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Recovering Non-Monotonicity Problems of Voting Rules

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Oy Kullanma Kurallarında Monotonluk İhlali Problemini Çözmek

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1) Social Choice Theory
2) Voting Rules
3) Monotonicity
4) Minimal Monotonic Extension
5) Plurality with a runoff

To my family...

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Abstract

A social choice rule (SCR) is monotonic if an elected alternative still gets elected when it is improved for at least one voter and the ranking among other alternatives remain the same. Monotonicity is rather weak but well-known to discriminate against scoring elimination rules, such as plurality with a run off and single transferable vote. We define the minimal monotonic extension of a SCR as its unique monotonic supercorrespondence that is minimal with respect to set inclusion. After showing the existence of the concept, we characterize, for every non-monotonic SCR, the alternatives that its minimal monotonic extension must contain. We also discuss measures of non-monotonicity based on the divergence between the minimal monotonic extension of a SCR and the SCR itself. As minimal monotonic extensions can entail coarse SCRs, we address the possibility of maximally refining them without violating monotonicity provided that this refinement does not diverge from the original SCR more than the divergence prescribed by the minimal monotonic extension itself. We call these refinements minimal monotonic adjustments and identify conditions over SCRs that ensure unique minimal monotonic adjustments. As an application of our general findings, we consider plurality with a runoff, charaterize its minimal monotonic extension as well as its (unique) minimal monotonic adjustment. Interestingly, this adjustment is not coarser than plurality with a run off itself, hence we suggest it as a monotonic rectification of plurality with a runoff.

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Özet

Sosyal Seçme Kuralı tarafından seçilen bir adayın, diğer adayların sıralamaları sabitken, seçmenlerin tercihlerinde yükselmesi eğer söz konusu adayın seçilme durumuna zarar vermiyorsa, bu Sosyal Seçme Kuralı'na monoton denir. Monotonluk şartının iki turlu çoğunluk kuralı ve tek seferli devredilebilir alternatif oylama kuralı gibi elemeli skorlama kuralları tarafından sağlanmadığı bilinmektedir. Öncelikle bir Sosyal Seçme Kuralı'nın minimal monoton genişletmesini, o Sosyal Seçme Kuralı'nın alt kümesi olması hususiyetiyle minimal olan tekil monoton eşleşmesi olarak tanımlıyoruz. Konseptin varlığını matematiksel olarak gösterdikten sonra, konseptimizi her monoton olmayan Sosyal Seçme Kuralı için karakterize ediyoruz. Ayrıca monoton olmama halinin ölçümü olarak bir Sosyal Seçme Kuralı'nın minimal monoton genişlemesiyle Sosyal Seçme Kuralı'nın kendisinin aralarındaki sapmayı öne sürüyoruz. Minimal monoton genişletmenin kalın ve ham bir Sosyal Seçme Kuralı'na sebep olabilmesi nedeniyle, minimal monoton genişlemenin sebep olduğu sapmadan fazla olmamak şartı ile, minimal monoton genişletmeyi monotonluk ihlali yaratmayacak şekilde maksimum inceltmeyi inceliyoruz. Bu inceltmelere minimal monoton ayarlama diyoruz ve bu ayarlamaları Sosyal Seçme Kuralı için tekil olacak şekilde tanımlıyoruz. Araştırmanın uygulaması olarak, iki turlu çoğunluk kuralını ele alıp, söz konusu kuralın minimal monoton genişletmesini ve minimal monoton ayarlamasını karakterize ediyoruz. İlginç olarak yapılan ayarlamanın kuralın kendisinden daha az kalın ve ham olması nedeniyle iki turlu çoğunluk kuralının monoton uyarlamasını öneriyoruz.

Anahtar Kelimeler Sosyal Seçim Teorisi, Monotonluk, Minimal Monoton Genişletme, Oylama Kuralları, Turlu Prosedürler

Introduction

Monotonicity conditions imposed over social choice rules (SCRs) elaborate the idea that when at least one voter changes his preferences in a certain "direction", the collective choice must also "move towards a similar direction". Depending on the precise meaning attributed to the two concepts within quotation marks, the literature admits a variety of monotonicity conditions, most of which have a normative appeal that rests on strategic concerns. These conditions are quite strong, the central example being the one identified by Maskin (1977, 1999) as a necessary (but not sufficient) condition for Nash implementation. Maskin monotonicity is quite strong condition than simple monotonicity. It implies that let F is an SCR, and two profiles R and R' and x is an alternative, if $x \in F(R)$, and x does not fall below any alternative in R' then $x \in F(R')$. In the article, Maskin showed that the number of voters is crucial for Nash Implementation. For instance, suppose that consider an environment where there are 2 voters and the SCR satisfies Pareto property, then F must be dictatorial for Nash implementation. The SCR in Nash equilibrium must be Maskin monotonic and monotonicity condition also is an almost sufficient condition for implementability. Suppose that consider an environment where there are 3 voters, if the SCR satisfies Maskin monotonicity and no veto power then it is implementable. For resolute SCRs, Maskin monotonicity coincides with the strong positive association which Muller and Satterthwaite (1977) show to be equivalent to strategy-proofness. In the article, they showed that if a voting rule satisfies strategy-proofness, then it must be satisfied by the strong positive association. Muller and Satterthwaite gave one important corollary: let n be the number of voters and n is greater or equal to 3, if one voting rule satisfies strong positive association and citizen's sovereignty condition, it must be dictatorial. Hence, by Gibbard (1973) and Satterthwaite (1975), only dictatorial or imposed resolute SCRs are Maskin monotonic. When resoluteness is not required, the class of Maskin monotonic SCRs expands but, as Jackson (2001) discusses, still excludes most of the interesting SCRs such as scoring rules and Condorcet extensions. Jackson stated that the result of elections affected by the entrance decision of candidates and they define a condition over the voting rules, namely, candidate stability. This condition requires that the incentive of candidates must not affect the election. They showed that only dictatorial procedures satisfy candidate stability if the candidate and voter's set are distinct. If candidates and voter's overlap, there are non-dictatorial procedures that satisfy the unanimity and stability of candidates. However, none of them satisfy the monotonicity condition, unlike scoring rules, Condorcet procedures. Jackson also defined 'top-paired monotonicity'. It implies that if preferences are uniform and there exist restricted 2 candidates top set, then the procedure must select restricted top set. They also indicated that no voting procedure that satisfies top-paired monotonicity, candidate stability and the unanimity. They concluded that strategic manipulation is possible for every non-dictatorial method.

There are three other prominent monotonicity conditions that can be expressed with reference to Maskin monotonicity: Danilov (1992) monotonicity characterizes Nash implementable SCRs, hence being stronger than Maskin monotonicity. Danilov used 'essential monotonicity' definition and showed that if essential monotonicity satisfied by the SCR, that SCR is implementable in Nash equilibrium. Essential monotonicity implies that improvement or non-essentially worsening positions for alternatives must not be detrimental for that alternative. Moore and Raphello (1998) showed that implementability condition can be represented as a system. (Eg. $C:(a,R)$). Danilov interested and answered that question by new condition, namely, essential monotonicity. Thanks to Danilov, we know that any essential monotone Social Choice Correspondence is monotone however, not every monotone Social Choice Correspondence is essentially monotonic, however generally, monotonicity implies essential monotonicity. Danilov gives a condition of that assertion. Together with those solutions, we know that the mechanism (mechanism of social choice) is essentially monotone. Condition alpha (Abreu and Sen (1990)) replaces Maskin monotonicity when the target is implementation via subgame perfect equilibria. In their article, condition alpha is parallel to monotonicity by Maskin analyze is necessary and (almost) sufficient condition for implementation. Condition alpha is considerably weaker than Maskin monotonicity. For instance, majority voting satisfies condition alpha, but not monotonicity. As Nunez and Sanver (2018) show, it is satisfied by several Condorcet extensions while failed by scoring rules. However, Sertel and Yılmaz (1999) showed that the majoritarian compromise satisfies condition alpha and no veto power, then majoritarian compromise is subgame perfect implementable. For subgame perfect implementability, Nunez and Sanver defined new condition: Condition β. Condition β is a weak and simple condition, it is only applied alternative, let say x, is ranked top or bottom (last) by every voter. Take any two profiles R and R' which coincides with each other with ranking position (top or bottom) of x. Then if x is selected in R, it must be selected in R' to satisfy Condition β. However, all scoring rules fail to satisfy Condition β, hence all scoring rules fail to be subgame perfect implementable. Nunez and Sanver also, however, showed that if there are 3 alternatives plurality with a runoff is subgame perfect implementable. One-way monotonicity is suggested by Sanver and Zwicker (2009) as a weaker form of strategy-proofness. It is also weaker than Maskin monotonicity, indeed satisfied by scoring rules while failed by Condorcet extensions.

