



Pseudofinite groups as fixed points in simple groups of finite Morley rank

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ABSTRACT

We prove that if the group of fixed points of a generic automorphism of a simple group of finite Morley rank is pseudofinite, then this group is an extension of a (twisted) Chevalley group over a pseudofinite field. On the way to obtain this result, we classify non-abelian definably simple pseudofinite groups of finite centralizer dimension (using the ideas of Wilson (1993) [29]).

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0. Introduction

A group of finite Morley rank is a group equipped with a rank function from the set of non-empty definable subsets to the non-negative integers. This rank function imitates the Zariski dimension in algebraic geometry (for details see [6]). Algebraic groups over algebraically closed fields are examples of groups of finite Morley rank in which case the Morley rank coincides with the Zariski dimension. Actually, the only known infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields. The *Algebraicity Conjecture* (stated independently by Gregory Cherlin and Boris Zilber in the 1970s) says that any infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. Although this conjecture is still open, important results have been obtained by adapting and generalizing ideas from the classification of finite simple groups. This approach was suggested by Alexandre Borovik. For a detailed information about the Borovik program, we refer the reader to [1]. The current status of the conjecture can be summarized as follows:

In the theory of groups of finite Morley rank, the Sylow 2-subgroups are known to be conjugate and each of them is a finite extension of the central product $U * T$ where U is a definable connected 2-group of bounded exponent and T is a divisible abelian 2-group. Therefore, depending on the structure of the connected components of Sylow 2-subgroups, the Algebraicity Conjecture breaks up into four cases.

- (a) Even type: $U \neq 1$ and $T = 1$ (identified with Chevalley groups over algebraically closed fields of characteristic 2).
- (b) Odd type: $U = 1$ and $T \neq 1$ (structural results are obtained on potential non-algebraic odd type groups).
- (c) Mixed type: $U \neq 1$ and $T \neq 1$ (no such groups exist).
- (d) Degenerate type: $U = 1$ and $T = 1$ (difficult case).

A new approach to the Algebraicity Conjecture originates from results and ideas of Hrushovski in [12], where some classes of structures, including simple groups of finite Morley rank, with generic automorphisms are considered. Hrushovski proved that the set of fixed points of a generic automorphism is a PAC structure with a definable measure (for details, see [12]). In the particular case of simple groups of finite Morley rank, the ultimate aim is to prove that the group of fixed points is pseudofinite, that is, an infinite model of the theory of finite groups. However, unlike pseudofinite fields (which are characterized by Ax in [2]), it is not known how to characterize pseudofinite groups.

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As it is mentioned by Hrushovski in [12], the Algebraicity Conjecture implies the following conjecture.

Principal Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism α . Then, the group of fixed points of α is pseudofinite.*

This paper is a first step to construct a bridge between the Algebraicity Conjecture and the Principle Conjecture from the other direction. More precisely, we aim to prove that the Principal Conjecture implies the Algebraicity Conjecture. This implication can be stated in the following form.

Intermediate Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism α . Assume that the group of fixed points of α is pseudofinite. Then, G is isomorphic to a Chevalley group over an algebraically closed field.*

We can state the main result as follows.

Theorem 3.1. *Let G be an infinite simple group of finite Morley rank and α be an automorphism of G such that the definable hull of $C_G(\alpha)$ is G . If $C_G(\alpha)$ is pseudofinite, then there is a definable (in $C_G(\alpha)$) normal subgroup S of $C_G(\alpha)$ such that S is isomorphic to a (twisted) Chevalley group over a pseudofinite field and $C_G(\alpha)$ embeds in $\text{Aut}(S)$.*

Here, $C_G(\alpha)$ denotes the group of fixed points of α in G . Note that the assumption on the definable hull of $C_G(\alpha)$ is satisfied by generic automorphisms of groups of finite Morley rank. Moreover, we will observe in the last section that under the assumptions of this theorem, degenerate-type groups cannot exist.

The first result we obtain on the way to prove Theorem 3.1 is the classification of non-abelian definably simple pseudofinite groups of finite centralizer dimension. For this classification, we first observe that a sizeable part of Wilson’s classification proof for simple pseudofinite groups in [29] works for non-abelian definably simple pseudofinite groups. Then, with the help of the finite centralizer dimension property, we show that such groups are elementarily equivalent to (twisted) Chevalley groups over pseudofinite fields. Moreover, thanks to the results obtained by Ryten [25], we can replace the elementary equivalence by an isomorphism and we obtain the following result.

Proposition 2.11. *Every non-abelian definably simple pseudofinite group of finite centralizer dimension is isomorphic to a (twisted) Chevalley group over a pseudofinite field.*

Note that in some cases a non-abelian definably simple group G is automatically simple. For example, this holds if the theory of G is supersimple [28, Proposition 5.4.10]. However, in our case we do not know whether the theory of a pseudofinite group of finite centralizer dimension is supersimple or not.

The structure of this paper can be outlined as follows.

In the first section, we give necessary background information about basic model theoretic concepts, ultraproducts, pseudofinite groups, groups of finite Morley rank, and we explain our terminology and notation.

