EULER–ZAGIER SUMS VIA TRIGONOMETRIC SERIES

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In this note, we study the evaluations of Euler sums via trigonometric series. It is a commonly believed conjecture that for an even weight greater than seven, Euler sums cannot be evaluated in terms of the special values of the Riemann zeta function. For an even weight, we reduce the evaluations of Euler sums into the evaluations of double series and integrals of products of Clausen functions. We also re-evaluate Euler sums of odd weight using a new method based on trigonometric series.

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1. INTRODUCTION

Given an integer $p \geq 1$, for each integer $n \geq 1$ the sum

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$$

is called a generalized harmonic number of order p. The generalized harmonic numbers of order 1 are just the classical harmonic numbers denoted by h_n. When $p \geq 2$, the sequence $(H_n^{(p)})_n$ is the sequence of partial sums of the special value $\zeta(p)$ of the Riemann zeta function, where the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$.

Euler showed that the special values of the harmonic zeta function

$$H(1, s) = \sum_{n=1}^{\infty} \frac{h_n}{n^s}$$

can be computed in terms of the special values of the Riemann zeta function. More precisely, he proved that

$$H(1, m) = \frac{1}{2}(m+2)\zeta(m+1) - \frac{1}{2}\sum_{k=2}^{m-1} \zeta(k)\zeta(m+1-k) \tag{1.1}$$

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for all integers $m \geq 2$, where the sum over k is meant to be zero when $m = 2$.

Euler sums are series of the form

$$H(p,q) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$$

for integers $p \geq 1$ and $q > 1$. For an Euler sum $H(p,q)$, the integer $w = p+q$ is called its weight.

Note that when we say an Euler sum $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function, we mean that $H(p,q)$ is an element of the ring

$$\Omega = \mathbb{Q}[\zeta(k) : k \geq 2]$$

generated by $\mathbb{Q}$ and all special zeta values. Note also that a typical element of Ω is of the form $p(\zeta(k_1), \zeta(k_2), ..., \zeta(k_n))$ for some positive integers $k_1, k_2, ..., k_n$ and for some polynomial $p(x_1, x_2, ..., x_n)$ in the polynomial ring $\mathbb{Q}[x_1, x_2, ..., x_n]$.

For $p > 1$, it can be seen that we have the following reciprocity relation:

$$H(p,q) + H(q,p) = \zeta(p)\zeta(q) + \zeta(p+q). \tag{1.2}$$

Hence, if $H(p,q)$ $(p > 1)$ can be evaluated in terms of the special values of the Riemann zeta function, then so can $H(q,p)$. In particular, if $p = q > 1$, then $H(p,q) = H(p,p)$ can be computed in terms of the special values of the Riemann zeta function. It is also known that

$$H(2,4) = \zeta(3)^2 - \frac{1}{3}\zeta(6).$$

For an odd weight $w = p+q$, the evaluation of $H(p,q)$ was attempted by Euler, and he anticipated that $H(p,q)$ can be computed in terms of the special values of the Riemann zeta function. There was a gap in Euler's approach, and it was Nielsen who first gave the complete proof of this in [8]. Later on, the evaluations of Euler sums for an odd weight were studied again, and we refer the reader to [2, 7]. Generalizations of Euler sums are called multiple zeta values, and they have applications to physics, see [11]. Moreover, a generalization of the evaluations of Euler sums for an odd weight to multiple zeta values can be found in [10].

For an even weight w, we do not know much about Euler sums when $p > 1$ and $p \neq q$ except the cases $H(2,4)$ and $H(4,2)$. Moreover, it is widely believed that for an even weight $w \geq 8$, Euler sums of weight w cannot be evaluated in terms of the special values of the Riemann zeta function. However, this is still an open conjecture.

For each integer $m \geq 1$, the polylogarithm function $Li_m(z)$ is defined by

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

Note that for $m \geq 2$, $Li_m(z)$ is well-defined for all complex numbers z such that $|z| \leq 1$ and for $m = 1$, $Li_m(z)$ is well-defined for all complex numbers z such that $|z| \leq 1$ and $z \neq 1$. Given $m \geq 1$, according to the parity of m, we name either one of the real and imaginary parts of $Li_m(e^{it})$ by $Sl_m(t)$ and the other one by $Cl_m(t)$. Precisely, we define

$$Sl_m(t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^m} & \text{if } m \text{ is even} \\ \displaystyle\sum_{n=1}^{\infty} \frac{\sin(nt)}{n^m} & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad Cl_m(t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\sin(nt)}{n^m} & \text{if } m \text{ is even} \\ \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^m} & \text{if } m \text{ is odd.} \end{cases}$$

The functions $Sl_m(t)$ and $Cl_m(t)$ are called Clausen functions. The evaluations of $Cl_m(t)$ in terms of well-known functions are not known when $m \geq 2$. However, $Sl_m(t)$ can be evaluated. In particular (see [1]), for all integers $m \geq 1$,

$$Sl_m(t) = \sum_{j=0}^{m} \mathbb{Q}\pi^{m-j}t^j \tag{1.3}$$

where by $\mathbb{Q}v$ we mean an element of $\{qv : q \in \mathbb{Q}\}$. Here, we give the Clausen polynomials $Sl_m(t)$ from $m = 1$ to $m = 4$ explicitly as follows:

$$Sl_1(t) = \frac{\pi}{2} - \frac{t}{2}, \; Sl_2(t) = \frac{\pi^2}{6} - \frac{\pi t}{2} + \frac{t^2}{4}, \; Sl_3(t) = \frac{\pi^2 t}{6} - \frac{\pi t^2}{4} + \frac{t^3}{12},$$

$$Sl_4(t) = \frac{\pi^4}{90} - \frac{\pi^2 t^2}{12} + \frac{\pi t^3}{12} - \frac{t^4}{48}.$$

For positive integers a, b, c, let

$$CL(a,b,c) = \frac{1}{\pi}\int_0^{2\pi} t^a Cl_b(t) Cl_c(t)\, \mathrm{d}t.$$

Our first result is the following and it states that for an even weight w, the evaluations of Euler sums are related to the evaluations of the integrals $CL(a,b,c)$. Thus, Euler sums of even weight have integral representations in terms of trigonometric series up to zeta values.

Theorem 1.1. i. *Let $p, q > 1$ be two odd integers. Then $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function and the integrals of the form $CL(a,b,c)$ where a is even, $a < q$ and $b + c = p + 1$.*

ii. *Let $p, q > 1$ be two even integers such that at least one of them is 2. Then $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function and the integrals of the form $CL(a,2,2)$ where a is even and $a < q$.*

In the same case, alternatively, $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function and the integrals of the form $CL(a,2,1)$ where $3 \leq a < q$ and a is odd.

For natural numbers a, b, c, let

$$\omega(a,b,c) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^a m^b (n+m)^c}$$

be the double series. It is also called Tornheim double series or harmonic double series as it was first studied by Tornheim [9]. It is known that (see [6]) the double series $\omega(a,b,c)$ converges if and only if $a+c>1$, $b+c>1$ and $a+b+c>2$. The evaluations of $\omega(a,b,c)$ are related to the evaluations of Euler sums and for this we direct the reader to [3] and [4]. Our second theorem states that $CL(a,b,c)$ can be computed in terms of the special values of the Riemann zeta function and double series. In particular, it relates the evaluations of Euler sums to the evaluations of double series.

THEOREM 1.2. *Let a, b, c be positive integers such that a is even, b and c have the same parity and $b+c \geq 4$. Then the integral $CL(a,b,c)$ can be evaluated in terms of the special values of the Riemann zeta function and double series.*

In particular, if $p, q > 1$ are two integers such that either both p and q are odd or both p and q are even and at least one of them is 2, then $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function and double series.

In light of Theorem 1.1 and Theorem 1.2, when we consider the class of Euler sums, the class of double series and the class of integrals of the form $CL(a,b,c)$, we observe that the computations of elements of these three classes have related difficulty.

The following theorem concerns the termwise integrability of certain trigonometric series and it plays an important role in the proofs of the theorems presented in this note. It is also of independent interest.

THEOREM 1.3. *Let $(a_n)_n$ be a sequence of real numbers such that $(a_n)_n$ is monotonically decreasing to 0 and the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges. Then for every integer $m \geq 0$, both series

$$\sum_{n=1}^{\infty} a_n t^m \sin(nt) \quad \text{and} \quad \sum_{n=1}^{\infty} a_n t^m \cos(nt)$$

can be integrated termwise over the interval $[0, 2\pi]$.

Our fourth theorem gives new evaluations of Euler sums based on trigonometric series. Note that the following result was already obtained using many different approaches as we mentioned before. However, our technique is different from them.

THEOREM 1.4. *Let $p, q > 1$ be two integers. Then $H(p,q)$ can be evaluated in terms of the special values of the Riemann zeta function when the weight $w = p+q$ is odd, or $p = q$, or $(p,q) = (2,4)$, or $(p,q) = (4,2)$.*

Short overview of the paper: In Section 2, we prove several lemmas that we use in the proofs of our Theorems. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2. We also prove Corollary 5.1 where we recompute the Euler sum $H(2,4)$ and we compute the integrals $CL(2,2,2)$ and $CL(3,2,1)$ in terms of the special values of the Riemann zeta function using our approach. In Section 6, we prove Theorem 1.4. In Section 7, we prove Corollary 7.1 using Theorem 1.1 and Theorem 1.2 and we also prove Corollary 7.2.

2. PRELIMINARIES

Recall the following two lemmas from [1]. In the first lemma, certain generating functions related to generalized harmonic numbers are evaluated in terms of polylogarithmic values. In the second one, we obtain two Fourier series expansions arising from the first lemma.