A normatively appealing monotonicity concept is simple monotonicity that requires an improved alternative in voters' preferences yet leaving the other rankings unchanged is never detrimental to the prospects for winning of the raised alternative

Simple monotonicity is perhaps the oldest known monotonicity condition in the literature. It has been expressed under different names during its relatively long history that predates modern social choice theory. For a comprehensive account, see Black (1958), Brams and Fishburn (2002) and comments on page 120 of Fishburn (1982), (including footnote 1). Throughout the paper we refer to simple monotonicity as monotonicity. Monotonicity is rather weak and satisfied by most voting rules of the literature. However, it discriminates against scoring elimination rules which contain well-known members such as plurality with a run off and single transferable vote (Smith (1973))[1]

Sen (1995) suggests to recover failures of Maskin monotonicity by minimally

[1]Fishburn (1982) discusses other examples of monotonicity failure.

extending SCRs to their Maskin monotonic super-correspondences - a concept whose existence is ensured. In a similar vein, we consider monotonic extensions of non-monotonic SCRs. The SCR that picks every alternative at every preference profile is trivially monotonic. As a result, every non-monotonic SCR admits at least one monotonic super-correspondences. A more interesting but less obvious point is whether a unique minimal super-correspondence that ensures monotonicity exists. We answer this question affirmatively and characterize, for every non-monotonic SCR, the alternatives that its minimal monotonic extension must contain.

The divergence between the minimal monotonic extension of a SCR and the SCR itself can be seen as a measure of the extent to which this SCR fails monotonicity. We propose such a measure which we discuss via examples of SCRs whose monotonicity failures exhibit different natures. As minimal monotonic extensions can entail coarse SCRs, we address the possibility of maximally refining them without violating monotonicity. To be sure, such a refinement must diverge from the original SCR, as otherwise the monotonic extension that is refined would not be minimal. As a matter of fact, we search for refinements which do not diverge from the original SCR more than the divergence prescribed by the minimal monotonic extension itself. We call these refinements minimal monotonic adjustments and identify conditions over SCRs that ensure unique minimal monotonic adjustments.

We study a popular election system which is used in several countries including France and Turkey for presidential elections, namely the plurality with runoff. First, candidates run against eachother and every voter casts her vote for her most preferred candidate (so we disregard strategic voting). If there is a tie between the most voted candidates, they go to the second round. Then every pair of candidates run against eachother and whoever gets the most votes in these 2-candidate races becomes a winner. Hence there may be more than one winner of the election. If there is only one candidate who gets the most votes in round one, then she pairs up with the second most voted candidate of round one and they are put to a vote in round two. Who- ever gets the most votes becomes the winner of the election. If they tie in this round, then they are both announced to be winners. If in round one, there is one most voted candidate and there are more than one second most voted candidates, then they all go to round two and winner of round one races against each one of the second most voted candidates of round one. All winners of these pairwise races are the winners of the election.

As an application of our general findings, we consider plurality with a runoff, a SCR that is well-known to fail monotonicity. We charactarize the minimal monotonic extension of plurality with a run off, as well as its (unique) minimal monotonic adjustment. Interestingly, this adjustment is not coarser than plurality with a run off itself, hence we suggest it as a rectification of plurality with a run off.

Empirically, potential monotonicity failure of plurality with runoff could be seen in many elections. For instance, The 2002 French Presidential Election. In the first round, Jacques Chirac got %19.88 of votes and finished first place in the first round and Jean-Marie Le Pen, however, who got %16.86 of votes in the first round and finished second place in the first round behind Lionel Jospin who got %16.18 in the first round. In the second round, Chirac got %82.2 of votes and won the election. However, before the first round, if Jacques Chirac had made propaganda to gain Le Pen's votes, Lionel Jospin might have finished the first round in second place. In that situation, no one can argue that the possibility of success for Chirac in the Jospin-Chirac competition of second round, same as the second round of Chirac-Le Pen competition, Chirac could have lost the election although he increased own votes in the first round by making propaganda for Le Pen voters.

Another empirical example of potential monotonicity failure could have seen in the last presidential election of France in 2017. In the first round Emmanuel macron got %24.01 of votes and finished first place in the first round and Marine Le Pen, however, got %21.30 of votes in the first round and finished second place in the first round behind François Fillon who got %20.01 and Jean-Luc Melenchon who got %19.58 in the first round. In the second round, Emmanuel Macron got %66.1of votes and won the election. However, like my previous example, before the first round, if Emmanuel Macron had made right-wing propaganda, to take Marine Le Pen and François Fillon's votes Jean-Luc Melenchon might have finished the first round in second place. In that situation, again, no one can argue that the possibility of success for Macron in the Macron-Le Pen competition of the second round, same as the second round of Macron-Melenchon competition's. Emmanuel Macron could have lost the election, although he increased his own votes in the first round by making propaganda for right-wing voters.

The thesis is organized as follows: Section 2 presents the basic notions and notation. Section 3 contains our general results on minimal monotonic extensions. Section 4 applies these results to derive specific findings about plurality with a runoff. Section 5 concludes.

2 Basic notions and notations

Consider a *society* N with $\#N = n \geq 2$ confronting a set of *alternatives* A with $\#A \geq 2$. Each voter $i \in N$ has a *preference* $P_i \in L(A)$ where $L(A)$ is the set of linear orders over A. So precisely one of $x\ P_i\ y$ and $y\ P_i\ x$ holds for any distinct $x, y \in A$ while $x\ P_i\ x$ fails for all $x \in A$. Moreover, $x\ P_i\ y$ and $y\ P_i\ z$ implies $x\ P_i\ z$ for all $x, y, z \in A$. We write $P \in L(A)^N$ for a *(preference) profile.* A social choice rule (SCR) is a mapping $F : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$. Given any $P, P' \in L(A)^n$ and $x \in A$, we say that P' is an *improvement for* x with respect to P if and only if we have

(i) $y\ P_i\ z \Longleftrightarrow y\ P'_i\ z\ \forall y, z \in A \backslash \{x\}, \forall i \in N$

and

(ii) $y\ P_i\ x$ and $x\ P'_i\ y$ for some $y \in A \backslash \{x\}$ and for some $i \in N$.

We write IMP $_x(P) \subseteq L(A)^N$ for the set of preference profiles which are improvements for x with respect to P. When P' is an improvement for x with respect to P, we equivalently say that P is a worsening for x with respect to P' and let $\mathrm{WOR}_x(P') \subseteq L(A)^N$ be the set of preference profiles which are worsenings for x with respect to P'.

An SCR $F : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ is monotonic if and only if given any $P \in L(A)^N$, any $x \in F(P)$ and any $P' \in \mathrm{IMP}_x(P)$, we have $x \in F(P')$.

Although quite desirable, not every SCR is monotonic. The next section suggests a solution to monotonicity violations.

3 Monotonic extension

3.1 Definition and characterization

Given two SCRs F, $G : L(A)^N \Longrightarrow 2^A \backslash \{\emptyset\}$, we say that G is an *extension* of F if and only if $F \subseteq G$. We write $F \subseteq G$ whenever $F(R) \subseteq G(R)$ $\forall R \in L(A)^N$ If such G is also monotonic, we call it a *monotonic extension* of F. We define the set of all monotonic extensions of F as

$\mu(F) = \{G : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ I G is a monotonic extension of F

Note that $\mu(F)$ is non-empty for any SCR F, as the SCR K defined as $K(P) = A$ $\forall P \in L(A)^N$ is a monotonic extension of every F.

Given any two SCRs F and G with $F(P) \cap G(P) \neq \emptyset$ $\forall P \in L(A)^N$, we define the SCR $F \cap G$ as $F \cap G(P) = F(P) \cap G(P)$ $\forall P \in L(A)^N$.

Proposition 3.1 Let F, $G : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ be two SCRs with $F \cap G$ $(P) \neq \emptyset$ $\forall P \in L(A)^N$. If F and G are both monotonic, then $F \cap G$ is also monotonic.

Proof: Let F and G be as in the statement of the lemma. Take any $P \in L(A)^N$, any $x \in F \cap G$ (P) and any $P' \in IMP_x(P)$. As $x \in F \cap G$ (P) implies $x \in F(P)$ and $x \in G(P)$ while F and G are both monotonic, we have $x \in F(P')$ and $x \in G(P')$, implying $x \in F \cap G$ (P'), establishing that $F \cap G$ is monotonic. Q.E.D.

Among all the monotonic extensions of a given SCR F, we are mainly interested in the smallest with respect to set inclusion. We define the *minimal monotonic extension* of F as the SCR $\overline{F} = \bigcap_{G \in \mu(F)} G$. Note that $\overline{F}$ is well-defined, as $H \cap G$ is well-defined for any $H, G \in \mu(F)$ while $\mu(F)$ is finite. The monotonicity of $\overline{F}$ follows Lemma 1 and, by construction, within $\mu(F)$, $\overline{F}$ is minimal with respect to set inclusion. Note that while $F \subseteq \overline{F}$, we have $F = \overline{F}$ if and only if F is monotonic.

We now present a general characterization result which specifies the alternatives that minimal monotonic extensions contain.

