\sigma^2(\alpha(\rho\kappa-1)^2 - \sigma_b^2]x^2(M(\gamma-1)x^{\gamma-2}) \Bigg\} = 0. \quad (3.19)$$

For ease in notation, we define

$$Z := \left[-\delta + b + (b-r)\alpha(\rho\kappa - 1)\gamma - \frac{1}{2}\frac{(b-r)^2}{\sigma^2}\frac{\gamma}{\gamma-1} - \frac{1}{2}[\sigma^2(\alpha(\rho\kappa-1)^2 - \sigma_b^2]\gamma(\gamma-1) \right]$$

and thus,

$$M = [-Z^{-1}]^{1-\gamma}. \tag{3.20}$$

Finally, the optimal control processes are obtained as,

$$c^*(x) = \left([-Z^{-1}]^{1-\gamma} x^{-\gamma}\right)^{-1/(1-\gamma)}; \tag{3.21}$$

$$f^*(x) = \frac{1}{1-\gamma}\frac{b-r}{\sigma^2} + \alpha(\rho\kappa - 1). \tag{3.22}$$

3.3 The Growth Optimal Economy

As we see from the optimal results, we observe that the optimal leverage strategy is independent of the rate of consumption. However, it is dependent on the main factors of the economy and the risk aversion parameter. Our next goal is then to show how such leverage strategy is related to the growth optimal leverage strategy. To this end, we consider the logarithmic specification of $X(\cdot)$ given in equation (3.10). Then, by the application of Itô's formula we obtain

$$\begin{aligned} \mathrm{d}\left(\log(X(t))\right) = {} & [(b_t - c_t) + (b_t + r_t)f_t - \frac{1}{2}f_t^2\sigma_1^2 + (1+f_t)^2\sigma_2^2 \\ & - 2f_t\left(1+f_t\right)\rho\sigma_1\sigma_2]\mathrm{d}t - f_t\sigma_1\mathrm{d}W_1(t) + \left(1+f_t\right)\sigma_2\mathrm{d}W_2(t). \end{aligned} \tag{3.23}$$

Given the specification above, the growth optimal leverage startegy f^* is the solution of the following optimization problem:

$$f^* = \arg\sup \left[(b_t - c_t) + (b_t - r_t) f_t - \frac{1}{2} f_t^2 \sigma_1^2 + (1 + f_t)^2 \sigma_2^2 - 2 f_t (1 + f_t) \rho \sigma_1 \sigma_2 \right]. \tag{3.24}$$

From above, the growth optimal leverage ratio which is given by

$$f^* = \frac{b_t - r_t}{\sigma^2} + \alpha(\rho\kappa - 1). \tag{3.25}$$

Therefore, (3.22) with $\gamma = 0$ gives the growth optimal leverage strategy. For the last step, we substitute (3.25) to (**??**) to find the growth optimal net worth process. Then, it follows that

$$\begin{aligned} X^*(t) = x \exp \Bigg\{ & \int_0^t \beta_s \mathrm{d}s - \int_0^t \left[\frac{b_s - r_s}{\sigma^2} + \zeta(\rho\Psi - 1) \right] \sigma_1 \mathrm{d}W_1(s) \\ & + \int_0^t \left[1 + \frac{b_s - r_s}{\sigma^2} + \zeta(\rho\Psi - 1) \right] \sigma_2 \mathrm{d}W_2(s) \Bigg\}, \end{aligned} \tag{3.26}$$

where

$$\beta_s = (b_s - c_s) - \frac{1}{2}\sigma_2^2 + \frac{1}{2}\left[\frac{b_s - r_s}{\sigma} + \sigma(\alpha(\rho\kappa - 1)) \right]^2. \tag{3.27}$$

Remark 3.3.1. *In our analysis, we let the consumption rate be given exogenously. In this regards, consumption is not a control strategy, and we have the restriction that $c(t) > 0$. On the other hand, we will focus on whether if the economy follows the growth optimal strategy, i.e. $f(t) - f^*(t)$ is relatively small. Here, we will borrow the resilience notion introduced in Yener et al. (2014) to differentiate the* *state of the economy. In particular, we will refer the state of* $\beta < 0$ *and* $f - f^* > 0$ *as a warning sign since the the economy depreciates in expectation and the country borrows more than its optimal leverage rate. For further details regarding the underlying idea please refer to Yener et al. (2014).*

Chapter 4

Application For Turkey

In this chapter, we will investigate the optimization problem of debt leverage ratio for the Turkish Case and analyse the form of the economy. We first provide the background for the data and the estimation procedures.