LEMMA 2.1 ([1, Lemma 1]). *For $p \geq 3$ odd and complex number z with $|z| \leq 1$ and $z \neq 1$, we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} z^n = Li_{p+1}(z) + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k a_k Li_{p-k}(z) Li_{k+1}(z),$$

where $a_k = 1$ for $0 \leq k \leq \dfrac{p-3}{2}$ and $a_{\frac{p-1}{2}} = \dfrac{1}{2}$. Moreover, we have

$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) z^n = 3Li_3(z) + Li_2(z) Li_1(z)$$

for $|z| \leq 1$ and $z \neq 1$.

LEMMA 2.2 ([1, Lemma 2]). *For $p \geq 3$ odd and $0 < t < 2\pi$, we have*

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} \sin(nt) \\ &= Cl_{p+1}(t) + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k a_k \big(Sl_{p-k}(t) Cl_{k+1}(t) + Sl_{k+1}(t) Cl_{p-k}(t) \big), \end{aligned}$$

where $a_k = 1$ *for* $0 \le k \le \dfrac{p-3}{2}$ *and* $a_{\frac{p-1}{2}} = \dfrac{1}{2}$. *Moreover, for* $0 < t < 2\pi$, *we have*

$$\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2}\right)\cos(nt) = 3Cl_3(t) + Sl_2(t)Cl_1(t) - Cl_2(t)Sl_1(t).$$

In the following two lemmas, we extract some new Fourier series expansions from Lemma 2.1. Note that, the next lemma gives an analogous result of the previous lemma.

Lemma 2.3. *For* $p \ge 3$ *odd and* $0 < t < 2\pi$, *we have*

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n}\cos(nt) = Sl_{p+1}(t) + \sum_{k=0}^{\frac{p-1}{2}} a_k\big(Cl_{p-k}(t)Cl_{k+1}(t) - Sl_{k+1}(t)Sl_{p-k}(t)\big),$$

where $a_k = 1$ *for* $0 \le k \le \dfrac{p-3}{2}$ *and* $a_{\frac{p-1}{2}} = \dfrac{1}{2}$. *Moreover, for* $0 < t < 2\pi$, *we have*

$$\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2}\right)\sin(nt) = 3Sl_3(t) + Sl_2(t)Sl_1(t) + Cl_2(t)Cl_1(t).$$

Proof. By equating the real parts of both sides of the equality which is obtained by taking $z = e^{it}$ for $0 < t < 2\pi$ in the first part of Lemma 2.1, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n}\cos(nt) &= Sl_{p+1}(t) + Cl_p(t)Cl_1(t) - Sl_p(t)Sl_1(t)\\ &\quad + \frac{1}{2}\Big(Cl^2_{\frac{p+1}{2}}(t) - Sl^2_{\frac{p+1}{2}}(t)\Big)\\ &\quad + \sum_{\substack{k=1\\ k:odd}}^{\frac{p-3}{2}}(-1)^k\big(Sl_{p-k}(t)Sl_{k+1}(t) - Cl_{p-k}(t)Cl_{k+1}(t)\big)\\ &\quad + \sum_{\substack{k=1\\ k:even}}^{\frac{p-3}{2}}(-1)^k\big(Cl_{p-k}(t)Cl_{k+1}(t) - Sl_{p-k}(t)Sl_{k+1}(t)\big)\end{aligned} \tag{2.1}$$

and the first equation of the lemma directly follows from Equation (2.1).

The second equation in this lemma comes when we compare the imaginary parts of both sides of the second equation in Lemma 2.1 for $z = e^{it}$ and $0 < t < 2\pi$. □

LEMMA 2.4. *For all* $0 \le t \le 2\pi$,

$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n^2} + \frac{2h_n}{n^3} \right) \cos(nt) = 3Sl_4(t) + \frac{1}{2} Sl_2(t) Sl_2(t) - \frac{1}{2} Cl_2(t) Cl_2(t).$$

Proof. As stated in Lemma 2.1, we have

$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) z^n = 3Li_3(z) + Li_2(z) Li_1(z) \tag{2.2}$$

for $|z| \le 1$ and $z \neq 1$. By dividing both sides of (2.2) by z and then integrating with respect to z, we get

$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n^2} + \frac{2h_n}{n^3} \right) z^n = 3 \int \frac{Li_3(z)}{z} \, dz + \int Li_2(z) \frac{Li_1(z)}{z} \, dz.$$

However, for all $m \ge 1$,

$$\int \frac{Li_m(z)}{z} dz = Li_{m+1}(z).$$

Then, using integration by parts, we obtain that

$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n^2} + \frac{2h_n}{n^3} \right) z^n = 3Li_4(z) + \frac{1}{2} Li_2(z) Li_2(z) \tag{2.3}$$

for all z with $|z| \le 1$. When we compare the real parts of both sides of (2.3) by taking $z = e^{it}$ for $0 \le t \le 2\pi$, the result follows. $\square$

3. PROOF OF THEOREM 1.3

Let us first consider the series

$$\sum_{n=1}^{\infty} a_n \sin(nt). \tag{3.1}$$

For all $k \ge 1$,

$$\sum_{n=1}^{k} \sin(nt) = \frac{\sin\left(\frac{kt}{2}\right) \sin\left(\frac{(k+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

Hence given $\delta > 0$ arbitrarily small, the sequence

$$\left(\sum_{n=1}^{k} \sin(nt) \right)_k$$

of partial sums is uniformly bounded by the reciprocal $1/\sin(\delta/2)$ on the closed interval $[\delta, 2\pi-\delta]$. Then since the real sequence $(a_n)_n$ is monotonically decreasing to 0, it follows from Dirichlet test for uniform convergence that the series (3.1) is uniformly convergent on $[\delta, 2\pi-\delta]$. Consequently, given any integer $m \geq 0$, the series

$$\sum_{n=1}^{\infty} a_n t^m \sin(nt)$$

is uniformly convergent, hence termwise integrable on $[\delta, 2\pi-\delta]$, that is to say

$$\int_{\delta}^{2\pi-\delta} \sum_{n=1}^{\infty} a_n t^m \sin(nt)\ \mathrm{d}t = \sum_{n=1}^{\infty} a_n \int_{\delta}^{2\pi-\delta} t^m \sin(nt)\ \mathrm{d}t.$$

Given $\delta \in \mathbb{R}$ and $n, m, k \in \mathbb{Z}$ such that n is positive and m is non-negative, if $k \geq 1$ with $m+2-2k \geq 0$, we define

$$\begin{aligned} & f_{(n,m,k)}(\delta) \\ = \ & (-1)^k \frac{m!}{(m+2-2k)!}\left[(2\pi-\delta)^{m+2-2k}\cos(n(2\pi-\delta)) - \delta^{m+2-2k}\cos(n\delta)\right] \end{aligned}$$

and if $k \geq 1$ with $m+1-2k \geq 0$, we define

$$\begin{aligned} & g_{(n,m,k)}(\delta) \\ & = (-1)^{k-1}\tfrac{m!}{(m+1-2k)!}\left[(2\pi-\delta)^{m+1-2k}\sin(n(2\pi-\delta)) - \delta^{m+1-2k}\sin(n\delta)\right]. \end{aligned}$$

Note that if $m+2-2k > 0$, then for all n,

$$f_{(n,m,k)}(0) = (-1)^k \frac{m!(2\pi)^{m+2-2k}}{(m+2-2k)!}.$$

On the other hand, $g_{(n,m,k)}(0) = 0$ for all n and for all k with $m+1-2k \geq 0$. If $\delta > 0$ is small enough,

$$\int_{\delta}^{2\pi-\delta} \sin(nt)\ \mathrm{d}t = \frac{1}{n}\left[\cos(n\delta) - \cos(n(2\pi-\delta))\right] = \frac{f_{(n,0,1)}(\delta)}{n} \tag{3.2}$$

and by partial integration

$$\int_{\delta}^{2\pi-\delta} t\sin(nt)\ \mathrm{d}t = \frac{f_{(n,1,1)}(\delta)}{n} + \frac{g_{(n,1,1)}(\delta)}{n^2}. \tag{3.3}$$

If $m \geq 2$, we get the following reduction formula using integration by parts:

$$\begin{aligned} & \int_{\delta}^{2\pi-\delta} t^m \sin(nt)\ \mathrm{d}t \\ = \ & \frac{f_{(n,m,1)}(\delta)}{n} + \frac{g_{(n,m,1)}(\delta)}{n^2} - \frac{m.(m-1)}{n^2}\int_{\delta}^{2\pi-\delta} t^{m-2}\sin(nt)\ \mathrm{d}t. \end{aligned}$$

So, if $m \geq 2$, using successive integration by parts, after finitely many steps we obtain

$$\int_{\delta}^{2\pi-\delta} t^m \sin(nt)\ \mathrm{d}t = \sum_{k=1}^{[\frac{m+2}{2}]} \frac{f_{(n,m,k)}(\delta)}{n^{2k-1}} + \sum_{k=1}^{[\frac{m+1}{2}]} \frac{g_{(n,m,k)}(\delta)}{n^{2k}}. \tag{3.4}$$

Note that Equations (3.2) and (3.3) show that Equation (3.4) holds when $m = 0$ and $m = 1$ as well, if the second sum over k is taken to be zero when $m = 0$.

