

KEMENY CONSISTENCY OF SOCIAL WELFARE FUNCTIONS

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Özet

Bu çalışmamızda sosyal refah fonksiyonlarıyla ilgili bir koşul, Kemeny istikrarı diye adlandırdığımız bir koşul tanıtıyoruz. Bu koşul, temel olarak Kemeny uzaklığını baz alır ve bizim yüksek-tercihler dediğimiz tercih şekliyle alakalıdır. Sosyal refah kurallarının istikrarlarına yönelik bir özellik olan bu uygunuğu sağlayan kuralların hangileri olduğu, sağlamayanların hangileri olduğu araştırılmıştır. Farklı yöntemlerle bu koşulu tanımladık ve inceledik. Sonuç olarak, skollama kurallarının hiç birinin bu koşulu sağlayamadığı tesbit edilmiştir. Öte yandan Condorcet tipi kuralların bu koşulu sağlayan bazı kuralları içerebileceği gözlemlenmiştir. Bununla birlikte genel olarak bu koşulu sağlayan kurallara ilişkin özellikler araştırılmıştır. Ve bu hususta bir eşleştirme yolunda bulgular elde edilmiştir. Bu bağlamda, bir güçlü tarafsızlık koşulunun Kemeny istikrarı koşulu için yeterli bir koşul olduğu gösterilmiştir. Bütün bunlarla ilgili olarak literatürde var olan çalışmalarla ilgili bilgiler derlenmiş ve takdim edilmiştir.

Abstract

We introduce a consistency condition for social welfare functions; called Kemeny stability. This notion rests upon the Kemeny distance for rankings and a preference scheme, hyper-preferences as we call. It's been studied what rules satisfy this stability condition and which rules don't. We considered different ways of proposing this condition. As a result of our research, it's been shown that no scoring rule satisfies the condition of Kemeny stability. On the other hand, we found out that there exists Condorcet type social welfare functions which satisfy our condition. On the other hand, general characteristics of the rules satisfying the condition also has been studied and some results are achieved. In this context, a strong neutrality condition is shown to be sufficient. A literature search related to all these aspects also accomplished and presented in this work.

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1 Introduction

Think of a collective decision situation where individuals are required to rank alternatives in a way to represent their preferences. By determining the restrictions on preference submission individuals are immediately forced to reveal their preferences in certain ways. For instance the preference scheme inquired is the major restriction one can think of; let's say individuals are asked to submit a complete ranking. Then he is left with only a set of admissible rankings. Henceforth he is to determine which of the rankings is the best one in reflecting his preferences. So to speak, individuals are to compare, or rank, these rankings according to their tastes. This is an alternative view to spectate the individual choice in collective decision milieu; individuals must not only rank the alternatives constructively but they may pick the most suitable ranking among others.

We consider a standard social choice framework where there are certain number of individuals, a certain set of alternatives some to be chosen among (or ranked) and certain types of preferences to be submitted. An aggregation procedure is used to reach a best social outcome for this setting (preference profile). Furthermore, diverging from standard fashion, we expose a certain way of extending the preferences submitted to a higher level, to what we call *hyper-preferences*, which are the rankings of rankings, reflecting one's preferences over orderings, extracted out of his choice among orderings. In a social choice context, this can be seen as having preferences over "social rankings". For instance, think of a society confronted with an alternatives set and to decide on a socially best ranking of alternatives. This is a standard framework where we have a profile of preferences and a social welfare function. Furthermore, as we assumed, this society is also attached to a hyper profile where each of its members submit a preference over social rankings. Since the aggregation procedure is already agreed on, we shall be concerned with the outcome of this new social choice setting and in deed, as surely engrossing, with the relation between the outcomes of each settings. In this paper we condition social welfare functions to be stable (in differing ways), in the sense that the outcomes of each settings should coincide.

Having said these, as we only have a revealed choice among orderings informationally, it's obvious that we shall have a proper way of gathering the information we look for, a hyper-preference, from what we have, a first-order preference. This also leads to a plethora of directions so that we narrow our aspect by restricting ourselves to one specific way of extending preferences to higher levels, which we induce from what is known in the literature as Kemeny distance (9). By doing so, imposing the condition called Kemeny stability, we prompt certain features over preferences which will be pointed in following sections.

Axiomatic social choice literature is somehow mute to include such dimensional structures of preference. However, a similar phenomenon, the notion of "meta-preference" has been discussed among philosophers and economists from mid-70s and on. Although not directly related, for the sake of completeness, we'll mention those discussions. The idea there, though the setting depending on the writer, was to study that humans have higher levels of volition when they confront preference (or choice) structures that surely affects the actual attitudes. Putting aside the question of the name-father, we can refer to Harry G. Frankfurt (1971) as the one who brought up the issue of second-order volition (3).

Following Frankfurt, Jeffrey (8) and Sen (12; 14) were the most influential discussants of the subject as a broader philosophical issue. Jeffrey (1974) surveyed ideas of higher order preferences in order to analyze the more complex real-life observations, as an attempt to expand the theory of preference. On the other hand Sen tried to conceptualize the subject where he attempts to conjoin morality (or other-regarding) into theory of preference and henceforth propose better constructions for prevailing models so as to solve puzzles such as Prisoners' Dilemma.

According to Frankfurt (1971), it's been understood that it's a characteristic -and more assertively, the must- of humans to have second order desires, wants or volition and this may be distinctive of what is observed in daily life experiences involving choices. He suggested that one without higher-order volition is deprived of the human essence. Later, in 1982, we see Albert O. Hirschman writing on the subject (6), this time conjoint with

his theory of "exit and voice" (5) for public actions. He touches the subject and the literature up to some point in a way to support his approach of "dynamic" preferences where he considers a meta-preference as what gives roots to the changes in preferences when faced with disappointment. On the other hand, as to our knowledge, Igersheim (7) is the closest contemporary economist approach to the phenomena. In her work, the aim was to solve Sen (11) and Gibbard's (4) libertarian paradoxes via introducing the concept of meta-preferences to the setting. Igersheim explicitly uses what we call a *hyper-preference* and construct them as we do up to a point where she does not impose any restriction on how to obtain meta-preferences although referring to possibilities of devising a method which respects individual motivations and determines certain meta-ranking for each ranking.

Constructed upon what is discussed above, our study in this paper can be seen in the line of the literature on choosing aggregation rules endogenously if looked at with a consequentialist view, labeling aggregation rules with outcomes. To give sense of the referred literature, Koray (10) imposes a condition for social choice functions which requires a function to select itself when it's now functions to be selected depending on individual's preferences over alternatives. Furthermore Barbera and Jackson (1) weaken the same condition and look for implications of restricted domains and obtains possibility results whereas Koray (2000) ends up at an Arrovian impossibility.

In section 2 we introduce the base notions and the setting. In section 3 we analyze the scoring rules while it comes to Condorcet type of aggregation rules in section 4. We conclude after showing a sufficient condition for Kemeny stability in section 5.

2 Basic Notions and Definitions

Let N be the set of individuals and A be the set of alternatives with cardinality m . The set of all linear orders over A is denoted by $\mathcal{L}(A)$ where the set of all complete weak orders over A is denoted by $\mathcal{R}(A)$. An order $P \in \mathcal{L}(A)$ is a *linear extension* of a weak order $R \in \mathcal{R}(A)$ if for any $a, b \in A$, $aPb \Rightarrow aRb$. A profile R_N is a collection of orders, or formally $R_N \in \mathcal{D}^N$ where $\mathcal{D} \subseteq \mathcal{R}(A)$. The set of all linear extensions of $R \in \mathcal{R}(A)$ is denoted by $\Delta(R)$. The set of all profiles of linear orders compatible with the profile of weak orders R_N is denoted by $\Delta(R_N)$, or formally $\Delta(R_N) = \times_{i \in N} (\Delta(R_i))$. A profile of linear orders is an element of $\mathcal{L}(A)^N$.

A *social welfare function* $\alpha : \mathcal{L}(A)^N \rightarrow \mathcal{R}(A)$ is a mapping from profiles of strict orders to weak orders. Given a social welfare function α , an α -*induced social choice correspondence* $f_\alpha : \mathcal{L}(A)^N \rightarrow 2^A$ is a mapping from profiles of orders to subsets of A such that $\forall P_N \in \mathcal{L}(A)^N, \forall a \in A, a \in f_\alpha(P_N) \Leftrightarrow a\alpha(P_N)b, \forall b \in A$.

What follows is the definition of the *Kemeny distance* which is quite central in our analysis.

Definition 2.1 *The Kemeny distance between two strict orders $R, Q \in \mathcal{L}(A)$, denoted by $d_K(R, Q)$ is the symmetric difference between R and Q ; or formally $d_K(R, Q) = |\{(a, b) \in R : (a, b) \notin Q\}|$.*

Take any $P \in \mathcal{L}(A)$. The *Kemeny hyper-preference of the preference P* , denoted by $\varepsilon_K(P) \in \mathcal{R}(\mathcal{L}(A))$, is a complete weak order over linear orders over A constructed via the Kemeny distance as follows: $(Q, R) \in \varepsilon_K(P)$ or $Q\varepsilon_K(P)R \Leftrightarrow d_K(P, Q) \leq d_K(P, R)$.

Definition 2.2 *The Kemeny hyper profile of the profile P_N , denoted by $\varepsilon_K(P_N)$, is the profile of Kemeny hyper-preferences of individuals with preferences in P_N , or formally, $\varepsilon_K(P_N) = \{\varepsilon_K(P_i)\}_{i \in N}$.*

We are now ready to introduce the concept of Kemeny stability, with the weakest version beforehand.