In the second section, first, we analyze the structure of abelian definably simple pseudofinite groups. Then, we classify non-abelian definably simple pseudofinite groups of finite centralizer dimension.

In the third section, we prove the main result (Theorem 3.1).

1. Preliminaries

This section covers some background material which will be necessary throughout this paper. We assume that the reader is familiar with the basic notions in model theory such as structure, language, formula and theory.

Two structures \mathfrak{M} and \mathfrak{N} , in a common language \mathcal{L} , are *elementarily equivalent* if they satisfy the same \mathcal{L} -sentences and we write $\mathfrak{M} \equiv \mathfrak{N}$. A theory is *complete* if all of its models are elementarily equivalent.

A *definable set* in a structure \mathfrak{M} is a subset $X \subseteq M^n$ (M denotes the underlying set of the structure \mathfrak{M}) which is the set of realizations of a first order formula in the language of the structure \mathfrak{M} .

We emphasize that, throughout this paper, we stay within the borders of first order logic and when we say definable, we mean definable possibly with parameters. We consider groups (resp. fields) as structures in the pure group (resp. field) language.

1.1. Ultraproducts and pseudofinite groups

We give some background information about ultraproducts and pseudofinite groups and list some related facts. Proofs of these facts can be found in the indicated references. We refer the reader to [4,8] for a detailed information about ultraproducts.

An *ultrafilter* \mathcal{U} on a non-empty set I is a proper subset of the power set of I such that \mathcal{U} is closed under finite intersections and taking supersets and for any subset A of I , we have $A \in \mathcal{U}$ if and only if $I \setminus A \notin \mathcal{U}$. For any $i \in I$, the set $\{X \subseteq I \mid i \in X\}$ forms an ultrafilter which is called the *principal ultrafilter generated by i* and any ultrafilter containing a finite set turns out to be a principal ultrafilter. Note that the existence of non-principal ultrafilters is guaranteed by Zorn’s lemma.

Now let $\prod_{i \in I} X_i$ denote the cartesian product of non-empty structures in a common language \mathcal{L} and let \mathcal{U} be an ultrafilter on I . Considering the elements of $\prod_{i \in I} X_i$ as functions from I to $\bigcup_{i \in I} X_i$, define a relation on $\prod_{i \in I} X_i$ as follows.

$$x \sim_{\mathcal{U}} y \quad \text{if and only if} \quad \{i \in I \mid x(i) = y(i)\} \in \mathcal{U}.$$

It is routine to check that $\sim_{\mathcal{U}}$ is an equivalence relation. The quotient of the cartesian product with respect to this equivalence relation is called the *ultraproduct* and we denote it by $\prod_{i \in I} X_i / \mathcal{U}$. When all structures are the same, then this ultraproduct is called *ultrapower*. We denote the equivalence class of $x \in \prod_{i \in I} X_i$ with respect to $\sim_{\mathcal{U}}$ by $[x]_{\mathcal{U}}$. An ultraproduct of L -structures has a natural (coordinatewise) L -structure.

The importance of the ultraproduct construction is expressed in Łoś's Theorem [17] which states that a first order formula is satisfied in the ultraproduct if and only if it is satisfied in the structures indexed by a set belonging to the ultrafilter. In particular, first order properties of the ultraproduct are determined by the first order properties of the structures in the ultraproduct together with the choice of the ultrafilter.

It can be observed that if \mathcal{U} is an ultrafilter on a set I which is a disjoint union of finitely many subsets I_1, \dots, I_m , then exactly one of I_j is in \mathcal{U} . Moreover, whenever $J \in \mathcal{U}$ then, the set $\mathcal{U}_J = \{X \cap J \mid X \in \mathcal{U}\}$ forms an ultrafilter on J and we have $\prod_{i \in I} X_i / \mathcal{U} \cong \prod_{j \in J} X_j / \mathcal{U}_j$. In particular, ultraproducts over principal ultrafilters are isomorphic to one of the structures in the cartesian product. Therefore, throughout this paper, the ultrafilters in concern will always be non-principal.

Remark 1.1. Throughout the text, we will use the well-known properties mentioned in the previous paragraph. Moreover, whenever we have $\prod_{i \in I} X_i / \mathcal{U} \cong \prod_{j \in J} X_j / \mathcal{U}_j$, we will abuse the language, and keep writing \mathcal{U} and I for the ultrafilter and the index set.

Ulrich Felgner introduced in [11] pseudofinite groups as infinite models of the theory of finite groups in accordance with Ax's characterization of pseudofinite fields [2]. By a suitable choice of an ultrafilter, it can be shown that any pseudofinite group is elementarily equivalent to a non-principal ultraproduct of finite groups (see [30]). As we will see in Section 2, the additive group of rational numbers, $(\mathbb{Q}, +)$, is a pseudofinite group. However, the additive group of integers, $(\mathbb{Z}, +)$, is not a pseudofinite group, because the first order statement

the map $x \mapsto x + x$ is one-to-one if and only if it is onto

does not hold in $(\mathbb{Z}, +)$ while it holds in every finite group.