If the integers $m \geq 0$ and $k \geq 1$ are fixed, then as functions of the variable δ, the families of functions $\{f_{(n,m,k)}\}_n$ and $\{g_{(n,m,k)}\}_n$ are continuous and uniformly bounded on $[0, 2\pi]$. Then since

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$$

it follows from Weirstrass M-test that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} f_{(n,m,k)}(\delta) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n}{n^{2k}} g_{(n,m,k)}(\delta) \tag{3.5}$$

are uniformly convergent on $[0, 2\pi]$. Hence each series in (3.5) defines a continuous function of δ on $[0, 2\pi]$ and for all $m \geq 0$,

$$\begin{aligned}
&\int_0^{2\pi} \sum_{n=1}^{\infty} a_n t^m \sin(nt)\ \mathrm{d}t \qquad (3.6)\\
&= \lim_{\delta \to 0^+} \int_{\delta}^{2\pi-\delta} \sum_{n=1}^{\infty} a_n t^m \sin(nt)\ \mathrm{d}t\\
&= \lim_{\delta \to 0^+} \sum_{n=1}^{\infty} a_n \int_{\delta}^{2\pi-\delta} t^m \sin(nt)\ \mathrm{d}t\\
&= \lim_{\delta \to 0^+} \sum_{k=1}^{[\frac{m+2}{2}]} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} f_{(n,m,k)}(\delta) + \lim_{\delta \to 0^+} \sum_{k=1}^{[\frac{m+1}{2}]} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k}} g_{(n,m,k)}(\delta)\\
&= \sum_{k=1}^{[\frac{m+2}{2}]} \lim_{\delta \to 0^+} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} f_{(n,m,k)}(\delta) + \sum_{k=1}^{[\frac{m+1}{2}]} \lim_{\delta \to 0^+} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k}} g_{(n,m,k)}(\delta)\\
&= \sum_{k=1}^{[\frac{m+2}{2}]} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} f_{(n,m,k)}(0) + \sum_{k=1}^{[\frac{m+1}{2}]} \sum_{n=1}^{\infty} \frac{a_n}{n^{2k}} g_{(n,m,k)}(0)\\
&= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+2}{2}]} \frac{f_{(n,m,k)}(0)}{n^{2k-1}}.
\end{aligned}$$

Then if $m = 0$, by Equation (3.6) we get

$$\int_0^{2\pi} \sum_{n=1}^{\infty} a_n \sin(nt)\ \mathrm{d}t = \sum_{n=1}^{\infty} a_n \frac{f_{(n,0,1)}(0)}{n} = 0$$

where

$$\sum_{n=1}^{\infty} a_n \int_0^{2\pi} \sin(nt)\ \mathrm{d}t = 0$$

as well. Therefore, integration and summation can be interchanged when $m = 0$. Now let $m \geq 1$. If m is odd, then $\left[\dfrac{m+2}{2}\right] = \left[\dfrac{m+1}{2}\right]$ and if m is even,

$$f_{(n,m,[\frac{m+2}{2}])}(0) = 0.$$

Then, continuing from Equation (3.6) and using the formula

$$\int_0^{2\pi} t^m \sin(nt)\ \mathrm{d}t = \sum_{k=1}^{[\frac{m+1}{2}]} (-1)^k \frac{m!(2\pi)^{m+2-2k}}{(m+2-2k)!n^{2k-1}} \tag{3.7}$$

which comes from successive integration by parts for all $m \geq 1$, we finally get

$$\begin{aligned}
\int_0^{2\pi} \sum_{n=1}^{\infty} a_n t^m \sin(nt)\ \mathrm{d}t &= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+2}{2}]} \frac{f_{(n,m,k)}(0)}{n^{2k-1}} \\
&= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+1}{2}]} \frac{f_{(n,m,k)}(0)}{n^{2k-1}} \\
&= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+1}{2}]} (-1)^k \frac{m!(2\pi)^{m+2-2k}}{(m+2-2k)!n^{2k-1}} \\
&= \sum_{n=1}^{\infty} \int_0^{2\pi} a_n t^m \sin(nt)\ \mathrm{d}t.
\end{aligned}$$

Secondly, we consider the series

$$\sum_{n=1}^{\infty} a_n \cos(nt). \tag{3.8}$$

Since

$$\sum_{n=1}^{k} \cos(nt) = \frac{\sin\left(\frac{kt}{2}\right)\cos\left(\frac{(k+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$$

for all $k \geq 1$, the sequence

$$\left(\sum_{n=1}^{k} \cos(nt)\right)_k$$

of partial sums is uniformly bounded on the interval $[\delta, 2\pi-\delta]$ for all $\delta>0$, but arbitrarily small. Then by Dirichlet test the series (3.8) is uniformly convergent on $[\delta, 2\pi-\delta]$ and given any integer $m \geq 0$, the series

$$\sum_{n=1}^{\infty} a_n t^m \cos(nt)$$

is uniformly convergent, hence termwise integrable on $[\delta, 2\pi-\delta]$, that is to say

$$\int_{\delta}^{2\pi-\delta} \sum_{n=1}^{\infty} a_n t^m \cos(nt)\ \mathrm{d}t = \sum_{n=1}^{\infty} a_n \int_{\delta}^{2\pi-\delta} t^m \cos(nt)\ \mathrm{d}t.$$

However, for all $m \geq 0$,

$$\int_{\delta}^{2\pi-\delta} t^m \cos(nt)\ \mathrm{d}t = \sum_{k=1}^{[\frac{m+2}{2}]} \frac{F_{(n,m,k)}(\delta)}{n^{2k-1}} + \sum_{k=1}^{[\frac{m+1}{2}]} \frac{G_{(n,m,k)}(\delta)}{n^{2k}}$$

where

$$\begin{aligned} &F_{(n,m,k)}(\delta) \\ &\quad = (-1)^{k-1} \tfrac{m!}{(m+2-2k)!} \left[(2\pi-\delta)^{m+2-2k} \sin(n(2\pi-\delta)) - \delta^{m+2-2k} \sin(n\delta)\right] \end{aligned}$$

for $m+2-2k \geq 0$ and

$$\begin{aligned} &G_{(n,m,k)}(\delta) \\ &\quad = (-1)^{k-1} \tfrac{m!}{(m+1-2k)!} \left[(2\pi-\delta)^{m+1-2k} \cos(n(2\pi-\delta)) - \delta^{m+1-2k} \cos(n\delta)\right] \end{aligned}$$

for $m+1-2k \geq 0$ and the second sum over k is taken to be zero for $m=0$. Following similar steps as in Equation (3.6), for all $m \geq 0$, we get

$$\int_{0}^{2\pi} \sum_{n=1}^{\infty} a_n t^m \cos(nt)\ \mathrm{d}t = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+1}{2}]} \frac{G_{(n,m,k)}(0)}{n^{2k}}. \tag{3.9}$$

We get from Equation (3.9) that if $m=0$,

$$\int_{0}^{2\pi} \sum_{n=1}^{\infty} a_n \cos(nt)\ \mathrm{d}t = 0$$

where

$$\sum_{n=1}^{\infty} a_n \int_{0}^{2\pi} \cos(nt)\ \mathrm{d}t = 0$$

as well and if $m=1$, we get

$$\int_{0}^{2\pi} \sum_{n=1}^{\infty} a_n t \cos(nt)\ \mathrm{d}t = \sum_{n=1}^{\infty} a_n \frac{G_{(n,1,1)}(0)}{n^2} = 0$$

where

$$\sum_{n=1}^{\infty} a_n \int_0^{2\pi} t\cos(nt)\ \mathrm{d}t = 0$$

as well by integration by parts. Now let $m \geq 2$. If m is even, then $[\frac{m+1}{2}] = [\frac{m}{2}]$ and if m is odd, then

$$G_{(n,m,[\frac{m+1}{2}])}(0) = 0.$$

Then, using the equation

$$\int_0^{2\pi} t^m \cos(nt)\ \mathrm{d}t = \sum_{k=1}^{[\frac{m}{2}]} (-1)^{k-1} \frac{m!(2\pi)^{m+1-2k}}{(m+1-2k)!n^{2k}} \tag{3.10}$$

which holds for every integer $m \geq 2$ and continuing from Equation (3.9), we get

$$\begin{aligned}
\int_0^{2\pi} \sum_{n=1}^{\infty} a_n t^m \cos(nt)\ \mathrm{d}t &= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m+1}{2}]} \frac{G_{(n,m,k)}(0)}{n^{2k}} \\
&= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m}{2}]} \frac{G_{(n,m,k)}(0)}{n^{2k}} \\
&= \sum_{n=1}^{\infty} a_n \sum_{k=1}^{[\frac{m}{2}]} (-1)^{k-1} \frac{m!(2\pi)^{m+1-2k}}{(m+1-2k)!n^{2k}} \\
&= \sum_{n=1}^{\infty} \int_0^{2\pi} a_n t^m \cos(nt)\ \mathrm{d}t
\end{aligned}$$

as desired.

4. PROOF OF THEOREM 1.1

Let us first assume that both p and q are odd. Let $q = 2r+1$. Let us multiply both sides of the first equation in Lemma 2.3 by t^{2r} to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} t^{2r} \cos(nt) \\
&= t^{2r} Sl_{p+1}(t) - \sum_{k=0}^{\frac{p-1}{2}} a_k t^{2r} Sl_{k+1}(t) Sl_{p-k}(t) + \sum_{k=0}^{\frac{p-1}{2}} a_k t^{2r} Cl_{p-k}(t) Cl_{k+1}(t),
\end{aligned} \tag{4.1}$$

where $a_k = 1$ for $0 \leq k \leq \dfrac{p-3}{2}$ and $a_{\frac{p-1}{2}} = \dfrac{1}{2}$, and then integrate both sides of the above equation over $[0, 2\pi]$.