Definition 2.3 *A social welfare function α is weakly Kemeny stable if for any $P_N \in \mathcal{L}(A)^N$, there exists a $P'_N \in \Delta(\varepsilon_K(P_N))$ such that $\Delta(\alpha(P_N)) \cap f_\alpha(P'_N) \neq \emptyset$.*

This condition is quite weak since it only requires that we should find at least one linear extension of the Kemeny hyper profile which preserves a compatible linear order of the initial societal preference as among the best.

Definition 2.4 *A social welfare function α is Kemeny stable if for any $P_N \in \mathcal{L}(A)^N$ and for any $P'_N \in \Delta(\varepsilon_K(P_N))$, $\Delta(\alpha(P_N)) \subseteq f_\alpha(P'_N)$.*

Stability requires the social welfare function to preserve the social outcome for the initial profile of preferences over alternatives as among bests when we move to linear hyper-preferences. One may propose many different types of stability. We also considered a number of them and a couple of them and some results related can be found in the Appendix A.

3 Scoring Rules

In this section we study the Kemeny stability of scoring rules. To begin with visual examples we show instabilities of Borda, plurality and negative plurality rules. Then we move on by analyzing the scoring rules in general.

We define a social welfare scoring function $\varsigma : \mathcal{L}(A)^N \rightarrow \mathcal{R}(A)$ for every finite set of alternatives A with at least three alternatives as follows. Let $\{S_\varsigma^m\}_{m \geq 3} = \{S_\varsigma^3, S_\varsigma^4, \dots\}$ be a sequence of scoring vectors, m being the number of alternatives in A where $S_\varsigma^m = (s_\varsigma^{1,m}, s_\varsigma^{2,m}, \dots, s_\varsigma^{m,m}) \in \mathbb{R}^m$ is such that;

- (i) $s_\varsigma^{m,m} = 0$,¹
- (ii) $s_\varsigma^{1,m} \geq s_\varsigma^{2,m} \geq \dots \geq s_\varsigma^{m,m}$ and
- (iii) $s_\varsigma^{1,m} > s_\varsigma^{m,m}$.

Letting $|A| = m$, the ς score of the alternative $x \in A$ in the profile P_N is $S_\varsigma^m(x, P_N) = \sum_{i \in N} s_\varsigma^{r_i(x, P_N), m}$ where $r_i(x, P_N)$ is the rank of x in P_i . Then ς , a scoring rule with the sequence $\{S_\varsigma^m\}_{m \geq 3} = \{S_\varsigma^3, S_\varsigma^4, \dots\}$ of scoring vectors defined as above over the alternatives set A is such that $\forall P_N \in \mathcal{L}(A)^N$, $\forall x, y \in A$, $x_\varsigma(P_N)y \iff S_\varsigma^m(x, P_N) \geq S_\varsigma^m(y, P_N)$.

3.1 Illustration

The Borda Rule

Definition 3.1 *The Borda rule is the social welfare scoring function β defined with a sequence $\{S_\beta^m\}_{m \geq 3} = \{S_\beta^3, S_\beta^4, \dots\}$ of scoring vectors where $S_\beta^m = (s_\beta^{1,m}, s_\beta^{2,m}, \dots, s_\beta^{m,m}) \in \mathbb{R}^m$ is such that $s_\beta^{m,m} = 0$ and $s_\beta^{k,m} = s_\beta^{k+1,m} + 1$, $\forall k \in \{1, 2, \dots, m-1\}$.*

Consider the following profile P_N ;

¹The conditions (ii) and (iii) are basically enough to define a scoring rule, but we choose to simplify things by assuming (i) also, without loss of generality.

3	2	1
<hr style="width: 100%;"/>		
a	b	c
c	c	a
b	a	b

The Borda result is a set $\Delta(\beta(P_N)) = \{cab, acb\}$. But if we consider the linear hyper profile P'_N ;

3	2	1
<hr style="width: 100%;"/>		
acb	bca	cab
cab	bac	acb
abc	cba	cba
bac	cab	bca
cba	abc	abc
bca	acb	bac

where we see that $\beta(cab, P_N) = 21 > 19 = \beta(acb, P_N)$ which leads to exclusion of the ranking acb from the choice set of P'_N under β . Hence this profile constitutes an example to see that the Borda rule fails stability.

The Plurality Rule

Definition 3.2 *The plurality rule is the social welfare scoring function π defined with a sequence $\{S_\pi^m\}_{m \geq 3} = \{S_\pi^3, S_\pi^4, \dots\}$ of scoring vectors where $S_\pi^m = (s_\pi^{1,m}, s_\pi^{2,m}, \dots, s_\pi^{m,m}) \in \mathbb{R}^m$ is such that $s_\pi^{k,m} = 0, \forall k > 1$ and $s_\pi^{1,m} = 1$.*

Consider the following profile P_N ;

3	2	1
<hr style="width: 100%;"/>		
a	b	c
c	a	a
b	c	b

The plurality social welfare function will end up at the ranking abc . However, it is easy to note that the plurality social choice rule will result in the ranking acb for any Kemeny hyper profile.

The Inverse Plurality Rule

Definition 3.3 The inverse plurality rule is the social welfare scoring function π' defined with a sequence $\{S_{\pi'}^m\}_{m \geq 3} = \{S_{\pi'}^3, S_{\pi'}^4, \dots\}$ of scoring vectors where $S_{\pi'}^m = (s_{\pi'}^{1,m}, s_{\pi'}^{2,m}, \dots, s_{\pi'}^{m,m}) \in \mathbb{R}^m$ is such that $s_{\pi'}^{k,m} = 1, \forall k > 1$ and $s_{\pi'}^{1,m} = 0$.

Consider the following profile P_N ;

3	3	2	3	4
<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>c</i>
<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>	<i>b</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>a</i>

which results *abc* as the unique linear inverse plural social preference. But observe the Kemeny hyper profile of this profile below;

3	2	3	3	4
<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>cab</i>	<i>cba</i>
<i>acb, bac</i>	<i>abc, cab</i>	<i>abc, bca</i>	<i>cba, acb</i>	<i>cab, bca</i>
<i>bca, cab</i>	<i>cba, bac</i>	<i>acb, cba</i>	<i>abc, bca</i>	<i>bac, acb</i>
<i>cba</i>	<i>bca</i>	<i>cab</i>	<i>bac</i>	<i>abc</i>

which shows that *abc* should be defeated by others in any linear extension of this profile, under inverse plurality rule. Hence the inverse plurality rule also fails Kemeny stability.

3.2 Scoring Rules in General

We narrow down the question of stability to three-alternative case, since if there exists no three dimensional vector which qualifies to belong to the sequence $\{S_{\zeta}^m\}_{m \geq 3}$ and yet define a scoring rule which satisfies stability, then there cannot be defined a scoring rule which is stable².

Theorem 3.1 *There exists no Kemeny stable scoring rule.*

²It is also to be noted that scoring rules fail any type of stability we explored, which are added in the appendix.

In the proof of the theorem, we prove the impossibility of a scoring vector for three alternatives to be stable. To do so, we juxtapose a series of lemmata, useful to follow. Since it is of a special interest, we prove here the first of the lemmata, which shows that general plurality rules fail stability.

Lemma 3.1 *There exists no scoring rule ς with $S_\varsigma^3 = (s_\varsigma^{1,m}, 0, 0)$ which is Kemeny stable.*

Proof. Consider the following profile Q_N with three alternatives a , b and c where there are $n_1 + n_2 + n_3$ individuals with shown preferences over these alternatives such that $n_1 > n_2 > n_3$.

n_1	n_2	n_3
a	b	c
c	c	a
b	a	b

Hence we have that $S_\varsigma^3(a, Q_N) = n_1 s_\varsigma^{1,m} > S_\varsigma^3(b, Q_N) = n_2 s_\varsigma^{1,m} > S_\varsigma^3(c, Q_N) = n_3 s_\varsigma^{1,m}$ which gives $\varsigma(Q_N) = abc$. Observing the linear Kemeny hyper profile Q'_N of Q_N below;

n_1	n_2	n_3
acb	bca	cab
cab	bac	acb
abc	cba	cba
bac	cab	bca
cba	abc	abc
bca	acb	bac

we see that $\varsigma(Q_N)$ is Pareto-dominated by cab in Q'_N , which implies that $\varsigma(Q_N) \notin f_\varsigma(Q'_N)$ hence shows the failure to satisfy any type of stability. ■

The rest of the proof of the Theorem 3.1 is in the Appendix B. We now turn to Condorcet type rules.

4 Condorcet Social Welfare Functions

In this section we explore the Kemeny stability of the Condorcet type social welfare functions. For the sake of precision we first introduce the refinements for our purposes regarding some prevailing definitions and notions.

Let $\mu(P_N)$ denote the majority tournament for P_N where $a\mu(P_N)b$ iff $|\{i \in N : aP_i b\}| \geq |\{i \in N : bP_i a\}|$. When $S \subseteq A$, the *Condorcet winner* of S according to the profile P_N , $CW(P_N|S) \in S$ is such that $CW(P_N|S)\mu(P_N)a, \forall a \in S$ and we abbreviate $CW(P_N|A)$ as $CW(P_N)$ and note that it is either a singleton or empty.