Throughout this paper, a Chevalley group over an arbitrary field K will be denoted by $X(K)$ where X stands for one of the symbols $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $D_n (n \geq 4)$, E_6 , E_7 , E_8 , F_4 , G_2 . These symbols come from the classification of finite dimensional complex simple Lie algebras. Chevalley constructed Chevalley groups over arbitrary fields associated to these symbols. We will not go into details about the construction of neither these groups nor the twisted versions of these groups which can be constructed when the field K has some additional properties (see [26]). We denote twisted Chevalley groups by $X(K)$ as well, where X denotes one of the symbols ${}^2A_n (n \geq 2)$, ${}^2D_n (n \geq 4)$, 3D_4 , 2E_6 , 2B_2 , 2F_4 , 2G_2 . It is known that (twisted) Chevalley groups over arbitrary fields are simple as abstract groups except for ${}^2A_2(\mathbb{F}_4)$, ${}^2B_2(\mathbb{F}_2)$, ${}^2F_4(\mathbb{F}_2)$ and ${}^2G_2(\mathbb{F}_3)$. We refer the reader to [7,26] for details about these groups.

The following result of Françoise Point shows that any infinite Chevalley group over an ultraproduct of finite fields is an example of a pseudofinite group.

Fact 1.2 (Point [23]). *Let $\{X(F_i) \mid i \in I\}$ be a family of (twisted) Chevalley groups of the same type X over finite or pseudofinite fields, and let \mathcal{U} be a non-principal ultrafilter on the set I . Then*

$$\prod_{i \in I} X(F_i) / \mathcal{U} \cong X \left(\prod_{i \in I} F_i / \mathcal{U} \right).$$

If Fact 1.2 is combined with Keisler–Shelah's Ultrapower Theorem [14] then the following result can be obtained.

Fact 1.3 (Wilson [29]). *Any group G which is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field is pseudofinite.*

1.2. Groups of finite Morley rank

We list some basic properties of groups of finite Morley rank which will be used in the sequel without references. The proofs can be found in [6] as well as in the indicated articles.

Fact 1.4 (Macintyre [18]). *A group of finite Morley rank has the descending chain condition on definable subgroups, that is, every descending chain of definable subgroups stabilizes after finitely many steps.*

The descending chain condition on definable subgroups is a strong property which allows one to define the notion of *definable hull*. For any subset X of a group of finite Morley rank G , there is a smallest definable subgroup of G containing X , which is called the *definable hull* of X and denoted by $d(X)$.

The following fact is a corollary of a result by Baldwin and Saxl [3].

Fact 1.5 ([1], Corollary 2.9). *For any subset X of a finite Morley rank group G , the centralizer $C_G(X)$ is a definable subgroup. Moreover, there is an integer n such that for any $Y \subseteq G$ there is $Y_0 \subseteq Y$ with $|Y_0| \leq n$ and $C_G(Y) = C_G(Y_0)$.*

A group with the property as in the moreover part of Fact 1.5 is said to have *finite centralizer dimension*. More precisely, for any integer $k \geq 0$, a group G has centralizer dimension k , written $\text{cdim}(G) = k$, if it has a proper descending centralizer chain of length k and has no such chain of length greater than k . By a proper descending centralizer chain of length k we mean the chain of the form

$$G = C_G(x_0) > C_G(x_1) > C_G(x_1, x_2) > \cdots > C_G(x_1, \dots, x_k) = Z(G).$$

It is well-known that the class of groups with finite centralizer dimension is closed under taking subgroups and finite direct products [21]. Moreover, for any integer $k \geq 0$, having centralizer dimension k is a first order property in the language of groups (see [9,16]).

2. Definably simple pseudofinite groups

The main result of this section is the characterization of non-abelian definably simple pseudofinite groups of finite centralizer dimension as (twisted) Chevalley groups over pseudofinite fields. First, we state the theorem of Wilson about the classification of simple pseudofinite groups and point out some results in the literature strengthening Wilson’s theorem. Then, we work on definably simple pseudofinite groups and after analyzing the structure of the abelian ones, we concentrate on the non-abelian case. We observe that the first part of Wilson’s classification proof works for this case. Then, using the finite centralizer dimension property we obtain our result.

There is a weaker version of simplicity of a group which arises in model theory. A group is called *definably simple* if it has no non-trivial definable proper normal subgroups. In the class of non-abelian groups of finite Morley rank, which includes non-abelian algebraic groups over algebraically closed fields, definably simple groups coincide with the simple ones [24] (note that Wagner’s result mentioned in the introduction is much more general). However, in general, definably simple groups need not be simple. Wilson proved that no ultraproduct of alternating groups is simple (unless it is finite) while it is definably simple (see [30]). Moreover, while no infinite abelian group is simple, some of them are definably simple (see Fact 2.2). Therefore, the distinction between the notions of *definably simple* and *simple* becomes important.

Felgner conjectured in [11] that simple pseudofinite groups are isomorphic to (twisted) Chevalley groups over pseudofinite fields. Although Felgner obtained important results, it was Wilson who classified the simple pseudofinite groups, however, only up to elementary equivalence.

Fact 2.1 (Wilson [29]). *Every simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field.*

As it is pointed out by Wilson in [29], the elementary equivalence can be replaced by an isomorphism for the untwisted case in view of results obtained by Simon Thomas in his dissertation [27]. Moreover, M. Ryten’s recent results in his thesis [25] ensure that Wilson’s theorem can be strengthened for both Chevalley and twisted Chevalley groups and hence Felgner’s conjecture is true.