Theorem 3.1 Given any SCR $F : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$, any $P \in L(A)^N$, and any $x \in A \backslash F(P)$, we have $x \in \overline{F}(P) \backslash F(P)$ if and only if $\exists P' \in \mathrm{WOR}_x(P)$ such that $x \in F(P')$.

Proof: To show the 'if part', take any SCR F, any $P \in L(A)^N$, any $x \in A \backslash F(P)$ and $P' \in WOR_x(P)$ with $x \in F(P')$. Consider any $G \in \mu(F)$. Since G is an extension of F, we have $x \in F(P') \subseteq G(P')$ and $x \in G(P')$, as G is monotonic. Thus, $x \in G(P')$ for any $G \in \mu(F)$, implying $x \in \overline{F}(R) = \bigcap_{G \in \mu(F)} G(R)$.

To show the 'only if' part, take any SCR F and suppose there exists $P^* \in L(A)^N$ with $x^* \in \overline{F}(P^*) \backslash F(P^*)$ while $x^* \in F(P)$ for no $P \in \mathrm{WOR}_{x^*}(P^*)$. Construct the SCR F^* defined at each $P^* \in L(A)^N$ as

$$F^*(P) = \left\{ \begin{array}{ll} \overline{F}(P) & if P \notin WOR_{x^*}(P^*) \\ \overline{F}(P) \backslash \{x^*\} & if P \in WOR_{x^*}(P^*) \cup \{P^*\} \end{array} \right\}.$$

We have $F^* \subseteq \overline{F}$ by construction while $x^* \in \overline{F}(P^*) \backslash F^*(P^*)$. So, $F^* \subsetneq \overline{F}$. We now show $F \subseteq F^*$. Take any $P \in L(A)^N$ and any $x \in F(P) \subseteq \overline{F}(P)$. If $P \notin \mathrm{WOR}_{x^*}(P^*)$, then $\overline{F}(P) = F^*(P)$, hence $x \in F^*(P)$. If $P \in \mathrm{WOR}_{x^*}(P^*)$ and $x \neq x^*$, then again $\overline{F}(P) = F^*(P)$, hence $x \in F^*(P)$. Finally the case $P \in \mathrm{WOR}_{x^*}(P^*)$ and $x = x^*$ contradicts our supposition that $x^* \notin F(P)\ \forall P \in \mathrm{WOR}_{x^*}(P^*)$, showing $F \subseteq F^*$.

Thus, F^* is both an extension of F and a proper subcorrespondence of $\overline{F}$. We complete the proof by showing that F^* is monotonic which contradicts the minimality of $\overline{F}$. Take any $P \in L(A)^N$, any $x \in F^*(P)$ and any $P' \in \mathrm{IMP}_x(P)$. Consider first the case where $P \in \mathrm{WOR}_{x^*}(P^*)$. Since $F^*(P) = \overline{F}(P) \setminus \{x^*\}$, we have $x \neq x^*$ implying $x \in \overline{F}(P)$ which implies $x \in \overline{F}(P')$ by the monotonicity of $\overline{F}$ while $x \neq x^*$ensures $x \in F^*(P')$. Now consider the case where $P \notin \mathrm{WOR}_{x^*}(P^*)$. So $x \in \overline{F}(P) = F^*(P)$ and $x \in \overline{F}(P')$ by the monotonicity of $\overline{F}$. If $P' \notin \mathrm{WOR}_{x^*}(P^*)$, then $\overline{F}(P') = F^*(P')$, thus $x \in F^*(P')$. If $P' \in \mathrm{WOR}_{x^*}(P^*)$, then $P^* \in \mathrm{IMP}_x(P)$ which implies $x \neq x^*$, as otherwise $P \notin \mathrm{WOR}_{x^*}(P^*)$ is contradicted. When $x \neq x^*$, we have $x \in \overline{F}(P) = F^*(P)$ and $x \in \overline{F}(P')$ by the monotonicity of $\overline{F}$. Again, $x \neq x^*$ ensures $x \in F^*(P') = \overline{F}(P')$, establishing the monotonicity of F^*, thus completing the proof. Q.E.D.

3.2 A Measure of Non-Monotonicity

For any SCR F, let $\delta_F(P) = \overline{F}(P) \backslash F(P)\ \forall P \in L(A)^N$ and $\Delta_F = \{P \in L(A)^N : \delta_F(P) \neq \emptyset\}$ be the set of profiles at which F and $\overline{F}$ differ. So δ_F is a well-defined SCR over the domain Δ_F. Note that δ_F and δ_F may considerably differ among SCRs, as we exemplify below:

Example 3.1 F picks the unique plurality winner (ties broken according to an exogeneous linear order) and, if any, the alternative ranked unanimously worst. Here $\#F(P) \in \{1,2\}$ and $\delta_F(P) = A \backslash F(P)$ at each $P \in L(A)^N$.

Example 3.2 F picks all alternatives, except the unique plurality winner (ties broken according to an exogeneous linear order). Here $\#F(P) = \#A - 1$ and $\delta_F(P) = 1$ at each $P \in L(A)^N$.

Example 3.3 F picks the unique plurality winner (ties broken according to an exogeneous linear order). Moreover, some fixed $x \in A$ is also picked whenever it is the second best for all voters. Here $\#F(P) \in \{1,2\}$ and $\delta_F(P) = \{x\}$ at each $P \in L(A)^N$ with $x \notin F(P)$ while $\delta_F(P) = \emptyset$ at each $P \in L(A)^N$ with $x \in F(P)$.

Example 3.4 Fix a pair of alternatives $x, y \in A$. F picks $A \backslash \{x,y\}$ if both x and y are ranked first at least by one voter and F picks A otherwise. Here $\#F(P) \in \{\#A-2, \#A\}$ and $\delta_F(P) = \{x,y\}$ at each $P \in L(A)^N$ with $F(P) \neq A$ while $\delta_F(P) = \emptyset$ at each $P \in L(A)^N$ with $F(P) = A$.

3.3 Refining the extension: minimal monotonic adjustments

Given a SCR F, a monotonic subcorrespondence $G \subseteq \overline{F}$ of $\overline{F}$ is said to respect the divergence of $\overline{F}$ from F iff G complies with F whenever $\overline{F}$ does so, i.e., $G(P) = F(P)\ \forall P \in L(A)^N \backslash \Delta_F$. In case, G does not admit a monotonic proper subcorrespondence that respects the divergence of $\overline{F}$ from F (i.e., any $H \subseteq G$ that respects the divergence of $\overline{F}$ is monotonic if and only if $H = G$), we call G a minimal monotonic adjustment of F.

We now discuss the existence and uniqueness of minimal monotonic adjustments. The set of profiles that cause a monotonicity problem at $P \in \Delta_F$ for $x \in \delta_F(P)$ is denoted $\Pi_x(P) = \{P' \in \mathrm{WOR}_x(P) : x \in F(P')\}$. As we will soon observe, $\Pi_x(P)$ may or may not be a subset of Δ_F, which leads to the following condition we impose over F.

Condition α: $\Pi_x(P) \not\subset \Delta_F$ $\forall P \in \Delta_F$, $\forall x \in \delta_F(P)$.

We now consider a second condition over F.

Condition β: $\delta_F(P) = \emptyset$ and $\delta_F(P') \neq \emptyset \Longrightarrow \delta_F(P') = \{x\}$ $\forall x \in A$, $\forall P \in L(A)^N$, $\forall P' \in \mathrm{IMP}_x(P)$.

Note that among the SCRs considered in Section 3.2, Example 3.1 fails both conditions while Example 3.3 satisfies them. Moreover, Conditions α and β are logically independent, as Example 3.4 illustrates a case where Condition α is satisfied but Condition β is failed while Example 3.2 illustrates a case where Condition β is satisfied but Condition α is failed.

Our next result shows that the conjunction of Conditions α and β implies the monotonicity of δ_F.

Lemma 3.1

F is an SCR that satisfies Conditions α and β, then $\delta_F : \Delta_F \longrightarrow 2^A \backslash \{\emptyset\}$ is monotonic. –

Proof: Let F satisfy Conditions α and β. Take any $P \in \Delta_F$, any $x \in \delta_F(P)$ and any $P' \in \mathrm{IMP}_x(P) \bigcap \Delta_F$. By Condition α, $\exists P'' \in \mathrm{WOR}_x(P)$ with $x \in F(P'')$ and $\delta_F(P'') = \emptyset$. Note that $P' \in \mathrm{IMP}_x(P'')$. As $P' \in \Delta_F$, $\delta_F(P') \neq \emptyset$. By Condition β, $\delta_F(P') = \{x\}$. Q.E.D.