A subset $S \subseteq A$ is called a *majority cycle for the profile P_N* if S can be written as $S = \{s_1, s_2, \dots, s_{\#S}\}$ where $s_i\mu(P_N)s_{i+1}, \forall i \in \mathbb{N}_{\#S-1}$ and $s_{\#S}\mu(P_N)s_1$. The *top-cycle of the subset S for the profile P_N* , $T(S, P_N)$, is a majority cycle such that $a\mu(P_N)b, \forall a \in T(S, P_N)$ and $\forall b \in S \setminus T(S, P_N)$. Now let $A_1 = T(A, P_N)$ and recursively define $A_i = T(A \setminus \bigcup_{j=1}^{i-1} A_j, P_N), \forall i \geq 2$. Hence for any profile of preferences P_N over any finite set of alternatives A , we have a unique ordered partition (A_1, A_2, \dots, A_K) , called *majoritarian partition*, of nonempty subsets of A , where $i < j \implies a\mu(P_N)b$ for all $a \in A_i$ and $b \in A_j$. Note that when A_i is a singleton, it is the Condorcet winner among the rest, or formally $A_i = CW(P_N | \bigcup_{j=i}^K A_j)$.

Definition 4.1 *Let P_N be a profile of linear preferences over A and (A_1, A_2, \dots, A_K) be the corresponding majoritarian partition. A social welfare function α is said to be Condorcet type if $\forall P_N \in \mathcal{L}(A)^N$ it is the case that $x\alpha(P_N)y, \forall x, y \in A_i$ and $i < j \implies x\alpha^*(P_N)y, \forall x \in A_i$ and $\forall y \in A_j$.*

In the case where the majority tournament is a linear order for the initial profile over alternatives, as the following proposition shows the Condorcet winner, if exists, of any (linear) hyper profile will coincide with it.

Proposition 4.1 *Let P_N be a profile of preferences over a finite set of alternatives A such that $\mu(P_N) \in \mathcal{L}(A)$. Then for any $P'_N \in \Delta(\varepsilon_K(P_N))$, it is true that $CW(P'_N) \in \{\emptyset, \mu(P_N)\}$.*

Proof. Let $A = \{a_1, \dots, a_m\}$ and P_N be a profile over A such that $\mu(P_N) = a_1 a_2 \dots a_m$. Let $Q = b_1 b_2 \dots b_m$ be the Condorcet winner for P'_N , a (linear) hyper profile of P_N , and suppose for a contradiction that $b_1 \neq a_1 = b_h$ for some $h \in \{2, \dots, m\}$. Now define $Q' = b_1 \dots b_{h-2} b_h b_{h-1} b_{h+1} \dots b_m$ and observe that $d_K(P_i, Q) = d_K(P_i, Q') + 1$ for all $i \in N$ with $b_h P_i b_{h-1}$. Since a majority of voters rank b_h over b_{h-1} , we have that $Q' \mu(\varepsilon_K(P_N)) Q$, which is the desired contradiction since it also applies to any linearization of $\varepsilon_K(P_N)$ and the iteration of the argument ensures $Q = \mu(P_N)$. ■

Perhaps not surprisingly, we discover a positive result for the sake of Kemeny stability when we anatomize the Condorcet social welfare functions. In fact, we introduce one example of the mentioned. But before that, it would be preparatory to show stability features of some well-known Condorcet type solution concepts.

To start with, Copeland solution is shown to fail Kemeny stability.

Definition 4.2 *Copeland solution, κ , is defined as $x\kappa(P_N)y \iff c(x, P_N) \geq c(y, P_N)$ where $c(a, P_N) = |z \in A : a\mu(P_N)z| - |z \in A : z\mu(P_N)a|$.*

Consider the following profile P_N ;

$$P_N = \begin{array}{ccccc} & 1 & 1 & 1 & 1 & 1 \\ & \hline a & a & b & b & c \\ b & c & c & a & a \\ c & b & a & c & b \end{array}$$

where we have $\Delta(\kappa(P_N)) = \{abc\}$ since abc is the linear pairwise majority solution. Now consider the linear hyper profile P'_N ;

$$P'_N = \begin{array}{ccccc} & 1 & 1 & 1 & 1 & 1 \\ & \hline abc & acb & bca & bac & cab \\ acb & cab & bac & bca & acb \\ bac & abc & cba & abc & cba \\ cab & cba & cab & acb & bca \\ bca & bac & abc & cba & abc \\ cba & bca & acb & cab & bac \end{array}$$

where $\kappa(P_N) = abc$ defeats bac, acb, cba in $\mu(P'_N)$ while another order, acb is defeating more, namely cab, bca, bac, cba . This shows the instability. Let us now turn to another solution concept, the Slater solution.

Definition 4.3 *Slater solution for a profile $P_N \in \mathcal{L}(A)^N$ is a subset $SL(P_N)$ of $\mathcal{L}(A)$ such that $P \in SL(P_N) \iff d_K(P, \mu(P_N)) \leq d_K(R, \mu(P_N)), \forall R \in \mathcal{L}(A)$.*

Consider the following profile of preferences P_N , where A is a set of 8 alternatives;

$$\begin{array}{c}
 \begin{array}{ccccc}
 i_1 & i_2 & i_3 & i_4 & i_5 \\
 \hline
 b & a & d & c & d \\
 c & b & a & a & b \\
 d & c & b & d & c \\
 P_N = & a & d & c & b & a & . \\
 a' & b' & d' & d' & c' \\
 b' & c' & a' & b' & a' \\
 c' & d' & b' & c' & d' \\
 d' & a' & c' & a' & b'
 \end{array}
 \end{array}$$

Now if we partition A such as $X = \{a, b, c, d\}$ and $Y = \{a', b', c', d'\}$ and consider the restrictions of the profile to these partitions, $P_N|_X$ and $P_N|_Y$, they will be similar in the sense that the alternatives denoted with and without primes would be treated in same way in the profiles. This implies that $\mu(P_N)|_X$ and $\mu(P_N)|_Y$ are also similar. Observe that

$$a\mu(P_N)b\mu(P_N)c\mu(P_N)d\mu(P_N)a,$$

$$c\mu(P_N)a$$

and

$$d\mu(P_N)b$$

whilst all alternatives in X beat all in Y . The Slater solution to the tournament is $SL(\mu(P_N)|_X) = \{cdab\}$ which implies also that $SL(\mu(P_N)|_Y) =$

$\{c'd'a'b'\}$, hence $SL(P_N) = \{cdabc'd'a'b'\}$. Now let $Q = dbcad'b'c'a'$ and observe the Kemeny distances between P_i and $SL(P_N)$ and between P_i and Q ;

P_i	$SL(P_N)$	Q
P_1	$3 + 4$	$2 + 5$
P_2	$4 + 3$	$5 + 2$
P_3	$3 + 3$	$2 + 2$
P_4	$1 + 3$	$4 + 0$
P_5	$3 + 1$	$0 + 4$

It follows that in the Kemeny hyper profile, Q is strictly preferred to $SL(P_N)$ by individual 3. Now it is to note that this implies the existence of a linear extension where Q Pareto dominates $SL(P_N)$. Since Slater solution is in the Pareto set, this observation concludes that Slater rule fails Kemeny stability.

4.1 Condorcet Rules in General

Towards the Kemeny stability characteristics of Condorcet social welfare functions we expose and prove a proposition. To do this, we need to introduce the following setting. Let $Q \in \mathcal{L}(A)$. We write $Q = (Q_1 \rightarrow Q_2 \rightarrow \dots \rightarrow Q_h)$ for some $h \in \mathbb{N}$ and $Q_i \in \mathcal{L}(S_i)$ where $\emptyset \neq S_i \subseteq A$ and $\{S_1, S_2, \dots, S_h\}$ partitions A . We call Q_i a *segment* of Q . Note that S_i s are allowed to be singletons.

Proposition 4.2 *Let P_N be a profile of linear orders over A , where $|A| = m$. Let $Q, Q' \in \mathcal{L}(A)$ be such that $Q = (Q_1 \rightarrow a \rightarrow Q_2 \rightarrow b \rightarrow Q_3)$ and $Q' = (Q_1 \rightarrow b \rightarrow Q_2 \rightarrow a \rightarrow Q_3)$ for some segments Q_1, Q_2 and Q_3 and singletons a, b . Then, for any individual $i \in N$, $aP_i b \implies Q \varepsilon_K^*(P_i) Q'$.*

Proof. First note that the Kemeny distance only considers symmetric differences. Hence we can restrict our attention to Q_2, a and b . Consider an individual $i \in N$ with $aP_i b$ and the restriction $P_i|_{S_2 \cup \{a, b\}} = \overline{P}_i$, where we have $d_K(\overline{P}_i, Q|_{S_2 \cup \{a, b\}}) = d_K(P_i, Q)$ and $d_K(\overline{P}_i, Q'|_{S_2 \cup \{a, b\}}) = d_K(P_i, Q')$.