Now, we concentrate on definably simple pseudofinite groups starting from the abelian ones. The following fact is a folklore.

Fact 2.2. *Let A be an infinite abelian group. The following statements are equivalent.*

- (a) A is definably simple.
- (b) A is torsion-free divisible.
- (c) $A \equiv \prod_{p \in I} C_p / \mathcal{U} \equiv (\mathbb{Q}, +)$ where \mathcal{U} is a non-principal ultrafilter on the set I of all prime numbers and C_p is the cyclic group of order p .

Proof. Assume that (a) holds. Then, any non-trivial element of finite order generates a non-trivial proper definable normal subgroup of A , so A is torsion-free. Moreover, if A is not divisible, then nA is a proper definable normal subgroup of A for some integer $n \geq 2$ which is not possible by our assumption. Hence we get (b). Assume that (b) holds. Since the theory of divisible abelian torsion-free groups is complete and all groups in part (c) are models of this theory, we get the elementary equivalences in (c). As $\prod_{p \in I} C_p / \mathcal{U}$ is a definably simple group we have the implication (c) \Rightarrow (a). \square

The classification of non-abelian definably simple pseudofinite groups will be obtained in two steps. First, we show that every non-abelian definably simple pseudofinite group is elementarily equivalent to an ultraproduct of non-abelian finite simple groups. This result follows from Wilson’s proof of Fact 2.1. However, for a self-contained proof, we include here the results obtained by Wilson and we observe that his arguments work in the case of non-abelian definably simple pseudofinite groups. In the second step, we proceed differently by using our assumption on centralizer chains.

Definition 2.3. Let σ denote the following first order sentence which was defined by Felgner in [11]:

$$\forall x \forall y \left[(x \neq 1 \wedge C_G(x, y) \neq 1) \rightarrow \bigcap_{g \in G} (C_G(x, y) C_G(C_G(x, y)))^g = 1 \right].$$

Above, G stands for an arbitrary group, $C_G(x, y)$ denotes the centralizer of $\{x, y\}$ in G and for any subset $X \subseteq G$, we have $X^g = gXg^{-1}$.

Throughout this paper, $\text{Soc}(G)$ denotes the subgroup generated by minimal normal subgroups of a group G , the so-called socle of G .

Now, we recall some facts from [29] which will be needed in the sequel.

Fact 2.4 (Wilson [29]). *There is an integer k such that each element of each finite non-abelian simple group G is a product of k commutators.*

Remark 2.5. A stronger version of this result is known as *Ore's conjecture* which states that every element of a finite non-abelian simple group is a commutator [22]. This old conjecture was followed by a stronger conjecture, which is attributed to John Thompson, stating that for every finite non-abelian simple group G , there exists a conjugacy class C such that $G = CC$. We will refer to Thompson's conjecture later in the text. For a detailed information about the status of these conjectures see the survey article [13]. The proof of Ore's conjecture has been recently completed (see [20]).

Fact 2.6 (Wilson [29]). *Let σ be the sentence from Definition 2.3.*

- (a) *If G is a non-abelian simple group, then $G \models \sigma$, that is, σ holds in G .*
- (b) *If G is finite and $G \models \sigma$, then $\text{Soc}(G)$ is a non-abelian simple group.*

Fact 2.7 (Wilson [29]). *Let G be a finite group with a non-abelian simple socle. If G is not simple then $G' \neq G$. Moreover, if every element of $\text{Soc}(G)$ is a product of k commutators, then every element of G' is a product of $k + 3$ commutators.*

Fact 2.8 (Wilson [29]). *Every simple pseudofinite group is elementarily equivalent to an ultraproduct of finite simple groups.*

In the following lemma, we observe that Wilson's proofs for Fact 2.6(a) and Fact 2.8 work for non-abelian definably simple pseudofinite groups.

Lemma 2.9. *If G is a non-abelian definably simple pseudofinite group and σ is the sentence from Definition 2.3, then the following statements hold.*

- (a) $G \models \sigma$.
- (b) $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ where each G_i is a non-abelian finite simple group and \mathcal{U} is a non-principal ultrafilter on a set I .

Proof. (a) Assume that σ does not hold in G . Then, there are $x, y \in G$ such that $x \neq 1$ and $C_G(x, y) \neq 1$ and

$$N := \bigcap_{g \in G} (C_G(x, y) C_G(C_G(x, y)))^g \neq 1.$$

Clearly, N is a definable normal subgroup of G . Since G is definably simple and N is non-trivial by our assumption, we get $N = G$. As a result we have, $C_G(x, y) C_G(C_G(x, y)) = G$. Therefore, $C_G(x, y)$ is normalized by G . Since $C_G(x, y)$ is a non-trivial definable subgroup of G , we have $C_G(x, y) = G$. This is a contradiction, since G cannot have central elements as a non-abelian definably simple group.