Theorem 3.2

$F : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ satisfies Conditions α and β, then F admits a unique minimal monotonic adjustment $\widetilde{F}$ defined as $\widetilde{F}(P) = F(P)$ $\forall P \in L(A)^N \backslash \Delta_F$ and $\widetilde{F}(P) = \delta_F(P)$ $\forall P \in \Delta_F$. –

Proof: Let F satisfy Conditions α and β. As $\widetilde{F}$ respects the divergence of $\overline{F}$ from F, we have $\widetilde{F}(P) = F(P)$ $\forall P \in L(A)^N \backslash \Delta_F$. We now show $\delta_F(P) \subseteq \widetilde{F}(P)$ $\forall P \in \Delta_F$. Take any $P \in \Delta_F$ and any $x \in \delta_F(P)$. As $\Pi_x(P) \not\subset \Delta_F$ by Condition α,

$\exists P' \in \text{WOR}_x(P)$ with $x \in F(P')$ while $P' \in L(A)^N \backslash \Delta_F$. Thus, $\widetilde{F}(P') = F(P')$, hence $x \in \widetilde{F}(P')$. Note that $P \in \text{IMP}_x(P')$ while $\widetilde{F}$ is monotonic, implying $x \in \widetilde{F}(P)$.

We complete the proof by showing that $\widetilde{F}$ is monotonic. Take any $P \in L(A)^N$, any $x \in \widetilde{F}(P)$ and any $P' \in \text{IMP}_x(P)$. We show $x \in \widetilde{F}(P')$ for the four exhaustive cases below, thus establishing the monotonicity of $\widetilde{F}$.

Case 1. $P, P' \in \Delta_F$. Thus $\widetilde{F}(P) = \delta_F(P)$ and $\widetilde{F}(P') = \delta_F(P')$. By Lemma 3.1, monotonicity of δ_F implies $x \in \widetilde{F}(P')$.

Case 2. $P \in \Delta_F$, $P' \in L(A)^N \backslash \Delta_F$. Thus $x \in \widetilde{F}(P) = \delta_F(P)$, so $\exists P'' \in \text{WOR}_x(P)$ with $x \in F(P'')$. By Condition α, we let $P'' \notin \Delta_F$, without loss of generality. Since $P' \in \text{IMP}_x(P)$ and $P \in \text{IMP}_x(P'')$, we have $P' \in \text{IMP}_x(P'')$. As $\delta_F(P'') = \emptyset$ and $x \in F(P'')$, we have $x \in F(P') = \widetilde{F}(P')$.

Case 3. $P, P' \in L(A)^N \backslash \Delta_F$. Thus $x \in F(P)$, implying $x \in F(P') = \widetilde{F}(P')$, as otherwise we would have $P' \in \Delta_F$.

Case 4. $P \in L(A)^N \backslash \Delta_F$, $P' \in \Delta_F$. By Condition β, $\delta_F(P') = \widetilde{F}(P') = \{x\}$, i.e., $x \in \widetilde{F}(P')$. Q.E.D.

4 An application: Plurality with a runoff

Let $r(x, P_i) = \#\{y \in A I y P_i x\} + 1$ be the rank of $x \in A$ at $P_i \in L(A)$. For any $x \in A$ and $P \in L(A)^N$, define $\tau(x, P) = \#\{i \in N \mid r(x, P_i) = 1\}$ as the number of voters who are rank x at the top. Let $\pi_1(P) = \{x \in A \mid \tau(x, P) \geq \tau(y, P)\ \forall y \in A\}$ be the set of plurality winners, i.e., the alternatives ranked at the top by the highest number of voters. Let $\tau^1(P) = \tau(x, P)$ for some $x \in \pi_1(P)$. Similarly, $\pi_2(P) = \{x \in A \backslash \pi_1(P) \mid \tau(x, P) \geq \tau(y, P)\ \forall y \in A \setminus \pi_1(P)\}$ and $\tau^2(P) = \tau(x, P)$ for some $x \in \pi_2(P)$.

Let

$$RO(P) = \begin{cases} \{\{x,y\}\}_{x \in \pi_1(P), y \in \pi_1(P) \backslash \{x\}} & if \# \pi_1(P) \geq 2 \\ \{\{x,y\}\}_{\{x\} = \pi_1(P), y \in \pi_2(P)} & if \# \pi_1(P) = 1 \end{cases}$$

be defined as the set of pairs of alternatives that go for a runoff at $P \in L(A)^N$. Note that, RO is not a SCR. However, we define the SCR $\rho : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ which picks at each $P \in L(A)^n$ the set of candidates that go for a runoff, i.e., $\rho(P) =$

$\{x \in A \mid \exists \{x,y\} \in RO(P)\}$.

Plurality with a runoff is the SCR $F_{PR} : L(A)^N \longrightarrow 2^A \backslash \{\emptyset\}$ defined for each $P \in L(A)^n$ as

$F_{PR}(P) = \{x \in A | \exists \{x,y\} \in \mathrm{RO(P)}$ and $\#\{i \in N | xP_i\, y\} \geq \#\{i \in \mathrm{N} — yP_i x\}\}$. It is well-known that F_{PR} is not monotonic

Consider 3-candidate election plurality with a runoff with 93 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 1: Add caption

42	27	24
a	b	c
b	c	a
c	a	b

Profile-2

Table 2: Add caption

46	23	24
a	b	c
b	c	a
c	a	b

Let $n = 93, A = \{a, b, c\}$ and R be a profile where 42 voters have the preference a $R_i b$ R_i c; 27 voters have the preference b $R_i c$ R_i a; and 24 voters have the preference c $R_i a$ R_i b. So $RO(R) = \{\{a, b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 46 voters have the preference a $R'_i b$ R'_i c; 23 voters have the preference b $R'_i c$ R'_i a; and 24 voters have the preference c $R'_i a$ R'_i b. Now $RO(R') = \{\{a, c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 3-candidate election plurality with a runoff with 186 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is winner of the first round in both profiles. In profile-1 candidate b finished second place in the first round therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 3: Add caption

84	54	48
a	b	c
b	c	a
c	a	b

Profile-2

Table 4: Add caption

92	46	48
a	b	c
b	c	a
c	a	b

let $n = 186, A = \{a,b,c\}$ and R be a profile where 84 voters have the preference a $R_i b$ R_i c; 54 voters have the preference b $R_i c$ R_i a; and 48 voters have the preference c $R_i a$ R_i b. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 92 voters have the preference a $R'_i b$ R'_i c; 46 voters have the preference b $R'_i c$ R'_i a; and 48 voters have the preference c $R'_i a$ R'_i b. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 3-candidate election plurality with a runoff with 279 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 5: Add caption

124	81	72
a	b	c
b	c	a
c	a	b

Profile-2

Table 6: Add caption

138	69	72
a	b	c
b	c	a
c	a	b

Let $n = 279$, $A = \{a,b,c\}$ and R be a profile where 124 voters have the preference $a\ R_i b\ R_i\ c$; 81 voters have the preference $b\ R_i c\ R_i\ a$; and 72 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 138 voters have the preference $a\ R'_i b\ R'_i\ c$; 69 voters have the preference $b\ R'_i c\ R'_i\ a$; and 72 voters have the preference $c\ R'_i a\ R'_i\ b$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 128 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 7: Add caption

55	30	28	15
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 8: Add caption

58	27	28	15
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 128, A = \{a,b,c,d\}$ and R be a profile where 55 voters have the preference $a\ R_ib\ R_i\ c\ R_id$ 30 voters have the preference $b\ R_ic\ R_i\ aR_id$; 28 voters have the preference $c\ R_ia\ R_i\ b.\ R_id$ and 15 voters have the preference $d\ R_ib\ R_i\ c\ R_ia$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 58 voters have the preference $a\ R'_ib\ R'_i\ cR'_id$; 27 voters have the preference $b\ R'_ic\ R'_i\ aR'_id$; 28 voters have the preference $c\ R'_ia\ R'_i\ b\ R'_id$ and 15 voters have the preference $d\ R'_ib\ R'_i\ cR'_ia$ Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 256 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 9: Add caption

110	60	56	30
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 10: Add caption

116	54	56	30
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 256$, $A = \{a,b,c,d\}$ and R be a profile where 110 voters have the preference $a\ R_i b\ R_i\ c\ R_i d$ 60 voters have the preference $b\ R_i c\ R_i\ aR_i d$; 56 voters have the preference $c\ R_i a\ R_i\ b.\ R_i d$ and 30 voters have the preference $d\ R_i b\ R_i\ c\ R_i a$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 116 voters have the preference $a\ R'_i b\ R'_i\ cR'_i d$; 54 voters have the preference $b\ R'_i c\ R'_i\ aR'_i d$; 56 voters have the preference $c\ R'_i a\ R'_i\ b\ R'_i d$ and 30 voters have the preference $d\ R'_i b\ R'_i\ cR'_i a$ Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 512 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 11: Add caption

220	120	112	60
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 12: Add caption

232	108	112	60
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 512$, $A = \{a,b,c,d\}$ and R be a profile where 220 voters have the preference $a\ R_i b\ R_i\ c\ R_i d$ 120 voters have the preference $b\ R_i c\ R_i\ aR_i d$; 112 voters have the preference $c\ R_i a\ R_i\ b.\ R_i d$ and 60 voters have the preference $d\ R_i b\ R_i\ c\ R_i a$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 232 voters have the preference $a\ R'_i b\ R'_i\ cR'_i d$; 108 voters have the preference $b\ R'_i c\ R'_i\ aR'_i d$; 112 voters have the preference $c\ R'_i a\ R'_i\ b\ R'_i d$ and 60 voters have the preference $d\ R'_i b\ R'_i\ cR'_i a$ Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

We first show that the minimal monotonic extension of plurality with a run off can only contain elements who went for a run off.