Let the upper contour of y in S_2 for P_i be $\mathcal{U}_y = \{x \in S_2 : xP_i y\}$. Hence we have $\mathcal{U}_a \subseteq \mathcal{U}_b$. We can write $\overline{P_i} = (V \rightarrow a \rightarrow V' \rightarrow b \rightarrow V'')$ for some $V \in \mathcal{L}(\mathcal{U}_a \cap \mathcal{U}_b)$, $V' \in \mathcal{L}(\mathcal{U}_b - (\mathcal{U}_a \cup \{a\}))$ and $V'' \in \mathcal{L}(S_2 - (\mathcal{U}_a \cup \mathcal{U}_b \cup \{a, b\}))$. Now let us call $|\mathcal{U}_a \cap \mathcal{U}_b| = \ell_1$, $|\mathcal{U}_b - (\mathcal{U}_a \cup \{a\})| = \ell_2$ and $|S_2 - (\mathcal{U}_a \cup \mathcal{U}_b \cup \{a, b\})| = \ell_3$. It follows that $d_K(P_i, Q') - d_K(P_i, Q) = \ell_1 + 2\ell_2 + \ell_3 + \ell_4 + 1 - (\ell_1 + \ell_3 + \ell_4) = 2\ell_2 + 1$, where $\ell_4 = d_K(P_i|_{A-S-\{a,b\}}, Q|_{A-S-\{a,b\}})$. Since $\ell_2 \geq 0$, we have that $d_K(P_i, Q') > d_K(P_i, Q)$ which implies $Q\varepsilon_K^*(P_i)Q'$. ■

An immediate corollary is the following.

Corollary 4.1 *Let P_N be a profile of linear orders over A . Let $Q, Q' \in \mathcal{L}(A)$ be such that $Q = (Q_1 \rightarrow a \rightarrow Q_2 \rightarrow b \rightarrow Q_3)$ and $Q' = (Q_1 \rightarrow b \rightarrow Q_2 \rightarrow a \rightarrow Q_3)$ for some segments Q_1, Q_2 and Q_3 and singletons a, b . Then $a\mu(P_N)b \implies Q\mu(\varepsilon_K(P_N))Q'$ and hence $Q\mu(P'_N)Q', \forall P'_N \in \Delta(\varepsilon_K(P_N))$.*

As mentioned before, the existence of a Condorcet social welfare function satisfying Kemeny stability will be shown by introducing a special social welfare function. But before that, we will prove that another Condorcet social welfare function, θ as defined in following, satisfies Kemeny stability of *type 3*³. This will make easy to spectate the discussion when the existence of a Condorcet type social welfare function which is Kemeny stable is established.

Definition 4.4 *Let P_N be a profile of linear orders over A and (A_1, A_2, \dots, A_K) be the corresponding majoritarian partition. The social welfare function θ is such that $\forall i, j \leq K$, $a\theta(P_N)b$ and $\neg\{b\theta(P_N)a\} \forall a \in A_i, \forall b \in A_j$ when $i < j$ while $a\theta(P_N)b$ and $b\theta(P_N)a \forall a, b \in A_i$.*

The social welfare function θ basically takes ordered majoritarian partition into account such that it considers the alternatives in the same cycle as indifferent while respecting the strict order between cycles. In what follows, it is proven after Lemma 4.1 that the social welfare function θ turns out to be Kemeny stable of *type 3*.

³See Appendix A for types of stability.

Lemma 4.1 Given a profile P_N over A , let $Q = (S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_k \rightarrow \dots \rightarrow S_K)$, where for $1 \leq k \leq K$, $S_k = a_1^k a_2^k \dots a_{H_k}^k$ is a Hamiltonian path⁴ of A_k . Then $Q \in T(\mathcal{L}(A), \varepsilon_K(P_N))$.

Proof. The proof is done by induction over the number of alternatives.

Induction Basis: $m=3$

Let P_N be a preference profile over $A = \{a, b, c\}$. Without loss of generality we either have $\mu(P_N) = \{abc\}$ or a cycle, $\mu(P_N) = abca$. Thanks to the Proposition 4.1, we have $abc\mu(\varepsilon_K(P_N))bac$, $abc\mu(\varepsilon_K(P_N))acb$ and $abc\mu(\varepsilon_K(P_N))cba$. Besides $bac\mu(\varepsilon_K(P_N))cab$ and $acb\mu(\varepsilon_K(P_N))bca$ we also have;

- (i) if $\mu(P_N) = abc$, then $cab\mu(\varepsilon_K(P_N))cba$;
- (ii) if $\mu(P_N) \neq abc$, then $bca\mu(\varepsilon_K(P_N))cba$.

We see that abc defeats either directly or indirectly every other order in $\varepsilon_K(P_N)$, so any Hamiltonian path in the cycle $abca$ is in $T(\mathcal{L}(A), \varepsilon_K(P_N))$.

Inductive Assumption: let the result hold for $m-1$

For any number of alternatives, $Q\mu(\varepsilon_K(P_N))(a_1^1 \rightarrow S)$ if and only if $Q|_{A-\{a_1^1\}} \mu(\varepsilon_K(P_N|_{A-\{a_1^1\}}))S$ where $S \in \mathcal{L}(A - \{a_1^1\})$. Then, by the inductive hypothesis, Q indirectly defeats all orders of A in $\varepsilon_K(P_N)$ having the form $(a_1^1 \rightarrow S)$ where $|A| = m$.

Now take any $a_h^k \neq a_1^1$, and let $Q_{-hk} = Q|_{A-\{a_h^k\}} = (a_1^1 \dots a_{h-1}^k a_{h+1}^k \dots a_{H_K}^k)$. By the induction hypothesis, we observe that Q_{-hk} defeats any other order of $(A - \{a_h^k\})$ in $\varepsilon_K(P_N|_{A-\{a_h^k\}})$. Hence one get that $(a_h^k \rightarrow Q_{-hk})$ indirectly defeats in $\varepsilon_K(P_N)$ any order of the form $(a_h^k \rightarrow Z)$ where $Q_{-hk} \neq Z \in \mathcal{L}(A - \{a_h^k\})$.

Suppose that $a_1^1 \mu(P_N) a_h^k$. By Proposition 4.1, we have that $Q\mu(\varepsilon_K(P_N))(a_h^k \rightarrow Q_{-hk})$ so that Q indirectly defeats all orders of A in $\varepsilon_K(P_N)$ having the form $(a_h^k \rightarrow S)$.

Suppose now that $a_h^k \mu(P_N) a_1^1$. Hence, $a_h^k \neq a_2^1$ and either $a_{h-1}^k \mu(P_N) a_h^k$ (if $h > 1$) or $a_{H_K-1}^{k-1} \mu(P_N) a_h^k$ (if $h = 1$). Suppose that $h > 1$. By Proposition 4.1, $Q\mu(\varepsilon_K(P_N))(a_{h-1}^k \rightarrow Q_{-(h-1)k})\mu(\varepsilon_K(P_N))(a_h^k \rightarrow Q_{-hk})$. Since $(a_h^k \rightarrow Q_{-hk})$

⁴A Hamiltonian path visits each element exactly once.

indirectly defeats in $\mu(\varepsilon_K(P_N))$ any order of the form $(a_h^k \rightarrow Z)$, then Q defeats in $\mu(\varepsilon_K(P_N))$ any order of the form $(a_h^k \rightarrow S)$, or formally $Q \in T(\mathcal{L}(A), \varepsilon_K(P_N))$. ■

Theorem 4.1 *The social welfare function θ is Kemeny stable of type 3.*

Proof. It suffices to show that $Q = (S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_k \rightarrow \dots \rightarrow S_K)$ as defined earlier, is in $f_\theta(\varepsilon_K(P_N))$, where for $1 \leq k \leq K$, $S_k = a_1^k a_2^k \dots a_{H_k}^k$ is a Hamiltonian path of A_k of majoritarian partition for P_N , because Q is among the compatible linear orders with $\theta(P_N)$ already. But θ is defined so that $a\theta(P_N)b$ and $\neg\{b\theta(P_N)a\} \forall a \in A_i, \forall b \in A_j$ when $i < j$ while $a\theta(P_N)b$ and $b\theta(P_N)a \forall a, b \in A_i, \forall i, j \leq k$; which means that for all profiles, anything in the top-cycle is put in the top indifference class of θ . Hence $\forall P \in \mathcal{L}(A)$, $P \in T(\mathcal{L}(A), \varepsilon_K(P_N)) \implies P\theta(\varepsilon_K(P_N))Z, \forall Z \in \mathcal{L}(A)$ and we also know that $Q \in \Delta(\theta(P_N)) \implies Q \in T(\mathcal{L}(A), \varepsilon_K(P_N))$ which completes the proof. ■

Now let us introduce a social welfare correspondence, $\bar{\theta}$, which is shown to be Kemeny stable (of type 1) and through which we show existence of a Kemeny stable Condorcet social welfare function.

Definition 4.5 *Let P_N be a profile of linear orders over A and (A_1, A_2, \dots, A_K) be the corresponding majoritarian partition. The social welfare correspondence $\bar{\theta}$ is such that $p \in \bar{\theta}(P_N) \iff p = (H_1 \rightarrow H_2 \rightarrow \dots \rightarrow H_k \rightarrow \dots \rightarrow H_K)$ where $H_k = a_1^k a_2^k \dots a_{H_k}^k$ is a Hamiltonian path in A_k .*

An immediate corollary to Theorem 4.2 is the following.

Corollary 4.2 *The social welfare correspondence $\bar{\theta}$ is Kemeny stable of type 3.*

Furthermore as we mentioned before, $\bar{\theta}$ is Kemeny stable of type 1 in deed.

Theorem 4.2 *The social welfare correspondence $\bar{\theta}$ is Kemeny stable.*

Proof. We know that Q , defined as in Lemma 4.1, is in $T(\mathcal{L}(A), \varepsilon_K(P_N))$ and it, by definition of $\bar{\theta}$, is in $\bar{\theta}(P_N)$. Now we have to show that there exists a Hamiltonian path $P = p_1 p_2 \dots p_L$ in $T(\mathcal{L}(A), \varepsilon_K(P_N))$ for some $L = |T(\mathcal{L}(A), \varepsilon_K(P_N))| \in \mathbb{N}$ such that $p_1 = Q$. But this is obvious to see because for any profile $\pi_N \in \mathcal{L}(A)^N$, $\forall x \in (T(\pi_N))$ there exists a Hamiltonian path in top-cycle that starts with x . ■

It is a corollary to this theorem that there exists a Condorcet type social welfare function which is Kemeny stable since it's always possible to define a singleton-valued selection of $\bar{\theta}$ which is Kemeny stable.