(b) As a non-abelian pseudofinite group, G is elementarily equivalent to $\prod_{i \in I} G_i / \mathcal{U}$ where each G_i is a non-abelian finite group and \mathcal{U} is a non-principal ultrafilter on a set I . Moreover, since $G \models \sigma$ by part (a), we may assume that $G_i \models \sigma$ for all $i \in I$ by Łoś's Theorem and Remark 1.1. It follows by Fact 2.6(b) that $\text{Soc}(G_i)$ is a non-abelian simple group for each $i \in I$. Now, let φ_i be a formula in the language of groups defining the set of products of $k + 3$ commutators where k is the integer given by Fact 2.4. By Fact 2.7, for any non-simple G_i , the formula φ_i defines a proper normal subgroup. Therefore, if G_i is not simple for almost all $i \in I$, then G has a proper definable normal subgroup by Łoś's Theorem. This is not possible as G is definably simple and hence G_i is a non-abelian finite simple group for almost all $i \in I$. Again, by referring to Remark 1.1, we can conclude that $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ where G_i is a non-abelian finite simple group for all $i \in I$. \square

Corollary 2.10. *Every definably simple pseudofinite group is elementarily equivalent to an ultraproduct of finite simple groups.*

Proof. Follows from Lemma 2.9(b) and Fact 2.2. \square

Now, we can prove our classification result.

Proposition 2.11. *Let G be a non-abelian definably simple pseudofinite group of finite centralizer dimension. Then G is isomorphic to a (twisted) Chevalley group over a pseudofinite field.*

Proof. By Lemma 2.9(b), we have $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ where each G_i is a non-abelian finite simple group. Since there are three families of non-abelian finite simple groups, without loss of generality, we assume that just one family occurs in the ultraproduct (see Remark 1.1). The possibilities are analyzed below.

Case 1. Sporadic Groups

Since there are finitely many sporadic groups, we may assume that all G_i 's are the same sporadic group H , that is, $G \equiv H^I / \mathcal{U}$. However, this forces G to be finite which is not the case.

Case 2. Alternating Groups

It is well-known that the centralizer dimension of alternating groups increases as the rank increases. More precisely, if we consider the centralizer chain of the form

$$C(1) > C((12)(34)) > \dots > C((12)(34), \dots, (k-3, k-2)(k-1, k)),$$

we can observe that the centralizer dimension of $\text{Alt}(n)$ is greater than $\frac{n}{4} - 1$. Therefore, the finite centralizer dimension property is satisfied only if there is a bound on the orders of the alternating groups in the ultraproduct. But then, the ultraproduct is finite and this case is eliminated as well.

Case 3. Groups of Lie type

In this case, all G_i 's are from one of the infinite families of classical groups $A_n, B_n, C_n, D_n, {}^2A_n, {}^2D_n$. In the proof of [19, Proposition 3.1], it is shown that if there is no bound on the Lie ranks of the groups in the ultraproduct, then the centralizer dimension cannot be finite.

Therefore, Lie ranks are bounded in the ultraproduct and we may assume that all G_i 's are the groups of the same Lie type with the fixed Lie rank n and over fields of increasing order.

We conclude that G is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field by Fact 1.2. Moreover, we can strengthen the elementary equivalence to an isomorphism by the results obtained by Ryten in [25]. \square

Corollary 2.12. *Every non-abelian definably simple pseudofinite group of finite centralizer dimension is simple.*

3. Proof of the main result

In this section, we will prove the following theorem.

Theorem 3.1. *Let G be an infinite simple group of finite Morley rank and α be an automorphism of G such that the definable hull of $C_G(\alpha)$ is G . If $C_G(\alpha)$ is pseudofinite, then there is a definable (in $C_G(\alpha)$) normal subgroup S of $C_G(\alpha)$ such that S is isomorphic to a (twisted) Chevalley group over a pseudofinite field and $C_G(\alpha)$ embeds in $\text{Aut}(S)$.*

Recall that, $C_G(\alpha)$ denotes the group of fixed points of α in G , that is, $C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$. Moreover, we would like to introduce a simplified notation for the double centralizers since they will play an important role in our arguments. For any group H and a subset X of H , we will denote the double centralizer $C_H(C_H(X))$ by $C_H^2(X)$ or even by $C^2(X)$ when the group in concern is specified clearly. Similarly the triple centralizer of X in H will be denoted by $C_H^3(X)$ or $C^3(X)$. It can be easily checked that $X \subseteq C_H^2(X)$ and $C_H^3(X) = C_H(X)$.

We fix G and α as in Theorem 3.1 from now on.

Remark 3.2. It can be observed that simple groups of finite Morley rank of degenerate type, satisfying the assumptions of Theorem 3.1, do not exist: it is known that simple groups of finite Morley rank of degenerate type do not have involutions [5]. Therefore, $C_G(\alpha)$ has no involutions. Since this is a first order property, $C_G(\alpha)$ is an ultraproduct of finite groups of odd orders which are solvable by Thompson's odd order theorem. Evgenii I. Khukro proved in [15] that such pseudofinite groups of finite centralizer dimension are solvable. However, then G is solvable as the definable hull of $C_G(\alpha)$ [1, I, Lemma 2.15]. This is not possible since G is simple.