Proposition 4.1 $\delta_{F_{PR}} \subseteq \rho$.

Proof: We first show that ρ is monotonic.

Take any $R \in L(A)^n$ and $x \in \varphi(R)$. Let R' be an improvement for x with respect to R. Then $\tau(x,R) \geq \tau(x,R')$ and $\tau(y,R) \leq \tau(y,R')$ $\forall y \in A \backslash \{x\}$. Therefore, x continues to be in $\pi_1(R')$ or $\pi_2(R')$. So $x \in \varphi(R')$. Hence φ is monotonic. Thus, $\rho \in \mu(F_{PR})$, implying $\overline{F_{PR}} \subseteq \rho$, whicn in turn implies $\delta_{F_{PR}} \subseteq \rho$.

The next lemma shows what type of elements from ρ are not included in $\overline{F_{PR}}$:

Proposition 4.2

For any $P \in L(A)^N$ and any $x \in \rho(P) \setminus F_{PR}(P)$, if $\tau(x,P) \leq \tau(y,P)$ $\forall \{x,y\} \in RO(P)$, then $x \notin \overline{F_{PR}}(P)$.

Proof: Take any $P \in L(A)^N$ and any $x \in \rho(P) \setminus F_{PR}(P)$ with $\tau(x,P) \leq \tau(y,P)$ $\forall \{x,y\} \in RO(P)$. Suppose $x \in \overline{F_{PR}}(P)$. By Proposition 2.1, $\exists P' \in WOR_x(P)$ with $x \in F_{PR}(P')$.

First observe that $\pi_1(P') \neq \{x\}$ does not hold because otherwise $P \in \text{IMP}_x(P')$ would imply $\pi_1(P) = \{x\}$ as well, which would contradict $\tau(x,P) \leq \tau(y,P)$ $\forall \{x,y\} \in RO(P)$. As a result, the following two cases are exhaustive:

Case 1: $x \in \pi_1(P')$ and $\#\pi_1(P') \geq 2$.

So $\tau(y,P') = \tau(z,P')$ $\forall \{y,z\} \in RO(P')$. Note that $P \in \text{IMP}_x(P')$ and $\tau(x,P) \leq \tau(y,P)$ $\forall \{x,y\} \in RO(P)$ imply $RO(P) = RO(P')$. As $x \in F_{PR}(P')$, $\exists \{x,y\} \in RO(P')$ with $\#\{i \in N \mid x\, P'_i\, y\} \geq \#\{i \in N \mid y\, P'_i\, x\}$. By $RO(P) = RO(P')$ we note $\{x,y\} \in RO(P)$ and by $P \in \text{IMP}_x(P')$ we have $\#\{i \in N \mid x\, P_i\, y\} \geq \#\{i \in N \mid y\, P_i\, x\}$, implying $x \in F_{PR}(P)$, contradicting $x \in \rho(P) \setminus F_{PR}(P)$.

Case 2: $x \in \pi_2(P')$ and $\pi_1(P') = \{y\}$ for some $y \in A \setminus \{x\}$.

So $\{x,y\} \in RO(P')$ and $\{x,z\} \notin RO(P')$ $\forall z \in A \setminus \{xy\}$. As $x \in F_{PR}(P')$, we have $\#\{i \in N \mid x\, P'_i\, y\} \geq \#\{i \in N \mid y\, P'_i\, x\}$. Note that $P \in \text{IMP}_x(P')$ implies $\{x,y\} \in RO(P)$ and $\#\{i \in N \mid x\, P_i\, y\} \geq \#\{i \in N \mid y\, P_i\, x\}$, implying $x \in F_{PR}(P)$, contradicting $x \in \rho(P)$ $\setminus F_{PR}(P)$.

Therefore, we conclude $x \notin \overline{F_{PR}}(P)$. Q.E.D.

Proposition 4.3 For any $P \in L(A)^N$ and any $x \in \rho(P) \setminus F_{PR}(P)$ with $\tau(x,P) > \tau(y,P)$ for some $\{x,y\} \in RO(P)$, we have $x \in \overline{F_{PR}}(P)$ if and only if $\exists z \in A \backslash \{x,y\}$ such that

1-) $\tau(x,P) - \tau(y,P) \geq \tau(y,P) - \tau(z,P)$

2-) $N_{x,z} \geq \tau(y,P) - \tau(z,P)$

3-) $\#(i \in N \mid xP_i z) - \#(i \in N \mid zP_i x) \geq 2(\tau(y,P) - \tau(z,P)$.

Proof: Take any $P \in L(A)^N$ and any $x \in \rho(P) \setminus F_{PR}(P)$ with $\tau(x,P) > \tau(y,P)$ for some $\{x,y\} \in RO(P)$.

To see the "only if" part, let $x \in \overline{F_{PR}}(P)$. So by proposition 3.1, $\exists\, P' \in WOR_x(P)$ such that $x \in F_{PR}(P')$. As $x \notin F_{PR}(P)$, $\#(i \in N \mid xP_i y) < \#(i \in N \mid yP_i x)$. As $P' \in WOR_x(P)$, $x \notin F_{PR}(P^)$, $\#(i \in N \mid xP_i' y) < \#(i \in N \mid yP_i' x)$. So $\exists z \in A \backslash \{x,y\}$ such that $\{x,z\} \in RO(P')$ while $\#(i \in N \mid xP_i' z) \geq \#(i \in N \mid zP_i' x)$ because otherwise $x \in F_{PR}(P')$ would not be possible. Moreover $\{x,z\} \notin RO(P)$ because otherwise we would have $x \in F_{PR}(P)$. Thus, noting $\{x,z\} \in RO(P') \backslash RO(P)$, we observe the existence of some $K \subseteq (i \in N \mid P_i \neq P_i')$. $r(x,P_i) = 1$ and $r(z,P_i) = 2\ \forall i \in K$.

Moreover, $\#K \geq \tau(y,P) - \tau(z,P)$ and $\#K \geq \tau(x,P) - \tau(y,P)$ which establishes (1) and (2).

As $\#(i \in N \mid xP_i' z) \geq \#(i \in N \mid zP_i' x)$,

we have $\#(i \in N \mid xP_i' z) - \#(i \in N \mid zP_i' x) \geq 2\#K$ which establishes (3).

To see "if" part, suppose $\exists Z \in A$ I $\{x,y\}$ that satisfies conditions (i), (ii) and (iii) in the statement of the lemma.

Take some K $\subseteq N_{x,z}$ with $\#K = \tau(y,P) - \tau(z,P)$. Condition (i) ensures the existence of K. Now, take $P' \in L(A)^N$ with $P_i' = P_i\ \forall i \in N \backslash K$ and $sP_i' t \iff sP_i t\ \forall$ s, t $\in A \backslash \{x,z\}\ \forall i \in K$ and $zP_i' xP_i s\ \forall s \in A \backslash \{x,z\}\ \forall i \in K$. Note that $P' \in WOR_x(P)$. Moreover, as $\#K = \tau(y,P) - \tau(z,P)$, we have $\tau(y,P') = \tau(z,P')$. Also, by condition (ii), $\tau(x,P') \geq \tau(z,P')$. Hence $\{x,z\} \in RO(P')$. By condition (iii) and the existence of # K $\#(i \in N \mid xP_i' z) \geq \#(i \in N \mid zP_i' x)$. Thus, $x \in F_{PR}(P')$ with by proposition 3.1 $x \in \overline{F_{PR}}(P)$. Q.E.D

Definition 4.1 Given any $P \in L(A)^N, x, y, z \in A$ there is a critical score:

$c(x,y,z;p) = min[(\zeta(x,P) - \zeta(y,P)), N_{x,z}, \#\{(i \in n : xP_i) - (i \in NzP_ix)\}/2]$

Theorem 4.1 Take any $P \in L(A)^N$ and any $x \in A \setminus \delta(P)$, $X \in \overline{\delta}(P)$ if and only if;

$\zeta(x,P) \geq \zeta(y,P)$ for some $x \in RO(P)$ and there exists $z \in A\setminus\{xy\}$ such that $c(x,y,z,P) \geq \zeta(y,P) - \zeta(z,P)$

Propositions 4.1, 4.2 and 4.3 lead the the following corollary:

Corollary

For each $P \in L(A)^N$, we have $\delta_{F_{PR}}(P) = x \in \rho(P)\setminus F_{PR}(P) : \tau(x,P) > \tau(y,P)$ for some $\{x,y\} \in RO(P)\text{WHILE}\exists z \in A\setminus\{x,y\} with \tau(x,P) - \tau(y,P) \geq \tau(y,P) - \tau(z,P)$ and $N_{x,z} \geq \tau(y,P) - \tau(z,P)$ AND $\#(i \in N \mid xP_iz) - \#(i \in N \mid zP_ix) \geq 2(\tau(y,P) - \tau(z,P))$ Note that $\#\delta_{F_{PR}}(P) \in \{0,\ 1\}$ at each $P \in L(A)^N$.