5 Concluding Remarks

So far we discussed two main path of social welfare correspondences; scoring rules and Condorcet type rules. We now show a sufficient condition for Kemeny stability and then conclude.

5.1 A sufficient condition

It's useful to note the following remark which basically says that any linearization of a Kemeny extension of a linear order preserves the Kemeny distance in the sense that if Q is strictly closer to P than R , then any linearization of Kemeny extension of P also ranks Q higher than R .

Remark 5.1 For all $Q, R, P \in \mathcal{L}(A)$, $Q\varepsilon_K^*(P)R \implies QP'R, \forall P' \in \Delta(\varepsilon_K(P))$.

At this point we visit another condition on social welfare functions, called "strong neutrality" and show its sufficiency for Kemeny stability. The reader is advised to be aware of the discussions Sen (13) puts forward on the overly welfarist aspect of this kind of neutrality (or independence) conditions⁵.

Definition 5.1 Let $N \in \mathbb{N}$ represent a society and P_N and Q_N be two profiles of linear preferences over A and B respectively. A social welfare function f is said to satisfy strong neutrality if whenever we have $xP_iy \iff wQ_iz, \forall i \in N$ for some $x, y \in A$ and $w, z \in B$ we have $xf(P_N)y \iff wf(Q_N)z$.

Definition 5.2 For all $p \in \mathcal{L}(A)^N$ with a_kpa_l for some $a_k, a_l \in A$, let p^{-kl} be the $a_k a_l$ -swap of p and defined as $p^{-kl} = (Q_1 \rightarrow a_l \rightarrow Q_2 \rightarrow a_k \rightarrow Q_3)$ where $p = (Q_1 \rightarrow a_k \rightarrow Q_2 \rightarrow a_l \rightarrow Q_3)$.

Theorem 5.1 A social welfare function is strongly neutral only if it is Kemeny stable.

Proof. Let for all finite X , $\alpha : \mathcal{L}(X)^N \rightarrow \mathcal{R}(X)$ be a strongly neutral social welfare function and let $p = (a_1 a_2 \dots a_m) \in \Delta(\alpha(P_N))$ where $A =$

⁵And we are also aware of the "strictness" of the condition although it's included in this study as an example of a sufficiency condition.

$\{a_1, a_2, \dots, a_m\}$. Then, $a_i \alpha(P_N) a_j$, for all $a_i, a_j \in A$ with $i < j \leq m$. Now take $k, l \in \{1, 2, \dots, m\}$ such that $a_k \alpha(P_N) a_l$. We can partition N with N_{kl} and N_{lk} such that $N_{kl} = \{i \in N : a_k P_i a_l\}$ and $N_{lk} = \{i \in N : a_l P_i a_k\}$ for some $k, l \in \{1, 2, \dots, m\}$. This constitutes a partition of N because $P_N \in \mathcal{L}(A)^N$. We know from the Proposition 4.2 that $\forall i \in N \ a P_i b \implies p \varepsilon_K^*(P_i) p^{-ab}$ for every p in $\mathcal{L}(A)$ such that apb . Due to the Remark 5.1, if we suppose $a_k p a_l$, we have that $i \in N_{kl} \implies p P'_i p^{-kl}, \forall P'_i \in \Delta(\varepsilon_K(P_i))$ since $i \in N_{kl} \implies p \varepsilon_K^*(P_i) p^{-kl}$.

$$\begin{array}{ccc}
\begin{array}{cc}
\hline
N_{kl} & N_{lk} \\
\hline
| & | \\
a_k & a_l \\
| & | \\
a_l & a_k \\
| & | \\
\hline
\end{array}
& \implies \varepsilon_K(P_N) : &
\begin{array}{cc}
\hline
N_{kl} & N_{lk} \\
\hline
| & | \\
p & p^{-kl} \\
| & | \\
p^{-kl} & p \\
| & | \\
\hline
\end{array}
\end{array}$$

Now since $a_k \alpha(P_N) a_l$, we have by strong neutrality that $p \alpha(P'_N) p^{-kl}$ for all linearization P'_N of $\varepsilon_K(P_N)$. Whence $p \in \Delta(\alpha(P_N))$ implies that p beats p^{-ij} in $\varepsilon_K(P_N)$ for all $i, j \in \{1, 2, \dots, m\}$ which exhausts the set of all linear orders over A . ■

An immediate corollary follows if we weaken the condition of neutrality as "a social welfare function f is said to satisfy strong neutrality if whenever we have $x P_i y \iff w Q_i z, \forall i \in N$ for some $x, y \in A$ and $w, z \in B$ we have $x f^*(P_N) y \implies w f(Q_N) z$ ".

Corollary 5.1 *A social welfare function satisfies neutrality only if it is Kemeny stable of type 3.*

5.2 Conclusion

We introduced a new notion of stability for social aggregation rules. This new notion, "Kemeny stability", requires an aggregation rule to preserve the socially best ranking among the best rankings when it is applied to the hyper-preference profile. By a hyper-preference we mean a ranking of

rankings of alternatives which is extracted from an individual's preferences over alternatives via the use of Kemeny distance. The condition could be exposed in different ways and we considered a couple of them.

We began by analyzing the Kemeny stability of scoring rules. It turned out that there exists no scoring rule which satisfies our requirement. While showing this, we particularly looked at some specific examples of scoring rules. Furthermore, we looked at Condorcet type social welfare functions. Consequently we showed that although not trivially, there may be found Condorcet type social welfare functions which are Kemeny stable. Finally we have shown a sufficient condition to satisfy the requirement.

One of the major things to note is that the use of the Kemeny distance is essential in our analysis. It is done so not only due to the reputation of Kemeny's distance in social choice literature but is also justified by the idea of hyper-preferences and our approach to the phenomena. The hyper-preference of an individual, in our context, represents his preferences over social outcomes and -as doing so is quite natural in such abstractions- individuals are assumed to be selfish. This only points to the fact that under Kemeny distance the best ranking for an individual who ranked alternatives as in p , is p itself. The further structure of Kemeny hyper-preferences is less debatable since it rests on the idea that if the 'same' is the best then the 'reverse' is the worst and going from the best to the worst in terms of alteration, one loses utility. We close this note by adding that however natural it appears to rest on Kemeny distance in constructing hyper-preferences, it is an open area of interesting research to look for alternative approaches.

Two more things which are dual; one is related to the way we look for stability (from the extension to rules or *vice versa*) and the other is the interpretations of the stabilities (different types of stability and restrictions over domains of hyper preferences; neutral *etc.*). For the former, it reveals itself from the (in)stability of scoring rules. We showed that there does not exist a stable scoring rule, if we stick to Kemeny distance. However, it is of interest to find out, for example, an 'extension' rule for which the Borda rule is stable when same methods used. This applies of course not only to scoring rules, but to any aggregation rule. For the latter we can example

neutrality restriction over the domain of hyper-preferences as we considered in the Appendix B. It may be of interest to look for significant restrictions.

Last but not least, the characterization of Kemeny stable (of any type) social welfare functions is still an open question. Among others we conjecture that the weakest stability condition we consider together with quasi independence of irrelevant alternatives condition *à la* Campbell (2) will characterize Kemeny stability.

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A Appendix: Types of Stability

In this appendix, we introduce different types of stability interpretations we consider. These are called *type 1*, *type 2*, *neutral* and *weak* stabilities. We show the interrelations of different types and put out some results related.

Definition A.1 *A social welfare function α is Kemeny stable (of type 1) if for any $P_N \in \mathcal{L}(A)^N$ and for any $P'_N \in \Delta(\varepsilon_K(P_N))$, $\Delta(\alpha(P_N)) \subseteq f_\alpha(P'_N)$.*

Having a profile of linear orders and a social welfare function, we induce a weak social preference. If we extend this profile via Kemeny distance, we obtain a new profile of preferences over linear orders. This profile consists of weak preferences of course. Generally, there may be many linear order profiles compatible with this weak preference profile. Type 1 stability requires the social welfare function to preserve the social outcome for the initial profile of preferences over alternatives among bests when we move to linear hyper-preferences.

Definition A.2 *A social welfare function α is Kemeny stable of type 2 if for any $P_N \in \mathcal{L}(A)^N$ and for any $P'_N \in \Delta(\varepsilon_K(P_N))$, $f_\alpha(P'_N) \subseteq \Delta(\alpha(P_N))$.*

Type 2 stability requires that if we want to apply the same social welfare function to all of these linearized profiles we should always observe orders compatible with the social preference induced from the initial profile over alternatives. Note that type 2 is even more demanding than type 1 when $\Delta(\alpha(\cdot))$ is singleton-valued for the initial profile. In deed, type 2 restricts the possible social outcomes set while type 1 only requires not to exclude the initial social outcomes.

Definition A.3 *A social welfare function α is Kemeny stable of type 3 if for any $P_N \in \mathcal{L}(A)^N$ and for any $P'_N \in \Delta(\varepsilon_K(P_N))$, $\Delta(\alpha(P_N)) \cap f_\alpha(P'_N) \neq \emptyset$.*

Instead of preserving all (if many) initial social outcomes as in the case of type 1 and type 2, the third type requires only that at least one would be satisfied by the social welfare function in the hyper-preference setting. When $\Delta(\alpha(P_N))$ is singleton-valued, of course, type 3 coincides with type 1.