Consider the real sequence whose n^{th} term is

$$b_n = \frac{H_n^{(p)}}{n}.$$

Since

$$0 < b_n < \frac{h_n}{n} < \frac{1+\log n}{n} \quad \text{and} \quad b_n - b_{n+1} = \frac{H_n^{(p)} - \frac{n}{(n+1)^p}}{n(n+1)} > 0$$

for all n, the sequence $(b_n)_n$ decreases monotonically to 0. Moreover, the series

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^2} = H(p,2)$$

is convergent. Hence, Theorem 1.3 applies to the sequence $(b_n)_n$ and using Theorem 1.3 and Equation (3.10), we get

$$\begin{aligned}
\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} t^{2r} \cos(nt)\ \mathrm{d}t &= \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} \int_0^{2\pi} t^{2r} \cos nt\ \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} \sum_{k=1}^{r} (-1)^{k-1} \frac{(2r)!(2\pi)^{2r+1-2k}}{(2r+1-2k)!n^{2k}} \\
&= \sum_{k=1}^{r} \mathbb{Q}\pi^{2r+1-2k} H(p, 2k+1). \qquad (4.2)
\end{aligned}$$

Note that for any non-negative integers a, b, c,

$$\int_0^{2\pi} t^a Sl_b(t)\ \mathrm{d}t = \int_0^{2\pi} t^a \sum_{i=0}^{b} \mathbb{Q}\pi^{b-i} t^i\ \mathrm{d}t = \sum_{i=0}^{b} \mathbb{Q}\pi^{b-i} \int_0^{2\pi} t^{a+i}\ \mathrm{d}t = \mathbb{Q}\pi^{a+b+1} \qquad (4.3)$$

and since

$$Sl_b(t)Sl_c(t) = \sum_{i=0}^{b} \mathbb{Q}\pi^{b-i} t^i \sum_{j=0}^{c} \mathbb{Q}\pi^{c-j} t^j = \sum_{k=0}^{b+c} \mathbb{Q}\pi^{b+c-k} t^k$$

we have

$$\int_0^{2\pi} t^a Sl_b(t) Sl_c(t)\ \mathrm{d}t = \mathbb{Q}\pi^{a+b+c+1}. \qquad (4.4)$$

In particular,

$$\int_0^{2\pi} t^{2r} Sl_{p+1}(t)\ \mathrm{d}t = \mathbb{Q}\pi^{p+q+1} \qquad (4.5)$$

and

$$\int_0^{2\pi}\sum_{k=0}^{\frac{p-1}{2}} a_k t^{2r} Sl_{k+1}(t)Sl_{p-k}(t)\ \mathrm{d}t = \mathbb{Q}\pi^{p+q+1}. \tag{4.6}$$

Hence by combining Equations (4.2), (4.5) and (4.6) via Equation (4.1), we get

$$\sum_{k=1}^{r}\mathbb{Q}\pi^{2r+1-2k}H(p,2k+1) = \mathbb{Q}\pi^{p+q+1} + \sum_{k=0}^{\frac{p-1}{2}}\mathbb{Q}\int_0^{2\pi} t^{2r}Cl_{p-k}(t)Cl_{k+1}(t)\ \mathrm{d}t.$$

When we divide both sides of the previous equation by π and substitute $6\zeta(2)$ for π^2, we get

$$\sum_{k=1}^{r}\mathbb{Q}\zeta(2)^{r-k}H(p,2k+1) = \mathbb{Q}\zeta(2)^{\frac{p+q}{2}} + \sum_{k=0}^{\frac{p-1}{2}}\mathbb{Q}CL(2r,p-k,k+1). \tag{4.7}$$

In order to prove the theorem in case both p and $q = 2r+1$ are odd and $p,q>1$, we fix p and proceed by induction on r. If $r=1$, by Equation (4.7)

$$H(p,3) = \mathbb{Q}\zeta(2)^{\frac{p+3}{2}} + \sum_{k=0}^{\frac{p-1}{2}}\mathbb{Q}CL(2,p-k,k+1).$$

Hence, $H(p,3)$ can be evaluated in terms of $\zeta(2)$ and the integrals of the form $CL(2,b,c)$ where $b+c=p+1$. Now assume the theorem holds for all k such that $1\le k<r$. By Equation (4.7),

$$H(p,2r+1) = \mathbb{Q}\zeta(2)^{\frac{p+2r+1}{2}} - \sum_{k=1}^{r-1}\mathbb{Q}\zeta(2)^{r-k}H(p,2k+1) + \sum_{k=0}^{\frac{p-1}{2}}\mathbb{Q}CL(2r,p-k,k+1).$$

However, by the induction hypothesis, for any $1\le k<r$, the sum $H(p,2k+1)$ can be evaluated in terms of the special values of $\zeta(s)$ and the integrals of the form $CL(a,b,c)$ where a is even, $a<2k+1$ and $b+c=p+1$. Then the theorem follows for $H(p,2r+1)$ by the last equation.

In case both p and q are even such that at least one of them is 2, by reciprocity it is enough to prove the result when $p=2$. If $p=q=2$, then by reciprocity

$$H(2,2) = \frac{1}{2}\left[\zeta(2)^2+\zeta(4)\right].$$

So let $p=2$ and $q=2r+2$ where $r\ge 1$. We multiply both sides of the equation given in Lemma 2.4 by $\dfrac{1}{\pi}t^{2r}$ and then integrate them over the interval $[0,2\pi]$ to get

$$\text{(4.8)}\quad \frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)t^{2r}\cos(nt)\ \mathrm{d}t$$
$$=\frac{3}{\pi}\int_0^{2\pi} t^{2r}Sl_4(t)\ \mathrm{d}t+\frac{1}{2\pi}\int_0^{2\pi} t^{2r}Sl_2(t)Sl_2(t)\ \mathrm{d}t-\frac{1}{2}CL(2r,2,2).$$

Since the trigonometric series appearing on the left-hand side of Equation (4.8) is uniformly convergent on $[0,2\pi]$,

$$\text{(4.9)}\quad \frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)t^{2r}\cos(nt)\ \mathrm{d}t$$
$$=\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)\int_0^{2\pi} t^{2r}\cos(nt)\ \mathrm{d}t$$
$$=\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)\sum_{k=1}^{r}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r+1-2k}}{(2r+1-2k)!n^{2k}}$$
$$=\frac{1}{\pi}\sum_{k=1}^{r}\left(\mathbb{Q}\pi^{2r+1-2k}H(2,2k+2)+\mathbb{Q}\pi^{2r+1-2k}H(1,2k+3)\right).$$

On the other hand,

$$\int_0^{2\pi} t^{2r}Sl_4(t)\ \mathrm{d}t=\mathbb{Q}\pi^{2r+5}\quad\text{and}\quad\int_0^{2\pi} t^{2r}Sl_2(t)Sl_2(t)\ \mathrm{d}t=\mathbb{Q}\pi^{2r+5}$$

by Equations (4.3) and (4.4). Hence,

$$\sum_{k=1}^{r}\mathbb{Q}\pi^{2r-2k}H(2,2k+2)=\sum_{j=1}^{r}\mathbb{Q}\pi^{2r-2j}H(1,2j+3)+\mathbb{Q}\pi^{2r+4}-\frac{1}{2}CL(2r,2,2)$$

and by substituting $6\zeta(2)$ for π^2, we obtain that

$$\text{(4.10)}\quad \sum_{k=1}^{r}\mathbb{Q}\zeta(2)^{r-k}H(2,2k+2)=\sum_{j=1}^{r}\mathbb{Q}\zeta(2)^{r-j}H(1,2j+3)+\mathbb{Q}\zeta(2)^{r+2}-\frac{1}{2}CL(2r,2,2).$$

We will now prove the theorem for $p=2$ and $q=2r+2$ by induction on $r\geq 1$. If $r=1$, by Equation (4.10)

$$H(2,4)=\mathbb{Q}H(1,5)+\mathbb{Q}\zeta(2)^3+\mathbb{Q}CL(2,2,2)$$

and since $H(1,5)\in\Omega$ by Equation (1.1), the result is valid for $r=1$. For the inductive step, assume that $H(2,2k+2)$ can be evaluated in terms of the special zeta values and the integrals of the form $CL(a,2,2)$ where a is even and $a<2k+2$ for all $1\leq k<r$. Then since

$$H(2,2r+2)=\sum_{k=1}^{r-1}\mathbb{Q}\zeta(2)^{r-k}H(2,2k+2)+\sum_{k=1}^{r}\mathbb{Q}\zeta(2)^{r-k}H(1,2k+3)$$

$$+ \mathbb{Q}\zeta(2)^{r+2} + \mathbb{Q}CL(2r,2,2)$$

by Equation (4.10) and $H(1,2k+3) \in \Omega$ for all $k = 1, ..., r$ by Equation (1.1), the result follows for r from the induction assumption, and we are done.

To obtain the alternative result in the same case, we multiply both sides of the second equation in Lemma 2.3 by t^{2r+1} and then integrate them over the interval $[0, 2\pi]$ to get

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) t^{2r+1} \sin(nt)\ \mathrm{d}t \tag{4.11}$$
$$= 3\int_0^{2\pi} t^{2r+1} Sl_3(t)\ \mathrm{d}t + \int_0^{2\pi} t^{2r+1} Sl_2(t) Sl_1(t)\ \mathrm{d}t + \int_0^{2\pi} t^{2r+1} Cl_2(t) Cl_1(t)\ \mathrm{d}t.$$

Since the sequences

$$\left(\frac{H_n^{(2)}}{n} \right)_n \quad \text{and} \quad \left(\frac{2h_n}{n^2} \right)_n$$

are both monotonically decreasing to zero and the series

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2h_n}{n^3}$$

are both convergent, Theorem 1.3 applies to the trigonometric series appearing on the left-hand side of Equation (4.11). So, we get

$$\begin{aligned}
&\int_0^{2\pi} \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) t^{2r+1} \sin(nt) \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) \int_0^{2\pi} t^{2r+1} \sin(nt)\ \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n} + \frac{2h_n}{n^2} \right) \sum_{k=1}^{r+1} (-1)^k \frac{(2r+1)!(2\pi)^{2r+3-2k}}{(2r+3-2k)! n^{2k-1}} \\
&= \sum_{k=1}^{r+1} \left(\mathbb{Q}\pi^{2r+3-2k} H(2,2k) + \mathbb{Q}\pi^{2r+3-2k} H(1,2k+1) \right).
\end{aligned} \tag{4.12}$$