Theorem 4.2 $F_{PR} : L(A)^N \longrightarrow 2^A\setminus\{\emptyset\}$ admits a unique minimal monotonic adjustment $\widetilde{F_{PR}}$ defined as $\widetilde{F_{PR}}(P) = F_{PR}(P)$ $\forall P \in L(A)^N\setminus\Delta_F$ and $\widetilde{F_{PR}}(P) = \delta_{F_{PR}}(P)$ $\forall P \in \Delta_F$.

Proof: We establish the proof through Theorem 3.2, by showing that F_{PR} satisfies both Conditions α and β. Condition β is satisfied because $\#\delta_{F_{PR}}(P) = 1$ for all $P \in \Delta_{F_{PR}}$. As for Condition α, take any $P \in \Delta_{F_{PR}}$. By Proposition 4.2, $\delta_{F_{PR}}(P) = \pi_1(P) = \{x\}$. By Theorem 3.1, $\exists P' \in \text{WOR}_x(P)$ with $x \in F(P')$. By Proposition 4.2, we can find $P' \in \Pi_x(P)$ such that $P' \notin \delta_F$. Therefore Condition α is satisfied. Q.E.D.

Present a discussion of $\widetilde{F_{PR}}$ and whether we can present it as an alternative SCR.

5 Conclusion

We know since Smith (1973) that all point run off procedures, including plurality with a run off used in Turkish and French presidential elections, may exhibit perverse reactions to changes in voters' preferences. To be more precise, under point run off procedures, it may happen that some voters' lifting a candidate in their preference rankings (everything else being the same) can be detrimental to the lifted candidate's chance of getting elected. We introduced the concept of a "minimal monotonic extension" which is the smallest monotonic super correspondence of a non-monotonic social choice rule. Also we introduced minimal monotonic adjustment of SCR's. . Up to now, we have been able to show that the concept is well defined. Moreover, the difference between a non-monotonic social choice rule and its minimal monotonic extension is a reflection of the degree of non-monotonicity of that social choice rule. Our open question is to compute the minimal monotonic extensions of every point run off procedure. This will enable us the compare these procedures according to how non-monotonic they are. Moreover, we will hence be identifying monotonic rectifications of these procedures.

Monotonicity conditions imposed over social choice rules (SCRs) elaborate the idea that when at least one voter changes his preferences in a certain ''direction", the collective choice must also ''move towards a similar direction". Depending on the precise meaning attributed to the two concepts within quotation marks, the literature admits a variety of monotonicity conditions, most of which have a normative appeal that rests on strategic concerns.These conditions are quite strong, the central example being the one identified by Maskin (1977, 1999) as a necessary (but not sufficient) condition for Nash implementation. For resolute SCRs, Maskin monotonicity coincides with strong positive association which Muller and Satterthwaite (1977) show to be equivalent to strategy-proofness. Hence, by Gibbard (1973) and Satterthwaite (1975), only dictatorial or imposed resolute SCRs are Maskin monotonic. When resoluteness is not required, the class of Maskin monotonic SCRs expands but, as Jackson (2001) discusses, still excludes most of the interesting SCRs such as scoring rules and Condorcet extensions. A normatively appealing monotonicity concept is simple monotonicity that requires an improved alternative in voters' preferences yet leaving the other rankings unchanged is never detrimental to the prospects for winning of the raised alternative. Simple monotonicity is perhaps the oldest known monotonicity condition in the literature. It has been expressed under different names during its relatively long history that predates modern social choice theory. I used to simple monotonicity as monotonicity. Monotonicity is rather weak and satisfied by most voting rules of the literature. However, it discriminates against scoring elimination rules which contain well-known members such as plurality with a runoff and single transferable vote (Smith (1973)). Thanks to Smith (1973) we know that all point run off procedures, including plurality with a run off used in French presidential elections, may exhibit perverse reactions to changes in voters' preferences. To be more precise, under point run off procedures, it may happen that some voters' lifting a candidate in their preference rankings (everything else being the same) can be detrimental to the lifted candidate's chance of getting elected. Sen (1995) suggests to recover failures of Maskin monotonicity by minimally extending SCRs to their Maskin monotonic super-correspondences - a concept whose existence is ensured. In a similar vein, I consider monotonic extensions of non- monotonic SCRs. The SCR that picks every alternative at every preference profile is trivially monotonic. As a result, every non-monotonic SCR admits at least one monotonic super-correspondences. A more interesting but less obvious point is whether a unique minimal super-correspondence that ensures monotonicity exists. We answered this question affirmatively and characterize, for every non-monotonic SCR, the alternatives that its minimal monotonic extension must contain. The divergence between the minimal monotonic extension of a SCR and the SCR itself can be seen as a measure of the extent to which this SCR fails monotonicity. We propose such a measure which we discuss via examples of SCRs whose monotonicity failures exhibit different natures. As minimal monotonic extensions can entail coarse SCRs, we address the possibility of maximally refining them without violating monotonicity. To be sure, such a refinement must diverge from the original

SCR, as otherwise the monotonic extension that is refined would not be minimal. As a matter of fact, we search for refinements which do not diverge from the original SCR more than the divergence prescribed by the minimal monotonic extension itself. We call these refinements minimal monotonic adjustments and identify conditions over SCRs that ensure unique minimal monotonic adjustments. Finally, we consider plurality with a runoff. We characterize the minimal monotonic extension of plurality with a run off, as well as its (unique) minimal monotonic adjustment. Interestingly, this adjustment is not coarser than plurality with a run off itself, hence we suggest it as a rectification of plurality with a run off. We introduce two concepts of interest in the treatment of SCRs that fail monotonicity. We start with the minimal monotonic extension of a SCR F which is the monotonic super correspondence $\overline{F}$ of F that is minimal with respect to set inclusion. Every non-monotonic F admits a unique minimal monotonic extension $\overline{F}$. However, $\overline{F}$ may be too coarse to be used as a SCR. As a result we suggest to refine it while preserving its monotonicity through the SCR $\widetilde{F}$ which we call the minimal monotonic adjustment of F. $\widetilde{F}$ agrees with F at profiles where $\overline{F}$ agrees with F. In other words, in case $\overline{F}$ leaves F intact, $\widetilde{F}$ does not distort F. On the other hand, when $\overline{F}$ adds alternatives to F, $\widetilde{F}$ refines $\overline{F}$ which should be at the expense of discarding certain outcomes prescribed by F. We are able to provide conditions under which a SCR admits a unique minimal monotonic adjustment. To illustrate the use of all this analysis, we consider a well-known non-monotonic SCR, namely plurality with a runoff and identify its minimal monotonic extension as well as its minimal monotonic adjustment.

Our findings inspire the derivation a formal index of non-monotonicity based on the shape that the minimal monotonic extension gets. This would allow to compare the degree of non-monotonicity of SCRs, such as plurality with a runoff and the single transferable vote, by computing their minimal monotonic extensions. Furthermore, in case the minimal monotonic extensions turn out to be narrow, one can consider to replace a non-monotonic SCR by its minimal monotonic adjustment (which will also be narrow) in order to restitute monotonicity, as we have done for plurality with a runoff. We leave all these as open questions raised by our analysis