Lemma 3.3. *Let H be a non-trivial subnormal subgroup of $C_G(\alpha)$. Then the following statements hold.*

- (a) $d(H) = G$.
- (b) $C_G(H) = 1$.

Proof. (a) Since H is a subnormal subgroup of $C_G(\alpha)$ there is a finite chain of subgroups of $C_G(\alpha)$ such that

$$1 \neq H \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{n-1} \trianglelefteq C_G(\alpha).$$

Basic properties of definable hull (see [6, Lemma 5.35]) ensure that if we take the definable hulls of each subgroup in the chain, we get the following subnormal chain

$$1 \neq d(H) \trianglelefteq d(H_1) \trianglelefteq \dots \trianglelefteq d(H_{n-1}) \trianglelefteq d(C_G(\alpha)) = G.$$

Since G is simple, $d(H_{n-1}) = G$ and inductively we get $d(H) = G$.

(b) Let H be a non-trivial subnormal subgroup of $C_G(\alpha)$. Since $H \leq C_G^2(H)$ and $C_G^2(H)$ is a definable subgroup of G (see Fact 1.5), we have $d(H) \leq C_G^2(H)$. However, $d(H) = G$ by part (a) and we get $C_G(H) \leq Z(G)$. As a simple group, G has trivial center and so, $C_G(H) = 1$. \square

Corollary 3.4. *There are no non-trivial subnormal subgroups of $C_G(\alpha)$ which are definable in G . In particular, $C_G(\alpha)$ has no non-trivial finite subnormal subgroups.*

Corollary 3.5. *There are no non-trivial abelian subnormal subgroups of $C_G(\alpha)$.*

Lemma 3.6. $C_G(\alpha)$ is elementarily equivalent to an ultraproduct of finite groups such that in each finite group, the centralizer of any non-trivial normal subgroup is trivial.

Proof. As a pseudofinite group, $C_G(\alpha) \equiv \prod_{i \in I} G_i / \mathcal{U}$ where G_i is a finite group for all $i \in I$ and \mathcal{U} is a non-principal ultrafilter on a set I . Let μ be the following sentence:

$$\forall x \forall y \quad [(x \neq 1 \wedge y \neq 1) \rightarrow \exists z [y, x^z] \neq 1].$$

It can be observed that μ holds in a group X if and only if the centralizer of any non-trivial normal subgroup of X is trivial. By Lemma 3.3(b), μ holds in $C_G(\alpha)$. Therefore, we have $\prod_{i \in I} G_i / \mathcal{U} \models \mu$ and the lemma follows by Łoś’s Theorem and Remark 1.1. \square

Remark 3.7. From now on, we fix a family of finite groups $\{G_i \mid i \in I\}$ and an ultrafilter \mathcal{U} on I such that $C_G(\alpha) \equiv \prod_{i \in I} G_i / \mathcal{U}$ and the centralizer of any non-trivial normal subgroup of G_i is trivial for each $i \in I$. Moreover, whenever we say “for almost all $i \in I$ ” we mean “for elements of a subset of I belonging to \mathcal{U} ”.

Lemma 3.8. One of the following statements holds:

- (a) $C_G(\alpha)$ is isomorphic to a (twisted) Chevalley group over a pseudofinite field.
- (b) For each $i \in I$, there is a unique minimal normal subgroup M_i of G_i which is necessarily a direct product of non-abelian conjugate simple groups.

Proof. (a) If G_i is simple for almost all $i \in I$, then $C_G(\alpha)$ is definably simple and hence isomorphic to a (twisted) Chevalley group over a pseudofinite field by Proposition 2.11.

(b) Assume that G_i is not simple for almost all $i \in I$. If G_i has two minimal normal subgroups M_i and N_i , then we have $M_i \cap N_i = 1$ and hence $[M_i, N_i] = 1$. Therefore, N_i centralizes M_i . However, this contradicts to Lemma 3.6 and hence, each G_i has a unique non-abelian minimal normal subgroup M_i . Therefore, $\text{Soc}(G_i) = M_i$ for all $i \in I$. Although it is well-known in finite group theory, we will recall the structure of such socles. Let S_i be a minimal normal subgroup of M_i . Since $M_i \trianglelefteq G_i$, the group $\langle S_i^x \mid x \in G_i \rangle$ is a normal subgroup of G_i contained in M_i . As M_i is a minimal normal subgroup of G_i , we get $\langle S_i^x \mid x \in G_i \rangle = M_i$. Moreover, $(S_i)^x$ is a minimal normal subgroup of M_i for each $x \in G_i$. Therefore, the normal subgroup $(S_i)^x \cap \langle S_i^y \mid y \in G_i \setminus \{x\} \rangle$ of M_i is either trivial or S_i^x is contained in $\langle S_i^y \mid y \in G_i \setminus \{x\} \rangle$. In the latter case, we can eliminate $(S_i)^x$ from the product. As a result we can conclude that

$$\text{Soc}(G_i) = M_i = S_i^{g_{i0}} \times S_i^{g_{i1}} \times \dots \times S_i^{g_{ik_i}},$$

where $g_{ij} \in G_i$ for $0 \leq j \leq k_i$. Note that if $S_i^{g_{ij}}$ has a proper non-trivial normal subgroup N_i for some $0 \leq j \leq k_i$, then N_i is normalized by M_i . This is not possible as $S_i^{g_{ij}}$ is a minimal normal subgroup of M_i . \square

In the following two lemmas, we observe that the structure of $\text{Soc}(G_i)$ can be simplified further.