On the other hand,

$$\int_0^{2\pi} t^{2r+1} Sl_3(t)\ \mathrm{d}t = \mathbb{Q}\pi^{2r+5} \quad \text{and} \quad \int_0^{2\pi} t^{2r+1} Sl_2(t) Sl_1(t)\ \mathrm{d}t = \mathbb{Q}\pi^{2r+5}$$

by Equations (4.3) and (4.4). Hence

$$\sum_{k=1}^{r+1} \mathbb{Q}\pi^{2r+3-2k} H(2,2k)$$

$$= \sum_{k=1}^{r+1} \mathbb{Q}\pi^{2r+3-2k} H(1,2k+1) + \mathbb{Q}\pi^{2r+5} + \int_0^{2\pi} t^{2r+1} Cl_2(t) Cl_1(t)\ dt.$$

By dividing both sides of the previous equation by π and substituting $6\zeta(2)$ for π^2, we get

$$(4.13)\quad \sum_{k=1}^{r+1} \mathbb{Q}\zeta(2)^{r+1-k} H(2,2k) = \sum_{k=1}^{r+1} \mathbb{Q}\zeta(2)^{r+1-k} H(1,2k+1) + \mathbb{Q}\zeta(2)^{r+2} + CL(2r+1,2,1).$$

We will now prove the alternative result for $p=2$ and $q=2r+2$ by induction on $r \geq 1$. If $r=1$, by Equation (4.13)

$$H(2,4) = \mathbb{Q}\zeta(2)H(2,2) + \mathbb{Q}\zeta(2)H(1,3) + \mathbb{Q}H(1,5) + \mathbb{Q}\zeta(2)^3 + \mathbb{Q}CL(3,2,1).$$

Then since $H(2,2) \in \Omega$ by reciprocity and the Euler sums $H(1,3), H(1,5) \in \Omega$ by Equation (1.1), the result is valid for $r=1$. For the inductive step, assume that for all $1 \leq k < r$, the Euler sum $H(2,2k+2)$ can be evaluated in terms of the special values of $\zeta(s)$ and the integrals of the form $CL(a,2,1)$ where a is odd and $a < 2k+2$. Then since

$$H(2,2r+2) = \sum_{k=1}^{r} \mathbb{Q}\zeta(2)^{r+1-k} H(2,2k) + \sum_{k=1}^{r+1} \mathbb{Q}\zeta(2)^{r+1-k} H(1,2k+1) + \mathbb{Q}\zeta(2)^{r+2} + \mathbb{Q}CL(2r+1,2,1)$$

by Equation (4.13) and $H(1,2k+1) \in \Omega$ for all $k = 1,...,r+1$ by Equation (1.1), the result also follows for r by the induction hypothesis.

5. PROOF OF THEOREM 1.2 AND COROLLARY 5.1

5.1. Proof of Theorem 1.2:

Let a, b, c be positive integers such that a is even, b and c have the same parity and $b + c \geq 4$.

If b, c are both even, since the series $Cl_b(t)$ and $Cl_c(t)$ are bounded and uniformly convergent,

$$CL(a,b,c) = \frac{1}{\pi} \int_0^{2\pi} t^a \sum_{m=1}^{\infty} \frac{\sin(mt)}{m^b} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^c}\ dt$$
$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \sin(mt)\sin(nt)\ dt.$$

Similarly, if $b, c > 1$ are both odd, the series $Cl_b(t)$ and $Cl_c(t)$ are bounded and uniformly convergent so that

$$\begin{aligned} CL(a,b,c) &= \frac{1}{\pi}\int_0^{2\pi} t^a \sum_{m=1}^{\infty}\frac{\cos(mt)}{m^b}\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^c}\ \mathrm{d}t \\ &= \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^c m^b}\int_0^{2\pi} t^a\cos(mt)\cos(nt)\ \mathrm{d}t. \end{aligned}$$

Now let $b \geq 3$ be odd. Given an integer $m \geq 1$ fixed, since

$$\cos(mt)\cos(nt) = \frac{\cos((m+n)t) + \cos((m-n)t)}{2} \tag{5.1}$$

we have

$$\sum_{n=1}^{\infty} t^a \frac{\cos(mt)}{m^b}\frac{\cos(nt)}{n} = \sum_{n=1}^{\infty}\frac{t^a\cos((m+n)t)}{2m^b n} + \sum_{n=1}^{\infty}\frac{t^a\cos((m-n)t)}{2m^b n}$$

and by reindexing the last two series above we get

$$\begin{aligned} &\sum_{n=1}^{\infty} t^a \frac{\cos(mt)}{m^b}\frac{\cos(nt)}{n} \\ &= \sum_{n=m+1}^{\infty}\frac{t^a\cos(nt)}{2m^b(n-m)} + \sum_{n=1}^{m}\frac{t^a\cos((m-n)t)}{2m^b n} + \sum_{n=1}^{\infty}\frac{t^a\cos(nt)}{2m^b(n+m)}. \end{aligned} \tag{5.2}$$

It follows from Theorem 1.3 that both series

$$\sum_{n=m+1}^{\infty}\frac{t^a\cos(nt)}{2m^b(n-m)} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{t^a\cos(nt)}{2m^b(n+m)}$$

are termwise integrable on $[0, 2\pi]$, so is the series

$$\sum_{n=1}^{\infty} t^a \frac{\cos(mt)}{m^b}\frac{\cos(nt)}{n}$$

by Equation (5.2). Consequently, given any integer $N \geq 1$, the series

$$t^a \sum_{m=1}^{N}\frac{\cos(mt)}{m^b}\sum_{n=1}^{\infty}\frac{\cos(nt)}{n}$$

is termwise integrable on $[0, 2\pi]$ and

$$\begin{aligned} &\int_0^{2\pi} t^a \sum_{m=1}^{\infty}\frac{\cos(mt)}{m^b}\sum_{n=1}^{\infty}\frac{\cos(nt)}{n}\ \mathrm{d}t \\ &= \int_0^{2\pi} t^a \sum_{m=1}^{N}\frac{\cos(mt)}{m^b}\sum_{n=1}^{\infty}\frac{\cos(nt)}{n}\ \mathrm{d}t \end{aligned}$$

$$+ \int_0^{2\pi} t^a \sum_{m=N+1}^{\infty} \frac{\cos(mt)}{m^b} \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} \, \mathrm{d}t$$

$$= \sum_{m=1}^{N} \sum_{n=1}^{\infty} \int_0^{2\pi} t^a \frac{\cos(mt)}{m^b} \frac{\cos(nt)}{n} \, \mathrm{d}t + O_a\left(\frac{1}{N^2}\right),$$

as

$$\sum_{m=N+1}^{\infty} \frac{\cos(mt)}{m^b} = O\left(\sum_{m=N+1}^{\infty} \frac{1}{m^3}\right) = O\left(\frac{1}{N^2}\right)$$

and $Cl_1(t)$ is absolutely integrable on $[0, 2\pi]$. Then letting N tend to infinity in the last equation, for $b \geq 3$ odd, we get

$$CL(a, b, 1) = \frac{1}{\pi} \int_0^{2\pi} t^a \sum_{m=1}^{\infty} \frac{\cos(mt)}{m^b} \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} \, \mathrm{d}t$$

$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^b} \int_0^{2\pi} t^a \cos(mt) \cos(nt) \, \mathrm{d}t.$$

Now since

$$\sin(mt)\sin(nt) = \frac{\cos((m-n)t) - \cos((m+n)t)}{2}$$

we have

$$\int_0^{2\pi} t^a \sin(mt)\sin(nt) \, \mathrm{d}t = \frac{1}{2} \int_0^{2\pi} t^a \cos((m-n)t) \, \mathrm{d}t - \frac{1}{2} \int_0^{2\pi} t^a \cos((m+n)t) \, \mathrm{d}t.$$

Hence, if b and c are both even,

$$CL(a, b, c) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \sin(mt)\sin(nt) \, \mathrm{d}t$$

$$= \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m-n)t) \, \mathrm{d}t$$

$$- \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m+n)t) \, \mathrm{d}t.$$

However,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m+n)t) \, \mathrm{d}t \tag{5.3}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^c m^b} \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!(m+n)^{2k}}$$

$$= \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!} \omega(c, b, 2k)$$

whereas

(5.4)
$$\sum_{\substack{m,n=1\\ m=n}}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m-n)t)\ \mathrm{d}t = \sum_{m=1}^{\infty} \frac{1}{m^{b+c}} \int_0^{2\pi} t^a\ \mathrm{d}t = \frac{(2\pi)^{a+1}}{a+1}\zeta(b+c)$$

and

(5.5)
$$\begin{aligned}
&\sum_{\substack{m,n=1\\ m>n}}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m-n)t)\ \mathrm{d}t \\
&= \sum_{\substack{m,n=1\\ m>n}}^{\infty} \frac{1}{n^c m^b} \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!(m-n)^{2k}} \\
&= \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!} \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n^c m^b (m-n)^{2k}} \\
&= \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{l^{2k} n^c (n+l)^b} \\
&= \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!} \omega(2k,c,b)
\end{aligned}$$

and similarly

(5.6)
$$\sum_{\substack{m,n=1\\ m<n}}^{\infty} \frac{1}{n^c m^b} \int_0^{2\pi} t^a \cos((m-n)t)\ \mathrm{d}t = \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a+1-2k}}{(a+1-2k)!} \omega(2k,b,c).$$