4.1 The Data & Estimation

The data used in this work is for the yearly periods that extend between 1980 and 2013. As mentioned previous the key variables in the study are:

1- Capital Stock;

2- Debt Stock;

3- GDP;

4- Investment;

5- Consumption.

For the capital stock, we retrieve the data from the Fred Economic Data website that provides the estimates of **?**. On the other hand, for 2-5 above, we retrieve the data from World Bank's World Development Indicator database. All data is denominated in US dollars. For the debt stock data, we use the total external debt stock values of Turkey as a proxy. On the other hand, both investment and consumption data is calculated by taking the percentage values of Turkish GDP (based on constant 2005 USD). Furthermore, the end of the observation period for the capital stock data (based on constant 2005 USD) is 2011. To calculate the capital stock in years 2012 and 2013, we added the investment rate $I(t)$ to the previous year's capital stock value. Finally, to ensure that the optimal leverage strategy does not exceed the lower boundary value (which is -1), we consider a coefficient θ and set its value by $\theta = -0.99$. In view of this, we adjust our findings accordingly. See for example Yener et al. (2015).

Because we are working with yearly data, we need to provide the estimation of the return on capital and the net effective rate of return on debt according. To this end, we consider the approach provided in Stein (2007) and Yener et al. (2014) and write

$$b(t) = \frac{\Delta Y(t)}{\Delta K(t)}; \quad r(t) = \frac{\Delta L(t) - C(t+1) - I(t+1) + Y(t+1)}{L(t)},$$

where, $\Delta H(t) = H(t+1) - H(t)$ is the difference operator for some function H suitable for our purposes. Furthermore, the deterministic part of the data, r_t, b_t and c_t are calculated by taking the three year moving averages of the related data.

After collecting the data, we then compute the growth optimal strategy f^* and the actual leverage strategy f. Then, β is calculated to estimate the resilience of the country. By using these two measure, we then provide an analysis of the Turkish economy.

4.2 The Analysis of Realized Leverage of Turkey

In the Figure 4.1, we observe that Turkey is a debtor country for the whole observation period. Mainly, the realized leverage of Turkey increases dramatically from 1980 to 2008.

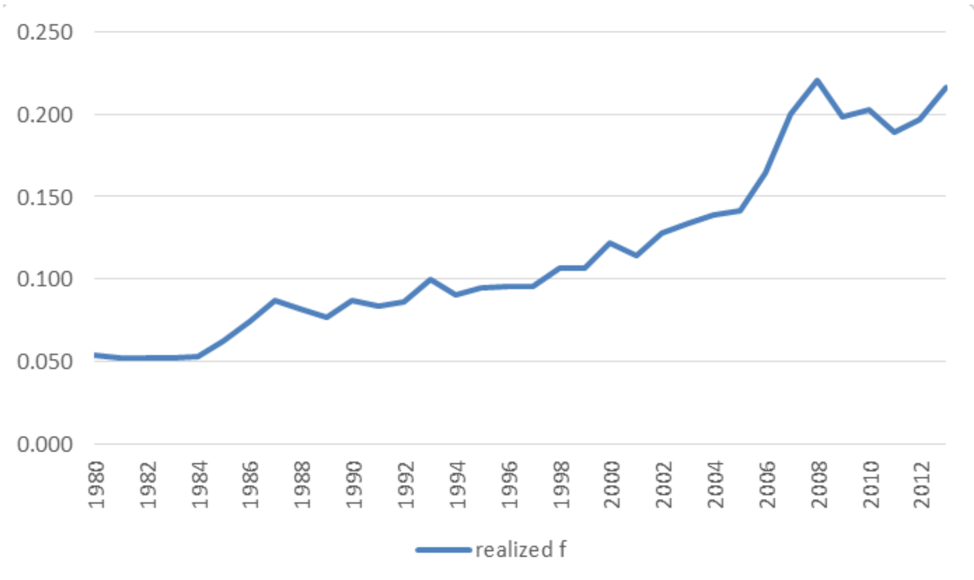

Figure 4.1: Realized Leverage

While, the increase was in a steady trend until 2005, it became much steeper after 2005, until 2008, a time that coincides with the global financial crisis; the realized leverage that was around 5% of Turkey's net worth became close to 22% in around year 2008. After that period, the realized leverage of Turkey declined to around 19% in year 2011, the increase back above 20% at the end of the observation period. It seems currently that the leverage ratio of Turkey is way above the levels it had encountered in its history.

To see whether such increase poses threats to the sustainably growth of Turkey, we then move to the next section to provide an analysis by using the non-resilience condition introduced in Yener et al. (2014)

4.3 The Analysis of Turkish Economy

We observe the time series paths of β and $f - f*$ in Figure 4.2. The solid (blue) line represents β, while the dashed line represents $f - f*$.