Definition 3.9. For any group H and integers $n \geq 1, k \geq 1$, let $\tau_{n,k}$ be the first order sentence which expresses the following statement. For all $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n \in H^k \setminus \{(1, \dots, 1)\}$ and for all $0 \leq i < j \leq n$, if

$$[C_H^2(\bar{x}_i), C_H^2(\bar{x}_j)] = 1,$$

then the product subgroup $C_H^2(\bar{x}_0) \dots C_H^2(\bar{x}_n)$ is not normal in H .

Lemma 3.10. The sentence $\tau_{n,k}$ holds in $C_G(\alpha)$ for any $n \geq 1$ and $k \geq 1$.

Proof. For the sake of clarity, we will prove it for $n = 1$ leaving the general case to the reader. Assume that $\tau_{1,k}$ does not hold in $C_G(\alpha)$ for some $k \geq 1$. This means that, there are non-trivial k -tuples \bar{x}, \bar{y} of elements of $C_G(\alpha)$ such that

$$[C^2(\bar{x}), C^2(\bar{y})] = 1 \text{ and } C^2(\bar{x})C^2(\bar{y}) \triangleleft C_G(\alpha),$$

where $C^2(\dots)$ denotes the double centralizer in $C_G(\alpha)$. It is clear that $C^2(\bar{x})$ and $C^2(\bar{y})$ are non-trivial since they contain the tuples \bar{x} and \bar{y} respectively. Therefore, $C^2(\bar{x})$ is a non-trivial subnormal subgroup of $C_G(\alpha)$ which is centralized by the non-trivial group $C^2(\bar{y})$. However, this contradicts to Lemma 3.3(b). \square

Lemma 3.11. $\text{Soc}(G_i)$ is a non-abelian simple group for almost all $i \in I$.

Proof. We continue using the same notation as in Lemma 3.8(b), that is, S_i denotes a simple minimal normal subgroup of $\text{Soc}(G_i)$. Let us work with a fixed $i \in I$ and consider the set $\{S_i^x \mid x \in G_i\}$. By eliminating the repeating conjugates of S_i we get

$$\{S_i^x \mid x \in G_i\} = \{S_i, S_i^{x_1}, \dots, S_i^{x_{m_i}}\}$$

for some $x_1, \dots, x_{m_i} \in G_i$. We will show that S_i is the unique element of this set and hence $\text{Soc}(G_i) = S_i$.

Note that all the centralizers mentioned below are taken in G_i and $x_0 = 1$.

Claim 1. For any $0 \leq j < k \leq m_i$, we have $[C^2(S_i^{x_j}), C^2(S_i^{x_k})] = 1$. Moreover, the product $C^2(S_i^{x_0})C^2(S_i^{x_1}) \dots C^2(S_i^{x_{m_i}})$ is a normal subgroup of G_i .

The elements of any distinct pair of conjugates of S_i commutes pairwise because $[S_i^{x_j}, S_i^{x_k}] \leq S_i^{x_j} \cap S_i^{x_k} = 1$ by minimality of conjugates of S_i in M_i . Therefore, $S_i^{x_j} \leq C(S_i^{x_k})$ and so we have $C^2(S_i^{x_j}) \leq C^3(S_i^{x_k})$. This means that $C^2(S_i^{x_j})$ centralizes $C^2(S_i^{x_k})$ and the first part of the claim follows. Since the components of the product commute pairwise and G_i permutes them by conjugation, normality of the product follows.

Now, $\prod_{i \in I} G_i / \mathcal{U}$ has finite centralizer dimension, let us say d , since $C_G(\alpha)$ has this first order property. Therefore, without loss of generality, we may assume $\text{cdim}(G_i) = d$ for all $i \in I$, by Łoś's Theorem and Remark 1.1.

Claim 2. For all $i \in I$, the group $C^2(S_i^{x_0})C^2(S_i^{x_1}) \dots C^2(S_i^{x_{m_i}})$ can be written as the product of at most d factors.

Let us fix $i \in I$. Since $S_i^{x_j}$ is a non-abelian group contained in $C^2(S_i^{x_j})$, for each $0 \leq j \leq m_i$, we can choose an element $c_{ij} \in C^2(S_i^{x_j}) \setminus Z(C^2(S_i^{x_j}))$. Let us consider the following chain of subgroups of G_i

$$C_{G_i}(c_{i0}) \geq C_{G_i}(c_{i0}, c_{i1}) \geq \dots \geq C_{G_i}(c_{i0}, c_{i1}, \dots, c_{im_i}).$$

Now, if

$$C^2(S_i^{x_j}) \not\subseteq C^2(S_i^{x_0})C^2(S_i^{x_1}) \dots C^2(S_i^{x_{j-1}}) \tag{*}$$

for some $1 \leq j \leq m_i$, then

$$C_{G_i}(c_{i0}, c_{i1}, \dots, c_{ij-1}) \supsetneq C_{G_i}(c_{i0}, c_{i1}, \dots, c_{ij-1}, c_{ij}).$$

This last inclusion is proper since $C^2(S_i^{x_j}) \subseteq C_{G_i}(c_{i0}, c_{i1}, \dots, c_{ij-1})$ but there is an element in $C^2(S_i^{x_j})$ which does not commute with c_{ij} . Since the situation in (*) cannot occur more than d times, the claim follows.