Therefore, if b and c are both even, we have

(5.7) $CL(a,b,c)$
$$\begin{aligned}
&= \frac{(2\pi)^a}{a+1}\zeta(b+c) \\
&+ \sum_{k=1}^{\frac{a}{2}} (-1)^{k-1} \frac{a!(2\pi)^{a-2k}}{(a+1-2k)!} \left[\omega(2k,b,c) + \omega(2k,c,b) - \omega(c,b,2k)\right] \\
&= \mathbb{Q}\zeta(2)^{\frac{a}{2}}\zeta(b+c) \\
&+ \sum_{k=1}^{\frac{a}{2}} \mathbb{Q}\zeta(2)^{\frac{a}{2}-k} \left[\omega(2k,b,c) + \omega(2k,c,b) - \omega(c,b,2k)\right].
\end{aligned}$$

On the other hand by Equation (5.1),

$$\int_0^{2\pi} t^a \cos(mt)\cos(nt)\ \mathrm{d}t = \frac{1}{2}\int_0^{2\pi} t^a \cos((m+n)t)\ \mathrm{d}t + \frac{1}{2}\int_0^{2\pi} t^a \cos((m-n)t)\ \mathrm{d}t.$$

So, if b, c are both odd and at least one of them is different from 1, by making use of equations (5.3), (5.4), (5.5) and (5.6) we obtain that

$$
\begin{aligned}
(5.8)\ CL(a,b,c) & \\
&= \frac{1}{\pi}\int_0^{2\pi} t^a \sum_{m=1}^{\infty}\frac{\cos(mt)}{m^b}\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^c}\ \mathrm{d}t \\
&= \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^b n^c}\int_0^{2\pi} t^a \cos(mt)\cos(nt)\ \mathrm{d}t \\
&= \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^b n^c}\int_0^{2\pi} t^a \cos((m+n)t)\ \mathrm{d}t \\
&+ \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^b n^c}\int_0^{2\pi} t^a \cos((m-n)t)\ \mathrm{d}t \\
&= \frac{(2\pi)^a}{a+1}\zeta(b+c) \\
&+ \sum_{k=1}^{\frac{a}{2}}(-1)^{k-1}\frac{a!(2\pi)^{a-2k}}{(a+1-2k)!}\left[\omega(2k,b,c)+\omega(2k,c,b)+\omega(c,b,2k)\right] \\
&= \mathbb{Q}\zeta(2)^{\frac{a}{2}}\zeta(b+c) \\
&+ \sum_{k=1}^{\frac{a}{2}}\mathbb{Q}\zeta(2)^{\frac{a}{2}-k}\left[\omega(2k,b,c)+\omega(2k,c,b)+\omega(c,b,2k)\right]
\end{aligned}
$$

as desired. The result of the theorem on Euler sums $H(p,q)$ directly follows from Theorem 1.1 and Equations (5.7), (5.8). So, we are done with the proof of Theorem 1.2.

We next recompute the Euler sum $H(2,4)$ and evaluate the integrals $CL(2,2,2)$ and $CL(3,2,1)$ in terms of the special values of the Riemann zeta function using our approach. For the computation of $H(2,4)$, we compute the double series $\omega(2,2,2)$ and for the computation of the double series $\omega(2,2,2)$, we use some arguments of [3]. We give the following proof in full details as it reflects the idea of Theorem 7.1.

Corollary 5.1. *We have the following evaluations:*

i. $H(2,4) = \zeta(3)^2 - \dfrac{1}{3}\zeta(6).$

ii. $CL(2,2,2) = \dfrac{44}{3}\zeta(6)$.

iii. $CL(3,2,1) = -22\zeta(6)$.

Proof. (i) Taking $r=1$ in Equation (4.8), we have

$$\begin{aligned}\text{(5.9)}\quad &\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)t^2\cos(nt)\ \mathrm{d}t\\ &= \frac{3}{\pi}\int_0^{2\pi}t^2 Sl_4(t)\ \mathrm{d}t+\frac{1}{2\pi}\int_0^{2\pi}t^2 Sl_2(t)Sl_2(t)\ \mathrm{d}t-\frac{1}{2}CL(2,2,2).\end{aligned}$$

Taking $r=1$ in Equation (4.9), we also have

$$\text{(5.10)}\qquad \frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)t^2\cos(nt)\ \mathrm{d}t = 8H(1,5)+4H(2,4).$$

On the other hand,

(5.11)

$$\frac{3}{\pi}\int_0^{2\pi}t^2 Sl_4(t)\ \mathrm{d}t = \frac{3}{\pi}\int_0^{2\pi}t^2\left(\frac{\pi^4}{90}-\frac{\pi^2t^2}{12}+\frac{\pi t^3}{12}-\frac{t^4}{48}\right)\ \mathrm{d}t = \frac{12}{945}\pi^6 = 12\zeta(6)$$

and

(5.12)

$$\frac{1}{2\pi}\int_0^{2\pi}t^2 Sl_2(t)Sl_2(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}t^2\left(\frac{\pi^2}{6}-\frac{\pi t}{2}+\frac{t^2}{4}\right)^2\ \mathrm{d}t = \frac{8}{945}\pi^6 = 8\zeta(6)$$

by direct computation. Taking $a=b=c=2$ in Equation (5.7), we get

$$\text{(5.13)}\qquad -\frac{1}{2}CL(2,2,2) = -\frac{4\pi^2}{6}\zeta(4)-\omega(2,2,2) = -4\zeta(2)\zeta(4)-\omega(2,2,2).$$

In order to compute the double series $\omega(2,2,2)$, for every integer $p>1$ and $x>0$ we define the function $f_p(x)$ by

$$f_p(x) = \sum_{n=1}^{\infty}\frac{1}{n^p(n+x)}.$$

It is well-known that

$$f_1(x) = \frac{1}{x}(\psi(x+1)+\gamma),$$

where ψ is the digamma function and γ is Euler's constant. Then,

$$\begin{aligned}f_2(x) &= \sum_{n=1}^{\infty}\frac{1}{n^2(n+x)} = \sum_{n=1}^{\infty}\frac{x+n-n}{n^2x(n+x)} = \sum_{n=1}^{\infty}\frac{1}{n^2x}-\sum_{n=1}^{\infty}\frac{1}{nx(n+x)}\\ &= \frac{1}{x}(\zeta(2)-f_1(x)) = \frac{1}{x}\zeta(2)-\frac{1}{x^2}(\psi(x+1)+\gamma).\end{aligned}$$

Therefore,

$$\frac{d}{dx}\left(-\sum_{n=1}^{\infty}\frac{1}{n^2(n+x)}\right)=\frac{d}{dx}\left(-\frac{1}{x}\zeta(2)+\frac{1}{x^2}(\psi(x+1)+\gamma)\right)$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{n^2(n+x)^2}=\frac{1}{x^2}\zeta(2)-\frac{2}{x^3}(\psi(x+1)+\gamma)+\frac{1}{x^2}\psi'(x+1).$$

Then since

$$\psi(m+1)+\gamma=h_m \quad \text{and} \quad \psi'(m+1)=\zeta(2)-H_m^{(2)}$$

for every positive integer m, we get

$$\begin{aligned}\omega(2,2,2)&=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1}^{\infty}\frac{1}{n^2(n+m)^2}\\&=\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{1}{m^2}\zeta(2)-\frac{2}{m^3}h_m+\frac{1}{m^2}(\zeta(2)-H_m^{(2)})\right)\\&=2\zeta(2)\zeta(4)-2H(1,5)-H(2,4).\end{aligned}\tag{5.14}$$

Now combining Equation (5.10), (5.11), (5.12), (5.13) and (5.14) via Equation (5.9), we get

$$6H(1,5)+3H(2,4)=20\zeta(6)-6\zeta(2)\zeta(4).\tag{5.15}$$

By Equation (1.1),

$$H(1,5)=\frac{7}{2}\zeta(6)-\zeta(2)\zeta(4)-\frac{1}{2}\zeta(3)^2.\tag{5.16}$$

Hence substituting Equation (5.16) in Equation (5.15), we get

$$H(2,4)=\zeta(3)^2-\frac{1}{3}\zeta(6).\tag{5.17}$$

(ii) Equations (5.14), (5.16) and (5.17) give $\omega(2,2,2)$ in terms of the special values of the Riemann zeta function by

$$\omega(2,2,2)=4\zeta(2)\zeta(4)-\frac{20}{3}\zeta(6).\tag{5.18}$$

Then from Equations (5.13) and (5.18), it follows that

$$CL(2,2,2)=8\zeta(2)\zeta(4)+2\omega(2,2,2)=16\zeta(2)\zeta(4)-\frac{40}{3}\zeta(6)=\frac{44}{3}\zeta(6).$$

(iii) Taking $r=1$ in Equation (4.11), we get

$$CL(3,2,1)=\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n}+\frac{2h_n}{n^2}\right)t^3\sin(nt)\ \mathrm{d}t\tag{5.19}$$

$$-\frac{3}{\pi}\int_0^{2\pi} t^3 Sl_3(t)\ \mathrm{d}t - \frac{1}{\pi}\int_0^{2\pi} t^3 Sl_2(t)Sl_1(t)\ \mathrm{d}t.$$

Taking $r=1$ in Equation (4.12) gives that

$$\begin{aligned}&\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n}+\frac{2h_n}{n^2}\right)t^3\sin(nt)\ \mathrm{d}t\\ =\ & -16\pi^2 H(1,3)+24H(1,5)-8\pi^2 H(2,2)+12H(2,4).\end{aligned}$$

When we write the Euler sums $H(1,3)$, $H(1,5)$, $H(2,2)$ and $H(2,4)$ in the last equation in terms of the special values of the Riemann zeta function using Equations (1.1), (1.2) and (5.17), we get

$$\frac{1}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n}+\frac{2h_n}{n^2}\right)t^3\sin(nt)\ \mathrm{d}t = -319\zeta(6). \tag{5.20}$$