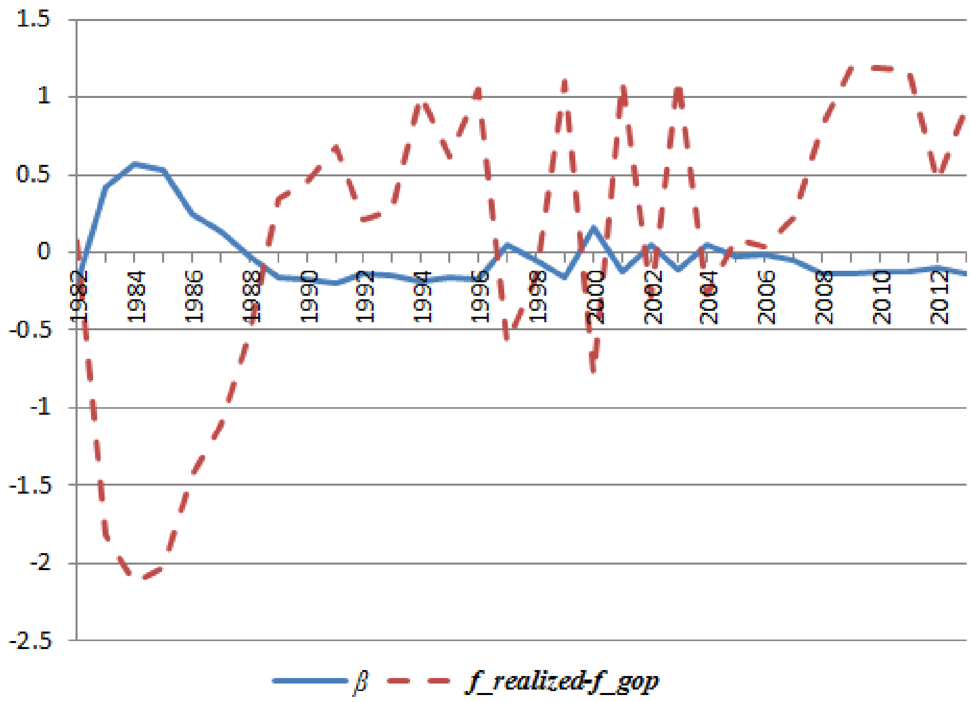

Figure 4.2: β and f-f*

We see that the Turkish economy became non-resilient in 1988 and the condition was persistent until 1997, a period when Turkish economy had economic hardships (i.e. the Turkish economic crisis in 1994) and had to ask the assistance of IMF . Within the following years, the time series paths of β and $f - f*$ followed a zigzagged pattern. Especially, the economy became non-resilient in years 1999, 2001 and 2003, the years that coincide with a major earth quake and a full blown financial crisis. However,in 2003, Turkey had considerably high GDP growth (5.27%) relative to its long-run GDP growth rate, although the economy is shown to be non-resilient during that time. Therefore, despite the high GDP growth in 2003, the country was non-resilient. But the condition was short-lived and Turkey became resilient again in 2004. Nevertheless, the resilience was also short-lived and since 2005 Turkish economy is non-resilient; hence the persistent non-resilience condition. In addition, its realized leverage levels increased to historically high levels. Within this view, Turkey therefore need to take precautionary measure in order to prevent a possible debt crisis that might be sparked due to external and/or internal financial and political events.

Chapter 5

Conclusion

In this thesis, we investigated the validity of the leverage strategy of Turkish economy under the view of Yener et al. (2014). First we modelled the economy by following Fleming and Stein (2004) and obtained the net worth process which is the controlled stochastic process of our dynamic programming problem. Given the model, we then solved the problem defined as a HARA type utility maximization problem from consumption.

Once we found the optimal problems, we then collected the data for the period between years 1980 and 2013 and applied with the motivation that provides measure regarding the strength of an economy. To decide on the economy's situation, we employed the notions introduced in Yener et al. (2014). In that framework, they first calculated the optimal leverage strategy, f^* that maximizes the expected growth of the economy.

Besides, Yener et al. (2014) identified another indicator, β, which is actually a trend of the net worth. These two indicators constitutes the resilience notion which describes the durability of the economy. Accordingly, if $f - f^* > 0$ and

$\beta < 0$, an economy is deemed to be non-resilient. That is, the economy borrowed excessively by giving up growth and even under the optimal strategy it still would not be able to grow.

Within this framework, we saw that Turkey's non-resilience condition was quite persistent since 2005. In fact, this persistent condition is also coupled with increasing realized leverage levels that were not previously seen in Turkish economic history. Therefore, Turkey needs policies that will alleviate the effects of both internal and external events that might spark a possible debt crisis for the country.

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