I would like to offer three aspects for open questions. As an expansion, first aspect of my open question is compute the minimal monotonic extensions of every point run off procedure. This will enable us the compare these procedures according to how non-monotonic they are. Moreover, we will hence be identifying monotonic rectifications of these procedures. Second aspect to developing my research project is empirical work of monotonicity failure. We can examine two-round runoff elections which are used by French presidential elections, parliament and local elections in France. We can search those election in the manner of which are exhibit potential monotonicity failure. For instance, the 2002 French Presidential Election. In the first round If Jacques Chirac got made propaganda to gain Le Pen votes, Lionel Jospin might have finished the first round in second place. In that situation, no one can argue that the possibility of success for Chriac in the Jospin-Chirac competition of second round, same as second round of Chirac-Le Pen competition's. Chirac could have lost the election although he increased own votes in the first-round by made propaganda for Le Pen voters. Together with widely empirical study, we will have opportunity to know that how monotonicity failure effects electoral system and importance of strategic voting. Third aspect for open questions is axiomatic properties of the voting rules and discrepancies among voting rules. For many decades, the literature on the election of committees or assemblies mostly dealt with one subject, the apportionment of the seats, either among different party lists, or among the different states or regions of a country. In American politics, the debates about the merits of the different apportionment methods have been running since the time of the founding fathers of the USA. In modern science, two mathematicians and economists, Young and Balinski (1982), made a complete mathematical analysis of apportionment methods. However, in the last ten years, social choice theory saw considerable progress on the analysis of committee elections (eg. Brams et al (2007), Elkind et al. (2017), Faliszewski et al (2017), Kamwa and Merlin (2018), Skowron et al. (2019)). Indeed, Since the publication of "Social Choice and Individual Values" by Arrow (1951), social choice theory offers a rigorous mathematical framework to analyze the properties of decision rules: the voting procedure is envisioned as a mathematical function from the set of preference profiles to the set of outputs, and we may seek which properties are satisfied by different voting procedures, and also discover which normative criteria are mutually compatible or not. In this framework, departing from the classical assumption that voters have to select party lists, numerous new rules were proposed, where the ballots could be as diverse as complete or partial rankings of the candidates, approval ballots (saying 'yes' or 'no' for each candidate), or evaluation of the candidates on a common scale. Thus, the final part of my project has two major objectives: First to better understand the axiomatic properties of the voting rules that have been specifically designed for committee elections. Secondly, to see to which extent different committee rules can lead to the selection of different sets of candidates for some voting patterns. Axiomatic Properties: In Ireland and Australia, the Single Transferable Vote (STV) method has been in use since the turn of the 20th century. In contrast to party-list elections, the ballots allow the voters to rank as many candidates as they wish, in decreasing order of preference. The determination of the elected k candidates proceeds in rounds. At each round, either one or more candidates are elected (which occurs when their number of supporting votes exceeds a predefined quota), or the candidate with lowest support is eliminated. Then, a system of transfer of votes to the remaining candidate's takes place, both from the ballots supporting a winner if the quota is exceeded, or from the ballots supporting the eliminated candidate. A problem with this family of voting rules is that they are not monotonic (Brams and Kilgour, Lepelley and Merlin): improving the rank of a candidate in the individual preference can be detrimental for him/ her (e.g. as I mentioned before transferring vote from Le Pen to Chirac in the 2002 French presidential elections may lead to the election of Jospin). A natural question would be to find the minimal monotonic

extension of these rules that is the minimal superset derived from STV that remains monotonic, and examines its properties. Discrepancies among voting rules: The literature on single-winner voting has come up with numerous results about the conditions (number of voters and candidates, domain restrictions) under which two rules are equal, may output sets of co-winners with an empty intersection, or may be such that the outcome of a rule is included in the outcome of the other. See for example Saari (1992), Ratliff (2003). Note that the question is much more complicated for the case of multi winner than for the single-winner case, due to the presence of the third parameter k, the number of candidates to select: for which values of m, for which values of m (number of candidates), n (number of voters) an k (number of winners) can two multi-winner rules differ? For which values of m, n and k will two multi-winner rules always have an empty intersection at some profile? It looks clear that, for a fixed number of voters and alternatives, as the number of winners k increases, the chance for two committee rules to pick common winners increases. It would be very interesting to know at which rate they do so. For instance, can we expect some relationships to emerge before we reach $k = m/2$ for some pairs of rules? In particular, the tools that have developed by Saari (1992) for the analysis of the discrepancies between scoring rules, and applied recently by Merlin et al.(2018) could be useful to tackle this kind of issues.

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7 Appendix: Monotonic Refinements

Intuitively if an SCR, F is not monotonic, our tendency would be to eliminate those alternatives that were chosen by F but cause non-monotonicity, to the point where F becomes monotonic. In other words, the first natural solution that comes to mind to resolve non-monotonicity is to *refine* F until it is monotonic. Below, we formalize this idea and present our results on when such a refinement is possible.

Given any two SCRs, F and G, we write $F \subseteq G$ if $F(R) \subseteq G(R)\ \forall R \in L(A)^n$. Other binary relations among SCRs are defined similarly. We say that G is a *refinement* of F if and only if $F \subseteq G$. In case G is also monotonic, we call it a *monotonic refinement* of F. First note that monotonic refinements do not have to exist. As an example, consider the following environment with $A = \{x,y\}$ and $N = \{1,2\}$. Let R and R' be such that yR_1x, xR_2y and xR'_1y, xR'_2y. Suppose that an SCR, F, is so that $F(R) = \{x\}$ and $F(R') = \{y\}$. Since R' is an improvement for x with respect to R and $F(R) = \{x\}$, $x \notin F(R')$ implies non-monotonicity of F. Since F is single valued at these profiles, there is no monotonic refinement for F.

Furthermore, even when a monotonic refinement exists, it does not have to be unique. Consider the SCR K, defined as $K(R) = A\ \forall R \in L(A)^n$. Take any x, $y \in A$ with $x \neq y$. Define G and G' as $G(R) = \{x\}\ \forall R \in L(A)^n$ and $G' = \{y\}\ \forall R \in L(A)^n$. Both G and G' are monotonic refinements of K and $G \neq G'$.

Despite the existence and uniqueness problems discussed above, we will now show that if a monotonic refinement exists for an SCR, F, then its *maximal monotonic refinement* exists and it is unique.

Definition Given an SCR F, $\widehat{F}$ is called the maximal monotonic refinement of F if $\widehat{F}$ is a monotonic refinement of F and for any monotonic refinement of G of F, we have $G \subseteq \widehat{F}$.

As discussed earlier, $\widehat{F}$ does not have to exist.

Lemma 7.1 *Consider an SCR, F. If G and G' are monotonic refinements of F, then so is $G \cup G'$.*

Proof Obviously, $G \cup G' \subseteq F$. To see monotonicity, take any profile $R \in L(A)^n$ and $x \in G \cup G'\ (R)$. Without loss generality, assume that $x \in G(R)$. Consider any improvement, R', of x with respect to R. Since G is monotonic, $x \in G(R')$, which means $x \in G \cup G'\ (R')$.

Given an SCR F, let $\kappa(F)$ denote the set of all monotonic refinements of F. In case $\kappa(F) \neq \emptyset$, there are finitely many SCRs in $\kappa(F)$, because both A and N are finite. Therefore, by repeated application of the previous lemma,

$$\widehat{F} = \bigcup_{G \in \kappa(F)}$$

is a monotonic refinement of F and by construction, any monotonic refinement of F is a subset of $\kappa(F)$. In other words, $\widehat{F}$ is the unique maximal monotonic refinement of F. In what follows, we discuss a new notion of monotonicity which will be a necessary and sufficient condition for having a monotonic refinement.

Definition An SCR, F is called almost monotonic if and only if for every $R \in L(A)^n$, $\exists\, x \in F(R)$ such that for all improvements, R', of x with respect to R, we have $x \in F(R')$.

Clearly, every monotonic SCR is also almost monotonic. However, as the following example illustrates, monotonicity and almost monotonicity are not equivalent.

Consider a society with two individuals, 1 and 2; and two alternatives, x and y. Listed in the table below are all possible preference profiles, where the descending order is in line with the order of preferences.

Table 13: Add caption

	R1		R2		R3		R4	
Individuals	1	2	1	2	1	2	1	2
Rankings	x	x	y	y	x	y	y	x
	y	y	x	x	y	x	x	y

Suppose that F is an SCR so that, $F(R_1) = F(R_2) = F(R_3) = \{y\}$ and $F(R_4) = \{x, y\}$. F is not monotonic because $x \in F(R_4)$ and R_1 is an improvement for x with respect to R_4; yet, $x \notin F(R_1)$. However, F is almost monotonic because $y \in F(R_i)$, $\forall\, i = 1, \ldots, 4$.

We now show that, in order for an SCR to have a monotonic refinement, and hence have a maximal monotonic refinement, it needs to be *almost monotonic* and vice versa.

Proposition 6.1 SCR, F, has a monotonic refinement if and only if it is almost monotonic.

Proof Suppose that F is almost monotonic. Define the set $u(R^*, x)$ as

$$u(R,x) = \{Q \in L(A)^n | Q \text{ is an improvement of } x \text{ with respect to } R\}.$$

Define the SCR, G as

$$G(R) = \{x | x \in F(R) and x \in F(Q) \forall Q \in u(R,x)\}$$

G is a non-empty valued SCR due to almost monotonicity of F. By construction, G is monotonic and obviously $G \subseteq F$. Therefore G is a monotonic refinement of F.

Conversely, assume that F has a monotonic refinement. Suppose that F is not almost monotonic, i.e., there exists $R^* \in L(A)^n$ such that $\forall x \in F(R^*)$, $\exists Q \in u(R^*, x)$ with $x \notin F(Q)$. Since this is true for all $x \in F(R^*)$, we can find a monotone refinement G of F such that $x \in G(R^*)$. Since G is monotone, $x \in G(Q) \subseteq F(Q)$ for all $Q \in u(R^*), x$, which is a contradiction. Therefore F is almost monotonic.

The next two corollaries discuss the relation between almost monotonicity and the existence of *proper* monotonic refinements *non-resolute* SCRs.

Definition An SCR F is called non-resolute if for some $R \in L(A)^N$, $\#F(R) \geq 2$.