By Claim 2, there are finitely many possibilities for the number of factors of the product groups $C^2(S_i^{x_0})C^2(S_i^{x_1}) \dots C^2(S_i^{x_{m_i}})$. By Remark 1.1, we can conclude without loss of generality that all of the product groups have the same number of factors, say $m + 1$, where $m \geq 0$.

Claim 3. $m = 0$.

Assume that $m \geq 1$ and $i \in I$. Since $\text{cdim}(G_i) = d$, there are $\bar{y}_0, \dots, \bar{y}_m \in (G_i)^d$ such that $C^2(S_i^{x_j}) = C^2(\bar{y}_j)$ for all $0 \leq j \leq m$. By the previous claims and the definition of m , the sentence $\tau_{m,d}$ from Definition 3.9 does not hold in G_i for all $i \in I$. Therefore, it does not hold in $C_G(\alpha)$ which contradicts to Lemma 3.10.

Now, we are ready to show that $\text{Soc}(G_i) = S_i$ for all $i \in I$. Let us fix $i \in I$. By Claim 3, there is $x \in G_i$ such that for all $y \in G_i$ we have $C^2(S_i^y) \subseteq C^2(S_i^x)$. If there is $y \in G_i$ such that $S_i^x \neq S_i^y$, then by Claim 1, $C^2(S_i^y)$ is abelian. This is not possible since $C^2(S_i^y)$ contains the non-abelian group S_i^y . Therefore, $S_i^x = S_i^y$ for all $y \in G_i$, that is $S_i \trianglelefteq G_i$. It follows that $\text{Soc}(G_i) = S_i$. \square

In order to simplify the notation, any ultraproduct $\prod_{i \in I} X_i / \mathcal{U}$ will be denoted by $(X_i)_{\mathcal{U}}$ in the rest of the paper.

Lemma 3.12. The ultraproduct $(\text{Soc}(G_i))_{\mathcal{U}}$ is a definable normal subgroup of $(G_i)_{\mathcal{U}}$ which is isomorphic to a (twisted) Chevalley group over a pseudofinite field.

Proof. Let us denote $\text{Soc}(G_i)$ by M_i for each $i \in I$. By Lemma 3.11, M_i is a non-abelian finite simple group for almost all $i \in I$. The ultraproduct $(M_i)_{\mathcal{U}}$ cannot be finite because this would yield a finite normal subgroup of $C_G(\alpha)$ which is not possible by Corollary 3.4. Therefore, $(M_i)_{\mathcal{U}}$ is a pseudofinite group. Since $(M_i)_{\mathcal{U}}$ has finite centralizer dimension (inherited from $(G_i)_{\mathcal{U}}$), by Proposition 2.11, we get that $(M_i)_{\mathcal{U}}$ is isomorphic to an ultraproduct of (twisted) Chevalley groups over finite fields. In order to prove the definability of $(M_i)_{\mathcal{U}}$, we need Thompson's Conjecture (see Remark 2.5). Erich W. Ellers and Nikolai Gordeev verified Thompson's Conjecture for finite simple groups of Lie type over fields with more than 8 elements [10]. Thompson's conjecture is applicable in our context since almost all of the fields in the ultraproduct have more than 8 elements. It follows that, for almost all $i \in I$, there is $r_i \in M_i$ such that $M_i = r_i^{M_i} r_i^{M_i}$. Since $r_i \in M_i$ and $M_i \trianglelefteq G_i$, it is clear that $r_i^{M_i} \subseteq r_i^{G_i} \subseteq M_i$. Therefore, $M_i = r_i^{G_i} r_i^{G_i}$ and M_i is uniformly definable in G_i . It follows that $(M_i)_{\mathcal{U}}$ is definable in $(G_i)_{\mathcal{U}}$ as $(r_i)_{\mathcal{U}}^{(G_i)_{\mathcal{U}}} (r_i)_{\mathcal{U}}^{(G_i)_{\mathcal{U}}}$. \square

Proof of Theorem 3.1. By Lemma 3.12, $(\text{Soc}(G_i))_{\mathcal{U}}$ is a definable normal subgroup of $(G_i)_{\mathcal{U}}$ which is isomorphic to a (twisted) Chevalley group over a pseudofinite field. Since $C_G(\alpha)$ is elementarily equivalent to $(G_i)_{\mathcal{U}}$, there is a definable normal subgroup S of $C_G(\alpha)$ which is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field. By Ryten [25], S is isomorphic to a (twisted) Chevalley group over a pseudofinite field. Moreover, since

$$S \trianglelefteq C_G(\alpha) \quad \text{and} \quad C_{C_G(\alpha)}(S) = 1,$$

$C_G(\alpha)$ embeds in $\text{Aut}(S)$ and the theorem follows. \square

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