One may also think of a regularity in extending the preferences. To give insight, one may propose that individuals having the same initial preferences over alternatives should also share the extended preference or more demandingly that the extension should be somehow neutrally done among individuals. The latter can be integrated in our puzzle as what follows.

Definition A.4 *A linear extension P'_N of a Kemeny hyper profile $\varepsilon_K(P_N)$ is said to be a neutral linear extension profile if any pair of preferences in P'_N is isomorphic in the sense that for each $P_i, P_j \in P_N$ there exists a permutation $\sigma_{ij} : A \rightarrow A$ such that $\sigma_{ij}(R)\varepsilon_K(P_i)\sigma_{ij}(Q) \Leftrightarrow R\varepsilon_K(P_j)Q, \forall R, Q \in \mathcal{L}(A)$ where $\forall T \in \mathcal{L}(A)$, $\sigma_{ij}(T)$ is the linear order such that $a\sigma_{ij}(T)b \Leftrightarrow \sigma_{ij}(a)T\sigma_{ij}(b)$.*

The neutral linear extensions subset of all linear extensions is denoted by $\Delta_\nu(\varepsilon_K(P_N))$. To clarify things, consider the following example. We have the following profile of preferences over alternatives.

$$\begin{array}{ccc} a & a & b \\ b & c & a \\ c & b & c \end{array}$$

Now we also have the corresponding Kemeny hyper profile along with a numerical representation;

$$\begin{array}{ccc|l} abc & acb & bac & (123) \\ acb, bac & abc, cab & bca, abc & (132), (213) \\ bca, cab & cba, bac & acb, cba & (231), (312) \\ cba & bca & cab & (321) \end{array} .$$

Then the following two linear extensions of the hyper profiles are the two of four possible neutral linear extensions.

$abc(123)$	$acb(123)$	$bac(123)$	and	$abc(123)$	$acb(123)$	$bac(123)$
$acb(132)$	$abc(132)$	$bca(132)$		$bac(213)$	$cab(213)$	$abc(213)$
$bac(213)$	$cab(213)$	$abc(213)$		$acb(132)$	$abc(132)$	$bca(132)$
$bca(231)$	$cba(231)$	$acb(231)$		$cab(312)$	$bac(312)$	$cba(312)$
$cab(312)$	$bac(312)$	$cba(312)$		$bca(231)$	$cba(231)$	$acb(231)$
$cba(321)$	$bca(321)$	$cab(321)$		$cba(321)$	$bca(321)$	$cab(321)$

So if we restrict our attention on stability to only such linearization, we have the following property.

Definition A.5 *A social welfare function α is neutral Kemeny stable if for any $P_N \in \mathcal{L}(A)^N$, $P'_N \in \Delta_\nu(\varepsilon_K(P_N)) \Rightarrow \alpha(P_N) \subseteq f_\alpha(P'_N)$.*

Furthermore we may propose the following weakening.

Definition A.6 *A social welfare function α is weakly Kemeny stable if for any $P_N \in \mathcal{L}(A)^N$, there exists a linear hyper profile $P'_N \in \Delta(\varepsilon_K(P_N))$ such that $\Delta(\alpha(P_N)) \cap f_\alpha(P'_N) \neq \emptyset$.*

This condition is quite weak since it only requires that we should find at least one linear extension of the Kemeny hyper profile which preserves one initial societal preference as among the best.

The following remarks are devoted to relationships between different types of stability.

Remark A.1

(a) *If $\Delta(\alpha(P_N))$ is singleton valued for all $P_N \in \mathcal{L}(A)$, or in other words, if $\alpha(P_N) \in \mathcal{L}(A), \forall P_N \in \mathcal{L}(A)^N$ then;*

type 1 \iff type 2 \iff type 3 \implies neutral \implies weak.

(b) *If otherwise, the relationships are as follows;*

(i) type 1 \implies type 3 \implies weak;

(ii) type 2 \implies type 3;

(iii) *type 1* \implies *neutral* \implies *weak*.

As an example, let's pick the Borda rule and look at stability characteristics of it. As we show next, the Borda rule happens to be weakly stable when the number of alternatives is three, although it is not neutrally stable.

Proposition A.1 *When $|A| = 3$ the Borda rule is weakly Kemeny stable.*

Proof. Consider the following generic profile P_N ;

n_1	n_2	n_3	n_4	n_5	n_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

The resulting Borda scores for each alternative are;

$$\beta(a) = 2(n_1 + n_2) + n_3 + n_5,$$

$$\beta(b) = 2(n_3 + n_4) + n_1 + n_6,$$

$$\beta(c) = 2(n_5 + n_6) + n_2 + n_4.$$

Now let us assume $\beta(a) \geq \beta(b) \geq \beta(c)$ without loss of generality. Hence the resulting Borda outcome is such that $abc \in \Delta(\beta(P_N))$. We will prove that for any combination of n_i s such that $i \in \{1, 2, \dots, 6\}$ and $n_i \in \mathbb{N}$ there exists an extension of the Kemeny hyper profile to a compatible hyper profile of linear orders on linear orders, such that the Borda outcome of this linear hyper profile puts the ranking abc at the top. To show this, let us consider the case where we put the ranking abc only upper where we needed to replace it to obtain a linear extension. Hence we can count the maximum possible Borda score of the ranking abc it can take in an extended profile. Then we will compare this score with the minimum possible (and compatible with the maximum possible Borda score of the ranking abc) scores of each other rankings and show that these minimum scores of other rankings cannot beat the score of abc .

Observe the followings, implied by our assumption that $abc \in \Delta(\beta(P_n))$;

$$\beta(a) \geq \beta(b)$$

$$2(n_1 + n_2) + n_3 + n_5 \geq 2(n_3 + n_4) + n_1 + n_6$$

$$n_1 + 2n_2 + n_5 \geq n_3 + 2n_4 + n_6$$

Similarly,

$$\beta(b) \geq \beta(c)$$

$$2n_3 + n_4 + n_1 \geq n_6 + 2n_5 + n_2$$

and

$$\beta(a) \geq \beta(c)$$

$$2n_1 + n_2 + n_3 \geq n_5 + 2n_6 + n_6$$

Observe the following Kemeny hyper profile for P_N .

n_1	n_2	n_3	n_4	n_5	n_6
<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>
<i>acb, bac</i>	<i>abc, cab</i>	<i>abc, bca</i>	<i>bac, cba</i>	<i>cba, acb</i>	<i>cab, bca</i>
<i>bca, cab</i>	<i>cba, bac</i>	<i>acb, cba</i>	<i>abc, cab</i>	<i>abc, bca</i>	<i>bac, acb</i>
<i>cba</i>	<i>bca</i>	<i>cab</i>	<i>acb</i>	<i>bac</i>	<i>abc</i>

The table below shows the situation where abc is only put upper in the linear extensions. The empty spots with "-" are not determined and are to be filled with one of the rankings from the indifference class below them.

n_1	n_2	n_3	n_4	n_5	n_6	β
<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>	5
-	<i>abc</i>	<i>abc</i>	-	-	-	4
<i>acb, bac</i>	<i>cab</i>	<i>bca</i>	<i>bac, cba</i>	<i>cba, acb</i>	<i>cab, bca</i>	3
-	-	-	<i>abc</i>	<i>abc</i>	<i>bac, acb</i>	2
<i>bca, cab</i>	<i>cba, bac</i>	<i>acb, cba</i>	<i>cab</i>	<i>bca</i>	-	1
<i>cba</i>	<i>bca</i>	<i>cab</i>	<i>acb</i>	<i>bac</i>	<i>abc</i>	0

Now it is easy to calculate the maximum score abc gets;

$$\beta^+(abc) = 5n_1 + 4(n_2 + n_3) + 2(n_4 + n_5).$$

And here are the minimum scores of all other rankings;

$$\beta^-(acb) = 3n_1 + 5n_2 + n_3 + 3n_5 + n_6$$

$$\beta^-(bac) = 5n_1 + n_2 + 5n_3 + 3n_4 + n_6$$

$$\beta^-(bca) = n_1 + 3n_3 + 5n_4 + n_5 + 3n_6$$

$$\beta^-(cab) = n_1 + 3n_2 + n_4 + 5n_5 + 3n_6$$

$$\beta^-(cba) = n_2 + n_3 + 3n_4 + 3n_5 + 5n_6$$

Case 1

Suppose acb is not beaten by abc in any extension. Then also for the extension where abc gets its maximum score and acb gets its minimum score, acb is not beaten by abc . This requires that $\beta^-(acb) > \beta^+(abc)$. Which means that $3n_1 + 5n_2 + n_3 + 3n_5 + n_6 > 5n_1 + 4(n_2 + n_3) + 2(n_4 + n_5)$, or equivalently $n_2 + 2n_5 + n_6 > 2n_1 + 3n_3 + 2n_4$. But adding $(n_1 + n_3 + n_4)$ to both sides of the inequality $\beta(b) > \beta(c)$ we obtain $2n_1 + 3n_3 + 2n_4 > (n_2 + 2n_5 + n_6) + (n_1 + n_3 + n_4)$ which is a contradiction. The other cases to check are left to the reader. ■

Proposition A.2 *The Borda rule is not neutral Kemeny stable.*

Proof. Consider the sample profile P_N with preference distributions as follows;

$$\begin{array}{ccc} 1 & 1 & 3 \\ \hline a & a & b \\ b & c & a \\ c & b & c \end{array}.$$

The Borda result is such that $\Delta(\beta(P_N)) = \{abc, bac\}$. However, considering the neutral extension

1	1	3	β
<i>abc</i> (123)	<i>acb</i> (123)	<i>bac</i> (123)	5
<i>acb</i> (132)	<i>abc</i> (132)	<i>bca</i> (132)	4
<i>bac</i> (213)	<i>cab</i> (213)	<i>abc</i> (213)	3
<i>cab</i> (312)	<i>bac</i> (312)	<i>cba</i> (312)	2
<i>bca</i> (231)	<i>cba</i> (231)	<i>acb</i> (231)	1
<i>cba</i> (321)	<i>bca</i> (321)	<i>cab</i> (321)	0

we see that $\beta(bac) = 20 > 18 = \beta(abc)$, hence prove that Borda rule is not neutral Kemeny stable since *abc*, a Borda result of initial profile is excluded in the Borda bests of a neutral Kemeny hyper profile. ■

B Appendix: Proof of Theorem 3.4

As Lemma 3.5 shows, we are left only with the scoring rules which have either $s^{1,3} = s^{2,3} > s^{3,3}$ or $s^{1,3} > s^{2,3} > s^{3,3}$. The former is a sort of negative plurality rule which is already shown to fail stability.