Definition For an SCR F, G is called a proper monotonic refinement of F if G is a monotonic refinement of F and there exists $R \in L(A)^N$ so that $F(R) \neq G(R)$.

Corollary 7.1.1 *Let F be a non-resolute SCR which is not monotonic. Then, F has a proper monotonic refinement if and only if F is almost monotonic.*

Proof Suppose that F is almost monotonic. By Proposition 3.1, F has a monotonic refinement, say G. Since F is not monotonic, $G \neq F$. Therefore G is a proper monotonic refinement of F.

Assume that F has a proper monotonic refinement, which is a monotonic refinement. By Proposition 3.1, F is almost monotonic.

The majority with a runoff SCR that we discuss in detail later is not almost monotonic. Therefore, to tackle with the problem of monotonicity, refinement is not an option for this SCR. This fact leads us to the notion of monotonic extensions, which is discussed in the previous section.

-some examples of monotonicity violations-

Consider 3-candidate election plurality with a runoff with 465 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 14: Add caption

210	135	120
a	b	c
b	c	a
c	a	b

Profile-2

Table 15: Add caption

230	115	120
a	b	c
b	c	a
c	a	b

Let $n = 465$, $A = \{a, b, c\}$ and R be a profile where 210 voters have the preference $a\ R_i b\ R_i\ c$; 135 voters have the preference $b\ R_i c\ R_i\ a$; and 120 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a, b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 230 voters have the preference $a\ R_i' b\ R_i'\ c$; 115 voters have the preference $b\ R_i' c\ R_i'\ a$; and 120 voters have the preference $c\ R_i' a\ R_i'\ b$. Now $RO(R') = \{\{a, c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 3-candidate election plurality with a runoff with 930 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is winner of the first round in both profiles. In profile-1 candidate b finished second place in the first round therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 16: Add caption

420	270	240
a	b	c
b	c	a
c	a	b

Profile-2

Table 17: Add caption

460	230	240
a	b	c
b	c	a
c	a	b

let $n = 930, A = \{a, b, c\}$ and R be a profile where 420 voters have the preference $a\ R_i b\ R_i\ c$; 270 voters have the preference $b\ R_i c\ R_i\ a$; and 240 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 460 voters have the preference $a\ R_i' b\ R_i'\ c$; 230 voters have the preference b $R_i' c\ R_i'\ a$; and 240 voters have the preference $c\ R_i' a\ R_i'\ b$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 3-candidate election plurality with a runoff with 1395 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 18: Add caption

620	405	360
a	b	c
b	c	a
c	a	b

Profile-2

Table 19: Add caption

690	345	360
a	b	c
b	c	a
c	a	b

Let $n = 1395, A = \{a,b,c\}$ and R be a profile where 620 voters have the preference $a\ R_i b\ R_i\ c$; 405 voters have the preference $b\ R_i c\ R_i\ a$; and 360 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 690 voters have the preference $a\ R'_i b\ R'_i\ c$; 345 voters have the preference $b\ R'_i c\ R'_i\ a$; and 360 voters have the preference $c\ R'_i a\ R'_i\ b$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 640 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 20: Add caption

275	150	140	75
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 21: Add caption

290	135	140	75
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 640$, $A = \{a, b, c, d\}$ and R be a profile where 275 voters have the preference $a\ R_i b\ R_i\ c\ R_i d$ 150 voters have the preference $b\ R_i c\ R_i\ aR_i d$; 140 voters have the preference $c\ R_i a\ R_i\ b.\ R_i d$ and 15 voters have the preference $d\ R_i b\ R_i\ c\ R_i a$ Thus, $RO(R) = \{\{a, b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 290 voters have the preference $a\ R'_i b\ R'_i\ cR'_i d$; 135 voters have the preference $b\ R'_i c\ R'_i\ aR'_i d$; 140 voters have the preference $c\ R'_i a\ R'_i\ b\ R'_i d$ and 15 voters have the preference $d\ R'_i b\ R'_i\ cR'_i a$. Now $RO(R') = \{\{a, c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 1280 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 22: Add caption

550	300	280	150
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 23: Add caption

580	270	280	150
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 1280$, $A = \{a,b,c,d\}$ and R be a profile where 550 voters have the preference $a\ R_ib\ R_i\ c\ R_id$ 300 voters have the preference $b\ R_ic\ R_i\ aR_id$; 280 voters have the preference $c\ R_ia\ R_i\ b.\ R_id$ and 150 voters have the preference $d\ R_ib\ R_i\ c\ R_ia$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 116 voters have the preference $a\ R_i'b\ R_i'\ cR_i'd$; 270 voters have the preference $b\ R_i'c\ R_i'\ aR_i'd$; 280 voters have the preference $c\ R_i'a\ R_i'\ b\ R_i'd$ and 150 voters have the preference $d\ R_i'b\ R_i'\ cR_i'a$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 2560 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 24: Add caption

1100	600	560	300
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 25: Add caption

1160	540	560	300
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 2560$, $A = \{a,b,c,d\}$ and R be a profile where 1100 voters have the preference $a\ R_ib\ R_i\ c\ R_id$ 600 voters have the preference $b\ R_ic\ R_i\ aR_id$; 560 voters have the preference $c\ R_ia\ R_i\ b.\ R_id$ and 300 voters have the preference $d\ R_ib\ R_i\ c\ R_ia$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 1160 voters have the preference $a\ R'_ib\ R'_i\ cR'_id$; 540 voters have the preference $b\ R'_ic\ R'_i\ aR'_id$; 560 voters have the preference $c\ R'_ia\ R'_i\ b\ R'_id$ and 300 voters have the preference d $R'_ib\ R'_i\ cR'_ia$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

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Consider 3-candidate election plurality with a runoff with 2790 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 26: Add caption

1260	810	720
a	b	c
b	c	a
c	a	b

Profile-2

Table 27: Add caption

1380	690	720
a	b	c
b	c	a
c	a	b

Let $n = 2790$, $A = \{a,b,c\}$ and R be a profile where 1260 voters have the preference $a\ R_i b\ R_i\ c$; 810 voters have the preference $b\ R_i c\ R_i\ a$; and 720 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 1380 voters have the preference $a\ R'_i b\ R'_i\ c$; 690 voters have the preference $b\ R'_i c\ R'_i\ a$; and 720 voters have the preference $c\ R'_i a\ R'_i\ b$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 3-candidate election plurality with a runoff with 8370 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is lifted from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 28: Add caption

3780	2430	2160
a	b	c
b	c	a
c	a	b

Profile-2

Table 29: Add caption

4140	2070	2160
a	b	c
b	c	a
c	a	b

Let $n = 8370, A = \{a,b,c\}$ and R be a profile where 3780 voters have the preference $a\ R_i b\ R_i\ c$; 2430 voters have the preference $b\ R_i c\ R_i\ a$; and 2160 voters have the preference $c\ R_i a\ R_i\ b$. So $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 4140 voters have the preference $a\ R'_i b\ R'_i\ c$; 2070 voters have the preference $b\ R'_i c\ R'_i\ a$; and 2160 voters have the preference $c\ R'_i a\ R'_i\ b$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.

Consider 4-candidate election plurality with a runoff with 163840 voters. In both profile no candidate has an absolute majority in the first round. Candidate a is plurality winner in both profiles. In profile-1 second most preferred candidates is b therefore candidates a and b proceed a second round and in second round candidates a is declared to winner. In profile-2, however, despite candidate a is improved from profile-1 to profile-2, candidate c is declared to winner. Candidates a and c proceed to second round, but second preference of voters who are ranked candidate b is c .Therefore, candidates a, although increased support in profile-2 with respect to profile-1 lost the election.

Profile-1

Table 30: Add caption

70400	38400	35840	19200
a	b	c	d
b	c	a	b
c	a	b	a
d	d	d	c

Profile-2

Table 31: Add caption

74240	32400	35840	19200
a	b	c	d
b	c	a	b
c	a	b	c
d	d	d	a

Let $n = 163840$, $A = \{a,b,c,d\}$ and R be a profile where 70400voters have the preference $a\ R_i b\ R_i\ c\ R_i d$ 38400 voters have the preference $b\ R_i c\ R_i\ aR_i d$; 35840 voters have the preference $c\ R_i a\ R_i\ b.\ R_i d$ and 3840 voters have the preference $d\ R_i b\ R_i\ c\ R_i a$ Thus, $RO(R) = \{\{a,b\}\}$ and $F_{PR}(R) = \{a\}$. Let R' be a profile where 74240 voters have the preference $a\ R'_i b\ R'_i\ cR'_i d$; 34560 voters have the preference $b\ R'_i c\ R'_i\ aR'_i d$; 35840 voters have the preference $c\ R'_i a\ R'_i\ b\ R'_i d$ and 3840 voters have the preference $d\ R_i'b\ R_i'\ cR_i'a$. Now $RO(R') = \{\{a,c\}\}$ and $F_{PR}(R')$ =c while $R' \in IMP_a(R)$.