On the other hand,

$$\frac{3}{\pi}\int_0^{2\pi} t^3 Sl_3(t)\ \mathrm{d}t = \frac{3}{\pi}\int_0^{2\pi} t^3\left(\frac{\pi^2 t}{6}-\frac{\pi t^2}{4}+\frac{t^3}{12}\right)\ \mathrm{d}t = -\frac{8}{35}\pi^6 = -216\zeta(6) \tag{5.21}$$

and

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} t^3 Sl_2(t)Sl_1(t)\ \mathrm{d}t &= \frac{1}{\pi}\int_0^{2\pi} t^3\left(\frac{\pi^2}{6}-\frac{\pi t}{2}+\frac{t^2}{4}\right)\left(\frac{\pi}{2}-\frac{t}{2}\right)\ \mathrm{d}t\\ &= -\frac{3}{35}\pi^6 = -81\zeta(6).\end{aligned} \tag{5.22}$$

Hence by combining Equations (5.19), (5.20), (5.21) and (5.22), we get

$$CL(3,2,1) = -22\zeta(6).$$

as desired. □

6. PROOF OF THEOREM 1.4

In case $p=q$, the reciprocity relation (1.2) gives $H(p,q)=H(p,p)$ directly in terms of the special values of the Riemann zeta function as

$$H(p,p)=\frac{1}{2}\left[\zeta(p)^2+\zeta(2p)\right].$$

By Corollary 5.1, the theorem holds for $(p,q)=(2,4)$. If $p\neq q$ and the theorem holds for $H(p,q)$, then by reciprocity it also holds for $H(q,p)$ as

$$H(q,p)=\zeta(p)\zeta(q)+\zeta(p+q)-H(p,q).$$

Hence the theorem also holds for $(p,q)=(4,2)$ and it remains to prove the theorem when p is odd and q is even.

So, assume $p \geq 3$ is odd and $q = 2r + 2$ is even. Multiplying both sides of the first equation of Lemma 2.2 by t^{2r+1}, we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} t^{2r+1} \sin(nt) \tag{6.1}$$
$$= t^{2r+1} Cl_{p+1}(t) + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k a_k t^{2r+1} \big(Sl_{p-k}(t) Cl_{k+1}(t) + Sl_{k+1}(t) Cl_{p-k}(t) \big),$$

where $a_k = 1$ for $0 \leq k \leq \dfrac{p-3}{2}$ and $a_{\frac{p-1}{2}} = \dfrac{1}{2}$. We will now integrate both sides of Equation (6.1) from 0 to 2π and equate them. Let us begin with the left-hand side. By making use of Theorem 1.3 and Equation (3.7), we get

$$\begin{aligned}
\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} t^{2r+1} \sin(nt)\ dt &= \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} \int_0^{2\pi} t^{2r+1} \sin(nt)\ dt \\
&= \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n} \sum_{k=1}^{r+1} (-1)^k \frac{(2r+1)!(2\pi)^{2r+3-2k}}{(2r+3-2k)!n^{2k-1}} \\
&= \sum_{k=1}^{r+1} \mathbb{Q}\pi^{2r+3-2k} H(p, 2k).
\end{aligned} \tag{6.2}$$

When we integrate the right-hand side of Equation (6.1), we encounter integrals

$$\int_0^{2\pi} t^{2r+1} Cl_{p+1}(t)\ dt \quad \text{and} \quad \int_0^{2\pi} t^{2r+1} Sl_a(t) Cl_b(t)\ dt$$

where a and b are positive integers with $a + b = p + 1$. Note that if $b > 1$ is an integer, then Theorem 1.3 clearly applies to the sequence $(b_n)_n$, where $b_n = 1/n^b$ so that $t^m Cl_b(t)$ is termwise integrable on $[0, 2\pi]$ for every non-negative integer m. So,

$$\begin{aligned}
\int_0^{2\pi} t^{2r+1} Cl_{p+1}(t)\ dt &= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \int_0^{2\pi} t^{2r+1} \sin(nt)\ dt \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{r+1} (-1)^k \frac{(2r+1)!(2\pi)^{2r+3-2k}}{(2r+3-2k)!n^{2k-1}} \\
&= \sum_{k=1}^{r+1} \mathbb{Q}\pi^{2r+3-2k} \zeta(p+2k).
\end{aligned} \tag{6.3}$$

Now let us compute integrals of the form

$$\int_0^{2\pi} t^{2r+1} Sl_a(t) Cl_b(t)\ dt$$

where a and b are positive integers such that $a+b=p+1$ is even. In case a and b are both even,

$$Sl_a(t)=\sum_{j=0}^{a}\mathbb{Q}\pi^{a-j}t^j \quad \text{and} \quad Cl_b(t)=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n^b}$$

and

$$\begin{aligned}
(6.4)\qquad & \int_0^{2\pi} t^{2r+1}Sl_a(t)Cl_b(t)\ \mathrm{d}t \\
&= \sum_{j=0}^{a}\mathbb{Q}\pi^{a-j}\int_0^{2\pi} t^{2r+j+1}Cl_b(t)\ \mathrm{d}t \\
&= \sum_{j=0}^{a}\mathbb{Q}\pi^{a-j}\sum_{n=1}^{\infty}\frac{1}{n^b}\int_0^{2\pi} t^{2r+j+1}\sin(nt)\ \mathrm{d}t \\
&= \sum_{j=0}^{a}\mathbb{Q}\pi^{a-j}\sum_{n=1}^{\infty}\frac{1}{n^b}\sum_{k=1}^{[\frac{2r+j+2}{2}]}(-1)^k\frac{(2r+j+1)!(2\pi)^{2r+j+3-2k}}{(2r+j+3-2k)!n^{2k-1}} \\
&= \sum_{j=0}^{a}\sum_{k=1}^{r+1+[\frac{j}{2}]}\mathbb{Q}\pi^{2r+a+3-2k}\zeta(b-1+2k) \\
&= \sum_{k=1}^{r+1+\frac{a}{2}}\mathbb{Q}\pi^{2r+a+3-2k}\zeta(b-1+2k).
\end{aligned}$$

In case a and b are both odd,

$$Cl_b(t)=\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^b}$$

and using Equation (3.10) we similarly get

$$(6.5)\qquad \int_0^{2\pi} t^{2r+1}Sl_a(t)Cl_b(t)\ \mathrm{d}t=\sum_{k=1}^{r+\frac{a+1}{2}}\mathbb{Q}\pi^{2r+a+2-2k}\zeta(b+2k).$$

Now combining equations (6.2), (6.3), (6.4) and (6.5), we get

$$\begin{aligned}
\sum_{k=1}^{r+1}\mathbb{Q}\pi^{2r+3-2k}H(p,2k) &= \sum_{k=1}^{r+1}\mathbb{Q}\pi^{2r+3-2k}\zeta(p+2k) \\
&\quad + \sum_{\substack{k=0 \\ k:even}}^{\frac{p-1}{2}}\ \sum_{i=1}^{r+\frac{p+1-k}{2}}\mathbb{Q}\pi^{2r+p+2-2i-k}\zeta(1+2i+k)
\end{aligned}$$

$$+\sum_{\substack{k=0\\k:even}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{k}{2}}\mathbb{Q}\pi^{2r+3-2i+k}\zeta(p+2i-k)$$
$$+\sum_{\substack{k=0\\k:odd}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{p-k}{2}}\mathbb{Q}\pi^{2r+p+3-2i-k}\zeta(2i+k)$$
$$+\sum_{\substack{k=0\\k:odd}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{k+1}{2}}\mathbb{Q}\pi^{2r+4-2i+k}\zeta(p-1+2i-k).$$

Dividing both sides of the above equation by π and substituting $6\zeta(2)$ for π^2, we get

$$\sum_{k=1}^{r+1}\mathbb{Q}\zeta(2)^{r+1-k}H(p,2k)=\sum_{k=1}^{r+1}\mathbb{Q}\zeta(p+2k)\zeta(2)^{r+1-k} \tag{6.6}$$
$$+\sum_{\substack{k=0\\k:even}}^{\frac{p-1}{2}}\sum_{i=1}^{r+\frac{p-k+1}{2}}\mathbb{Q}\zeta(1+2i+k)\zeta(2)^{r-i+\frac{p-k+1}{2}}$$
$$+\sum_{\substack{k=0\\k:even}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{k}{2}}\mathbb{Q}\zeta(p+2i-k)\zeta(2)^{r-i+\frac{k+2}{2}}$$
$$+\sum_{\substack{k=0\\k:odd}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{p-k}{2}}\mathbb{Q}\zeta(2i+k)\zeta(2)^{r-i+\frac{p-k+2}{2}}$$
$$+\sum_{\substack{k=0\\k:odd}}^{\frac{p-1}{2}}\sum_{i=1}^{r+1+\frac{k+1}{2}}\mathbb{Q}\zeta(p-1+2i-k)\zeta(2)^{r-i+\frac{k+3}{2}}$$

where the right-hand side is always in terms of special Riemann zeta values for all integers $r\geq 0$.

Now to prove the theorem for all odd $p\geq 3$ and even $q=2r+2$, we fix p and proceed by induction on r. For $r=0$, the left-hand side of Equation (6.6) is a non-zero rational multiple of $H(p,2)$, hence $H(p,2)$ can be evaluated in terms of the special values of the Riemann zeta function. For the inductive step, assume the theorem holds for all non-negative integers $s<r$. By Equation

(6.6),

$$H(p,2r+2) = -\sum_{k=1}^{r} \mathbb{Q}\zeta(2)^{r+1-k} H(p,2k) + \text{RHS} \tag{6.7}$$

where RHS stands for the right-hand side of Equation (6.6). By the induction hypothesis, $H(p,2k)$ can be evaluated in terms of special zeta values for all $1 \le k \le r$, so can $H(p,2r+2)$ by Equation (6.7).