The lemmas are introduced so that once the last one is established, they will constitute an incompatibility with the definition of a scoring vector.

Lemma B.1 *If ζ is Kemeny stable, then S_ζ^6 is such that $s_\zeta^{2,6} = s_\zeta^{3,6}$ and $s_\zeta^{4,6} = s_\zeta^{5,6}$.*

Proof. Consider the following profile P_N and suppose (i) $s_\zeta^{1,3} > 2s_\zeta^{2,3}$;

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & a & a & a \end{array}$$

Since $S_\zeta^3(a, P_N) = 2s_\zeta^{1,3}$, $S_\zeta^3(b, P_N) = s_\zeta^{1,3} + 2s_\zeta^{2,3} = S_\zeta^3(c, P_N)$, we have $\zeta(P_N) = \{abc, acb\}$. Consider the following two hyper profiles;

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & cba \\ bac & cab & cba & bca \\ P'_N = acb & abc & bac & cab \\ bca & cba & cab & bac \\ cab & bac & abc & acb \\ cba & bca & acb & abc \end{array} \quad \begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & cba \\ acb & abc & bac & cab \\ P''_N = bac & cab & cba & bca \\ cab & bac & abc & acb \\ bca & cba & cab & bac \\ cba & bca & acb & abc \end{array}$$

we have

$$S_\zeta^6(abc, P'_N) = S_\zeta^6(acb, P'_N) = s_\zeta^{1,6} + s_\zeta^{3,6} + s_\zeta^{5,6}$$

$$S_\zeta^6(abc, P''_N) = S_\zeta^6(acb, P''_N) = s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{4,6}$$

and also

$$S_\zeta^6(bca, P'_N) = S_\zeta^6(cba, P'_N) = s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{4,6}$$

$$S_\zeta^6(bca, P''_N) = S_\zeta^6(cba, P''_N) = s_\zeta^{1,6} + s_\zeta^{3,6} + s_\zeta^{5,6}$$

where $\{\zeta(P'_N), \zeta(P''_N)\} \supseteq \{abc, acb\}$. Hence $s_\zeta^{1,6} + s_\zeta^{3,6} + s_\zeta^{5,6} \geq s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{4,6}$ and $s_\zeta^{1,6} + s_\zeta^{3,6} + s_\zeta^{5,6} \leq s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{4,6}$ by stability imply that $s_\zeta^{3,6} + s_\zeta^{5,6} = s_\zeta^{2,6} + s_\zeta^{4,6}$. Equivalently and by the definition of a scoring vector, we have $0 \geq s_\zeta^{3,6} - s_\zeta^{2,6} = s_\zeta^{4,6} - s_\zeta^{5,6} \geq 0$. So we have $s_\zeta^{2,6} = s_\zeta^{3,6}$ and $s_\zeta^{4,6} = s_\zeta^{5,6}$.

The case (ii) that $s_\zeta^{1,3} > 2s_\zeta^{2,3}$ will lead similar argument for the same profiles; the check is left to reader. Now let us turn to the case (iii) where $s_\zeta^{1,3} = 2s_\zeta^{2,3}$. Consider the profile Q ;

$$\begin{array}{ccc} 2 & 1 & 1 \\ \hline a & c & b \\ c & b & a \\ b & a & c \end{array}$$

where we have $\zeta(Q_N) = \{bca\}$, since $S_\zeta^3(a, Q_N) = 4s_\zeta^{2,3}$, $S_\zeta^3(b, Q_N) = 6s_\zeta^{2,3}$ and $S_\zeta^3(c, Q_N) = 5s_\zeta^{2,3}$. Observe that for any $Q' \in \Delta(\varepsilon_K(Q_N))$ we have

$$2s_\zeta^{2,6} + 2s_\zeta^{4,6} \geq S_\zeta^6(bca, Q'_N) \geq 2s_\zeta^{3,6} + 2s_\zeta^{5,6}$$

and

$$2s_\zeta^{1,6} + 2s_\zeta^{4,6} \geq S_\zeta^6(cba, Q'_N) \geq 2s_\zeta^{1,6} + 3s_\zeta^{5,6}.$$

And stability requires $2s_\zeta^{3,6} + 2s_\zeta^{5,6} \geq 2s_\zeta^{1,6} + 3s_\zeta^{5,6}$, which implies $0 \geq 2s_\zeta^{3,6} - 2s_\zeta^{1,6} \geq 3s_\zeta^{4,6} - 2s_\zeta^{5,6} \geq 0$. But then $3s_\zeta^{4,6} = 2s_\zeta^{5,6}$ implies that $s_\zeta^{4,6} = s_\zeta^{5,6} = 0$ which in turn implies by definition of a scoring vector and $S_\zeta^6(bca, Q'_N) \geq S_\zeta^6(cba, Q'_N)$ that $s_\zeta^{2,6} = s_\zeta^{3,6} (= s_\zeta^{1,6})$ also in deed. ■

Lemma 2 basically shows that to satisfy stability a scoring rule should ignore how a tie is broken in the Kemeny hyper profile.

Lemma B.2 *Let ζ be Kemeny stable. Then S_ζ^6 verifies that $s_\zeta^{1,6} = s_\zeta^{2,6} \Rightarrow s_\zeta^{5,6} > 0 (= s_\zeta^{6,6})$.*

Proof. Consider the following profile Q_N ;

n	m	n	n
a	a	b	c
b	c	a	b
c	a	c	a

We have the following scores for the alternatives;

$$S_\zeta^3(a, Q_N) = (n + m)s_\zeta^{1,3} + ns_\zeta^{2,3}$$

$$S_\zeta^3(b, Q_N) = ns_\zeta^{1,3} + 2ns_\zeta^{2,3}$$

$$S_\zeta^3(c, Q_N) = ns_\zeta^{1,3} + ms_\zeta^{2,3}$$

Now let $B < (\frac{s_\zeta^{2,3}}{s_\zeta^{1,3}})A$. Then we have $S_\zeta^3(b) > S_\zeta^3(a) > S_\zeta^3(c)$ hence $\Delta(\zeta(Q)) = \{bac\}$. The stability of ζ requires that $S_\zeta^6(bac, Q'_N) \geq S_\zeta^6(abc, Q'_N)$ for each profile Q'_N of linear hyper-preferences. Consider the following linear hyper profile, Q'_N ;

	n	m	n	n
	abc	acb	bac	cba
	bac	abc	abc	cab
$Q'_N =$	acb	cab	bca	bca
	bca	cba	cba	acb
	cab	bac	acb	bac
	cba	bca	cab	abc

where we need $2ns_\zeta^{1,6} + (n + m)s_\zeta^{5,6} \geq (2n + m)s_\zeta^{1,6} \Leftrightarrow s_\zeta^{1,6} \leq (\frac{n+m}{m})s_\zeta^{5,6}$ for $S_\zeta^6(bac, Q'_N) \geq S_\zeta^6(abc, Q'_N)$ by the use of Lemma B.3. Since we have $s_\zeta^{1,6} > 0$ it follows that $s_\zeta^{5,6} > 0$. ■

Lemma B.3 *Let ζ be Kemeny stable. Then it verifies that $s_\zeta^{1,6} = s_\zeta^{2,6} \Rightarrow 3s_\zeta^{2,3} = 2s_\zeta^{1,3}$.*

Proof. Suppose for a contradiction that $3s_\zeta^{2,3} < 2s_\zeta^{1,3}$ and consider the profile P_N below:

$$\begin{array}{cccc}
2 & 1 & 1 & 1 \\
\hline
a & a & c & b \\
b & c & b & c \\
c & b & a & a
\end{array}$$

We have that

$$\begin{aligned}
S_\zeta^3(a, P_N) &= 3s_\zeta^{1,3} \\
S_\zeta^3(b, P_N) &= s_\zeta^{1,3} + 3s_\zeta^{2,3} \\
S_\zeta^3(c, P_N) &= s_\zeta^{1,3} + 2s_\zeta^{2,3}
\end{aligned}$$

which implies that $\Delta(\zeta(P_N)) = \{abc\}$. From Lemma B.2 and the assumption of this lemma we have that $s_\zeta^{1,6} = s_\zeta^{2,6} = s_\zeta^{3,6}$, $s_\zeta^{4,6} = s_\zeta^{5,6}$ and furthermore by Lemma B.3 that $s_\zeta^{5,6} > s_\zeta^{6,6} = 0$. It is easy to check that for a linear hyper profile P'_N , $S_\zeta^6(abc, P'_N) \geq S_\zeta^6(acb, P'_N) \Rightarrow 3s_\zeta^{1,6} + s_\zeta^{5,6} \geq 3s_\zeta^{1,6} + 2s_\zeta^{5,6}$ which contradicts with the Lemma B.3.