7. SOME COROLLARIES

Our first corollary states that for an even weight w, if some Euler sums of even weight up to a certain weight can be evaluated in terms of the special values of the Riemann zeta function, then all Euler sums of even weight up to that weight can be evaluated in terms of the special values of the Riemann zeta function. In other words, if the conjecture on Euler sums of even weight that we mentioned before is true for some $H(p,q)$ with an even weight $w = p+q \ge 8$, then there is another pair (p_1,q_1) where $p_1+q_1 \le w$, $p_1 \notin \{p,q\}$ and the conjecture also holds for $H(p_1,q_1)$. The following result may also follow from what Euler did, and for rigorous proofs for Euler's results, we direct the reader to [5].

Corollary 7.1. *Let $w \ge 8$ be an even integer. For any even integer ρ with $8 \le \rho \le w$, suppose that*

$$|\{H(p,q) : p+q=\rho\}| - |\{H(p,q) \in \Omega : p+q=\rho\}| \le 2,$$

where $\Omega = \mathbb{Q}[\zeta(k) : k \ge 2]$. Then, $H(p,q) \in \Omega$ for all p,q such that $p+q$ is even and $p+q \le w$.

Proof. Let $r \ge 2$. By Equations (4.8) and (4.9),

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2}+\frac{2h_n}{n^3}\right)\sum_{k=1}^{r}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r+1-2k}}{(2r+1-2k)!n^{2k}} = \mathbb{Q}\pi^{2r+4} - \frac{1}{2}CL(2r,2,2).$$

Denote a typical element of the ring $\Omega = \mathbb{Q}[\zeta(k) : k \ge 2]$ by ZV_i (zeta value) for some i. Note that $\pi^h \in \Omega$ for any even integer $h \ge 0$. Then since $\pi^{2r+4} \in \Omega$ and $H(1,m) \in \Omega$ for every integer $m > 1$, we get

$$\begin{aligned}(-1)^{r-1}2(2r)!H(2,2r+2) + \frac{1}{\pi}\sum_{k=2}^{r-1}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r+1-2k}}{(2r+1-2k)!}H(2,2+2k) \\ = ZV_1 - \frac{1}{2}CL(2r,2,2).\end{aligned} \tag{7.1}$$

By Equation (5.7),

$$CL(2r,2,2) = ZV_2 + (-1)^{r-1}(2r)!\left[2\omega(2r,2,2) - \omega(2,2,2r)\right] + \sum_{k=1}^{r-1}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r-2k}}{(2r+1-2k)!}\left[2\omega(2k,2,2) - \omega(2,2,2k)\right]. \tag{7.2}$$

By [3, Corollary 2.4] and Equation (1.1),

$$\omega(a,b,c) = ZV_3 + \frac{(-1)^{a-1}}{(a-1)!}\sum_{k=1}^{c-1}\frac{(a+c-k-2)!}{(c-k-1)!}H(k+1, a+b+c-(k+1)).$$

This means that $\omega(a,b,c)$ can be evaluated in terms of the special values of the Riemann zeta function and Euler sums of weight $a+b+c$. Hence,

$$\omega(2r,2,2) = ZV_4 - H(2,2r+2) \tag{7.3}$$

and

$$\omega(2,2,2r) = ZV_5 - (2r-1)H(2,2r+2) - \sum_{k=2}^{2r-1}(2r-k)H(k+1, 2r+4-(k+1)). \tag{7.4}$$

Combining Equations (7.1), (7.2), (7.3) and (7.4), we get

$$\begin{aligned}
&(-1)^{r-1}2(2r)!H(2,2r+2) + \frac{1}{\pi}\sum_{k=2}^{r-1}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r+1-2k}}{(2r+1-2k)!}H(2,2+2k) \\
= \;& ZV_6 + (-1)^r\frac{(2r)!}{2}(2r-3)H(2,2r+2) \\
+ \;& \frac{1}{2}(-1)^r(2r)!\sum_{k=2}^{2r-1}(2r-k)H(k+1,2r+4-(k+1)) \\
- \;& \frac{1}{2}\sum_{k=1}^{r-1}(-1)^{k-1}\frac{(2r)!(2\pi)^{2r-2k}}{(2r+1-2k)!}\left[2\omega(2k,2,2) - \omega(2,2,2k)\right].
\end{aligned} \tag{7.5}$$

Observe that the coefficients of $H(2,2r+2)$ on opposite sides of the above equation have opposite signs, so they cannot cancel each other.

Now we proceed by induction on weight $w = 2r+4$. If we take $r=2$ in Equation (7.5), one sees that

$$-48H(2,6) = ZV_7 + 12H(2,6) + 24H(3,5).$$

This yields that $H(2,6) \in \Omega$ if and only if $H(3,5) \in \Omega$. So, by reciprocity of Euler sums (1.2), we have the assertion for $w = 8$. Suppose the theorem holds for $w-2 = 2r+2 \geq 8$. We will show that it also holds for $w = 2r+4$. So, we suppose that for any even integer ρ with $8 \leq \rho \leq w$, at most two of

$H(p,q)$ such that $p+q=\rho$ is not in $\mathbb{Q}[\zeta(k):k\geq 2]$. Then by the induction hypothesis all Euler sums $H(p,q)$ of weight $\leq w-2$ are in Ω. Note also that if $H(p,q)\in\Omega$, then $\pi^h H(p,q)\in\Omega$ for any even $h\geq 0$. Then by Equation (7.5), we arrive at the equality

$$\alpha_r H(2,2r+2) = ZV_8 + \frac{1}{2}(-1)^r(2r)!\sum_{k=2}^{2r-1}(2r-k)H(k+1,2r+4-(k+1)) \tag{7.6}$$

for some non-zero integer α_r. In the last equation, only Euler sums $H(p,q)$ of weight w with $p\leq 2r$ occur and their coefficients are non-zero.

If all $H(p,q)$ of weight w with $p\leq 2r$ are in Ω possibly except for $H(2,2r+2)$ and $H(2r+2,2)$, since $H(2r+2,2)$ does not occur on Equation (7.6) and $\alpha_r\neq 0$, we get that $H(2,2r+2)\in\Omega$ by Equation (7.6). By reciprocity, $H(2r+2,2)\in\Omega$ as well. The same idea applies if all $H(p,q)$ of weight w with $p\leq 2r$ are in Ω possibly except for $H(3,2r+1)$ and $H(2r+1,3)$. Now suppose that all $H(p,q)$ of weight w with $p\leq 2r$ are in Ω possibly except for $H(p,q)$ and $H(q,p)$ where $4\leq p\leq 2r$. Observe that both $H(p,q)$ and $H(q,p)$ occur in Equation (7.6) with non-zero distinct coefficients, say α, β. Then by Equation (7.6), $\alpha H(p,q)+\beta H(q,p)$ is in Ω and by reciprocity, $H(p,q)+H(q,p)$ is also in Ω. This yields that both $H(p,q)$ and $H(q,p)$ are in Ω. Hence, if for any even integer ρ with $8\leq\rho\leq w$, at most two Euler sums of weight ρ are not in Ω, then actually all Euler sums of even weight $\leq w$ are in Ω. □

For instance, if we take w to be 8, then we see that $H(2,6)$ is not in Ω if and only if $H(3,5)$ is not in Ω. For $w=10$, we obtain that if $H(2,8)$ is not in Ω, then at least one of $H(2,6)$, $H(4,6)$ and $H(3,7)$ is also not in Ω. Similarly, if $H(4,6)$ is not in Ω, then at least one of $H(2,6)$, $H(2,8)$ and $H(3,7)$ is also not in Ω; and if $H(3,7)$ is not in Ω, then at least one of $H(2,6)$, $H(2,8)$ and $H(4,6)$ is also not in Ω.

The Catalan constant is defined by the series

$$G=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^2}.$$

It is still not known whether G is rational or not. However, it is commonly believed that G is even transcendental.

Our second corollary states that if G is algebraic then we obtain a new transcendental number related to the generalized harmonic series.

Corollary 7.2. *Either the Catalan constant G is transcendental or at least one of the series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}^{(2)}}{n^2},\qquad \sum_{n=1}^{\infty}(-1)^n\frac{h_{2n}}{n^3}$$

is transcendental.

Proof. Take $t = \frac{\pi}{2}$ in Lemma 2.4. Note that $\cos\left(\frac{n\pi}{2}\right) = (-1)^{\frac{n}{2}}$ if n is even and zero otherwise. Thus,

$$\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^2} + \frac{2h_n}{n^3}\right)\cos\left(\frac{n\pi}{2}\right) = \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{2n}^{(2)}}{n^2} + \frac{h_{2n}}{n^3}\right).$$

On the other hand,

$$3Sl_4\left(\frac{\pi}{2}\right) = 3\left[\frac{\pi^4}{90} - \frac{\pi^2}{12}\left(\frac{\pi}{2}\right)^2 + \frac{\pi}{12}\left(\frac{\pi}{2}\right)^3 - \frac{1}{48}\left(\frac{\pi}{2}\right)^4\right] = -\frac{7}{3840}\pi^4,$$

$$\frac{1}{2}Sl_2\left(\frac{\pi}{2}\right)Sl_2\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi^2}{6} - \frac{\pi^2}{4} + \frac{\pi^2}{16}\right)^2 = \frac{1}{4608}\pi^4$$

and

$$-\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)Cl_2\left(\frac{\pi}{2}\right) = -\frac{1}{2}G^2$$

as

$$G = Cl_2\left(\frac{\pi}{2}\right).$$

Hence,

$$\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{2n}^{(2)}}{n^2} + \frac{h_{2n}}{n^3}\right) = q\pi^4 - \frac{1}{2}G^2$$

for some non-zero rational number q and since π is transcendental, we obtain the corollary. □

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