Now suppose for a contradiction this time that $3s_\zeta^{2,3} > 2s_\zeta^{1,3}$ and consider the profile Q_N below;

$$\begin{array}{ccc}
2 & 1 & 1 \\
\hline
a & c & c \\
b & a & b \\
c & b & a
\end{array}$$

where we have that;

$$\begin{aligned}
S_\zeta^3(a, Q_N) &= 2s_\zeta^{1,3} + s_\zeta^{2,3} \\
S_\zeta^3(b, Q_N) &= 3s_\zeta^{2,3} \\
S_\zeta^3(c, Q_N) &= 2s_\zeta^{1,3}
\end{aligned}$$

which implies $\Delta(\zeta(Q_N)) = \{abc\}$. Now it is to observe that for a linear hyper profile Q'_N , $S_\zeta^6(abc, Q'_N) \geq S_\zeta^6(acb, Q'_N) \Rightarrow 2s_\zeta^{1,6} + s_\zeta^{5,6} \geq 2s_\zeta^{1,6} + 2s_\zeta^{5,6}$ which contradicts with the Lemma B.2. Hence we are left with $3s_\zeta^{2,3} = 2s_\zeta^{1,3}$ which ends the proof. ■

In deed, the following lemma shows that we can't have $s_\zeta^{1,6} = s_\zeta^{2,6}$.

Lemma B.4 *Let ς be Kemeny stable. Then we have $s_{\varsigma}^{1,6} > s_{\varsigma}^{2,6}$.*

Proof. Suppose otherwise, or that $s_{\varsigma}^{1,6} = s_{\varsigma}^{2,6}$. Then by Lemmata B.1 and B.2,

$$\begin{aligned} s_{\varsigma}^{1,6} &= s_{\varsigma}^{2,6} = s_{\varsigma}^{3,6}, \\ s_{\varsigma}^{4,6} &= s_{\varsigma}^{5,6} > s_{\varsigma}^{6,6} = 0 \end{aligned}$$

and

$$3s_{\varsigma}^{2,3} = 2s_{\varsigma}^{1,3}.$$

But observe the profile P_N ;

$$\begin{array}{cccc} 2 & 1 & 1 & 1 \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & b \end{array}$$

where for any stable rule ς , we have

$$S_{\varsigma}^3(a, P_N) = 2s_{\varsigma}^{1,3} + 2s_{\varsigma}^{2,3} > S_{\varsigma}^3(b, P_N) = s_{\varsigma}^{1,3} + 3s_{\varsigma}^{2,3} > S_{\varsigma}^3(c, P_N) = 2s_{\varsigma}^{1,3}.$$

Hence $\Delta(\varsigma(P_N)) = \{abc\}$. But it's quick to observe that for a linear hyper profile P'_N we have $S_{\varsigma}^6(abc, P'_N) \geq S_{\varsigma}^6(bac, P'_N) \Rightarrow 3s_{\varsigma}^{1,6} + s_{\varsigma}^{5,6} \geq 3s_{\varsigma}^{1,6} + 2s_{\varsigma}^{5,6}$ which is the contradiction we desired, since Lemma B.2 showed that $s_{\varsigma}^{5,6} > 0$.

■

So we have that $s_{\varsigma}^{1,6} > s_{\varsigma}^{2,6} = s_{\varsigma}^{3,6} \geq s_{\varsigma}^{4,6} = s_{\varsigma}^{5,6} > s_{\varsigma}^{6,6} = 0$ necessarily for Kemeny stability.

Lemma B.5 *Let ς be Kemeny stable and that we have $s_{\varsigma}^{1,3} \geq s_{\varsigma}^{2,3} > 0$. Then it is true that $s_{\varsigma}^{1,6} = s_{\varsigma}^{2,6} + s_{\varsigma}^{5,6}$.*

Proof. Consider the profile P_N ;

$$\begin{array}{cccc} 2 & 2 & 1 & 1 \\ \hline a & a & c & b \\ b & c & b & c \\ c & b & a & a \end{array}$$

where we have;

$$\begin{aligned} S_\zeta^3(a, P_N) &= 4s_\zeta^{1,3} \\ S_\zeta^3(b, P_N) &= s_\zeta^{1,3} + 3s_\zeta^{2,3} \\ S_\zeta^3(c, P_N) &= s_\zeta^{1,3} + 3s_\zeta^{2,3}. \end{aligned}$$

Observe that we have $\Delta(\zeta(P_N)) = \{abc, acb\}$. Furthermore $S_\zeta^6(abc, P'_N) = S_\zeta^6(acb, P'_N) = 2s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{5,6}$ and $S_\zeta^6(bac, P'_N) = 3s_\zeta^{2,6} + 3s_\zeta^{5,6}$ by Lemma B.1. So it's required that $s_\zeta^{1,6} \geq s_\zeta^{2,6} + s_\zeta^{5,6}$.

Now consider the profile Q_N ;

$$\begin{array}{ccc} 2 & 1 & 1 \\ \hline a & a & b \\ c & b & c \\ b & c & a \end{array}$$

where we have;

$$\begin{aligned} S_\zeta^3(a, Q_N) &= 3s_\zeta^{1,3} \\ S_\zeta^3(b, Q_N) &= s_\zeta^{1,3} + s_\zeta^{2,3} \\ S_\zeta^3(c, Q_N) &= 3s_\zeta^{2,3}. \end{aligned}$$

If (i) $s_\zeta^{1,3} > 2s_\zeta^{2,3}$ we get $\Delta(\zeta(Q_N)) = \{abc\}$. Furthermore $S_\zeta^6(abc, Q'_N) = S_\zeta^6(acb, Q'_N) = s_\zeta^{1,6} + 2s_\zeta^{2,6} + s_\zeta^{5,6}$ and $S_\zeta^6(abc, Q'_N) = 2s_\zeta^{1,6} + s_\zeta^{2,6}$. So it's required that $s_\zeta^{1,6} \leq s_\zeta^{2,6} + s_\zeta^{5,6}$.

If (ii) $s_\zeta^{1,3} > 2s_\zeta^{2,3}$, consider the following profile T_N ;

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ \hline a & b & b & c \\ b & a & c & a \\ c & c & a & b \end{array}$$

where we have;

$$S_\zeta^3(a, T_N) = s_\zeta^{1,3} + 3s_\zeta^{2,3}$$

$$S_\zeta^3(b, T_N) = 2s_\zeta^{1,3} + s_\zeta^{2,3}$$

$$S_\zeta^3(c, T_N) = 2s_\zeta^{1,3} + s_\zeta^{2,3}.$$

hence $\Delta(\zeta(T_N)) = \{abc, acb\}$ by (ii). Observe $S_\zeta^6(abc, T'_N) = s_\zeta^{1,6} + s_\zeta^{2,6} + 3s_\zeta^{5,6}$ and $S_\zeta^6(cab, T'_N) = 2s_\zeta^{1,6} + 2s_\zeta^{5,6}$. Stability requires $s_\zeta^{1,6} + s_\zeta^{2,6} + 3s_\zeta^{5,6} \geq 2s_\zeta^{1,6} + 2s_\zeta^{5,6}$, hence $s_\zeta^{2,6} + s_\zeta^{5,6} \geq s_\zeta^{1,6}$.

If (iii) $s_\zeta^{1,3} = 2s_\zeta^{2,3}$ then by the last argument in the proof of Lemma B.1 we have a contradiction with Lemma B.4 that $s_\zeta^{1,6} > s_\zeta^{2,6}$.

Hence, to sum up, we have $s_\zeta^{2,6} + s_\zeta^{5,6} = s_\zeta^{1,6}$. ■

We are now ready to state and prove the theorem by showing contradiction of these findings.

Theorem 3. 1 *There exists no Kemeny stable scoring rule.*

Proof. Consider the following profile P_N ;

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & a & b \\ c & b & c & a \end{array}$$

where we observe;

$$S_\zeta^3(a, P_N) = 2s_\zeta^{1,3} + s_\zeta^{2,3}$$

$$S_\zeta^3(b, P_N) = s_\zeta^{1,3} + 2s_\zeta^{2,3}$$

$$S_\zeta^3(c, P_N) = s_\zeta^{1,3} + s_\zeta^{2,3}.$$

Hence $\Delta(\zeta(T_N)) = \{abc, bac\}$. Furthermore we have that $S_\zeta^6(abc, P'_N) = s_\zeta^{1,6} + s_\zeta^{2,6} + s_\zeta^{5,6}$ and $S_\zeta^6(bac, P'_N) = s_\zeta^{1,6} + s_\zeta^{2,6} + 2s_\zeta^{5,6}$. So we must have $S_\zeta^6(abc, P'_N) = S_\zeta^6(bac, P'_N)$. But this implies $s_\zeta^{5,6} = 0$ which contradicts with Lemma B.4 since $s_\zeta^{5,6} = 0 \implies s_\zeta^{1,6} = s_\zeta^{2,6}$ by Lemma B.5. ■