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An application of Stochastic Optimal Control Theory to Portfolio Optimization in Fictitious Markets

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An application of Stochastic Optimal Control Theory to Portfolio Optimization in Fictitious Markets

Rastgele Süreçlerde Eniyileme Teorisi'nin Portföy Yönetimine İkincil Piyasalarda Uygulanması

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Abstract

In this thesis, we extend Browne (1999) into a case where borrowing is prohibited. We study optimal investment of a portfolio manager who is evaluated against a benchmark such as S&P 500 by considering three related objectives. The market we studied consists of N risky assets and a risk-free banking account, and the benchmark is modelled as a stochastic process driven by N common and an uncommon factor. To overcome the case with borrowing constraints, we constructed an auxiliary market by employing the approach of Cvitanic and Karatzas (1992). This involves the introduction of fictitious parameters. Then, we show how optimal solutions can be found by using the techniques of stochastic optimal control in a market parametrized with such parameters. In this way, we obtain the optimal investment strategy of an investor seeking to beat an exogenously given benchmark under markets constrained due to borrowing prohibition. Via our solutions, we also show that constrained case has its own exclusive parameters, factors and structure.

Özet

Tez olarak sunulan bu almada, Browne (1999)'da yapılan çalışma borçlanma engelinin olduğu durumu kapsayacak şekilde genişletilmiştir. Çalışmada, S&P 500 gibi bir kıstasa karşı başarısı üzerinden değerlendirilen bir portfolyo yöneticisinin uygulaması gereken etkin yatırım stratejisini konu alınmıştır. Alım satım işlemlerinin yapıldığı piyasa N riskli varlık ve bir risksiz tahvili içeriyor olarak kabul edilmiş; kıstas ise N ortak faktörden etkilenen stokastik bir süreç olarak modellenmiş ve kıstasın ayrıca piyasadan bağımsız bir faktörden etkilenmesine izin verilmiştir. Borçlanma kısıtlarının olduğu durumda çözüm elde edebilmek için Cvitanic and Karatzas (1992)'deki yaklaşımdan esinlenerek imgesel parametreler yardımıyla İkincil Piyasa oluşturulmuştur. Böylece, bahsi geçen parametreler kullanlarak oluşturulan bu piyasada stokastik etkin kontrol yöntemleri ile nasıl çözüm elde edilebilineceği gösterilmiştir. Bu yolla, dışsal faktörler ile yönlendirilen kıstasa karşı başarılı olmayı amaçlayan ve borçlanma engeli olan bir portföy yöneticisinin uygulaması gereken etkin yatırım stratejisi bulunmuştur. Sonuçlarda ayrıca borçlanma kısıtının bulunduğu durumun kendine özgü ve ayrı parametreleri, yatırım biçimi ve onu etkileyen faktörlerinin olduğu tespit edilmiştir.

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Contents

Chapter 1

Introduction

This thesis extends Browne (1999) into a case where borrowing is prohibited. For the extension we consider the method of Cvitanic and Karatzas (1992) and solve the problem by employing the techniques of stochastic optimal control. This method is extensively used in the continuous time portfolio optimization literature. Mainly Merton (1969) introduced this method finding the closed form solution for optimal portfolio selection problem in a continuous time model. Later in Merton (1971), he extends the previous framework by considering a more generalized utility function. In this approach, prices are modeled as stochastic processes and the wealth process is defined to be a controlled stochastic process, which is directed by investment strategies or control vector. Thus, the object is to attain the optimal investment strategy that controls the wealth process which maximizes the expected utility.

A large literature came after Merton (1971). For example, among the notable studies, Davis and Norman (1990) solved the problem with single asset and a riskless banking account under transaction costs by considering HARA type utility function. Zariphopoulou (1992) extended the work by considering two risky assets and Shreve and Soner (1994) studied the problem with transaction costs in bond market. In another paper, the framework is considered by Grossman and Zhou (1993) to find the optimal investment strategies for controlling drawdowns. The approach was even considered in the field of international finance by a relatively recent paper, see Fleming and Stein (2004) that offers a method for debt crisis prediction. Especially, as a more relevant extension to our study here, the approach of Merton (1971) is also considered under the case of investment constraints. Some notable studies for the type of constraint considered in the thesis are Zariphopoulou (1994) and Vila and Zariphopoulou (1997) for trading in a market with a risky and a riskless asset under limited borrowing, and Fleming and Zariphopoulou (1991) for short-selling constraints.

In another vein, Pliska (1986) proposed an alternative method, the so called *martingale approach*, which is based on stochastic calculus and convex analysis. The method is decomposed into two parts; first, the optimal terminal wealth is derived, then the strategy that achieves the optimal terminal wealth is found via the martingale representation technique. This approach as well is considered under various market frictions such as transaction costs (see Cvitanic and Karatzas (1996)) and investment constraints (see Cvitanic and Karatzas (1992)). Mainly, Cvitanic and Karatzas (1992) introduces a duality method based techniques that allow to maximize utility in an incomplete market. Briefly, the technique deals with incompleteness by completing the market via the introduction of fictitious parameters under an auxiliary market. In this way, the fictitious parameters act as Lagrange multipliers and optimal solution may in turn be found upon the specification of them.

By considering the approach of Cvitanic and Karatzas (1992) under the stochastic optimal control framework, we solve Browne (1999) in a market where investors are constrained due to borrowing prohibition. In that study, Browne solves the optimal investment strategies of an investor aiming to beat a stochastic benchmark under three different objectives. These objectives are, in turn, concerned with survival, goal reaching, and reward problems. The portfolio dynamics of the investor, on the other hand, consist of investment in N risky asset and one risk free asset. The dynamics of the benchmark is similar to those of the investor's portfolio, however, it also involves and uncommon exogenously given risky factor. Due to the existence of an uncommon factor the market becomes in a sense incomplete. Yet via straightforward application of the techniques of stochastic optimal control Browne provides analytical solutions of all problems.

In this work, we assume that markets are incomplete not only due to the existence of uncommon factor, but also due to borrowing prohibition. In order to solve the problems, we first construct an auxiliary market as in Cvitanic and Karatzas (1992) in order to relax the constraints, and then solve the problem as if there are no constraints at all. Once the results are found under the auxiliary market, we then proceed to specify the results under the constrained market. The method follows from Yener (2014) that used this technique to solve the three problems under borrowing prohibition in a market consisting of a riskless and multiple risky assets that are modelled as geometric Brownian motion. Mainly, the study simplifies the constant withdrawal rate of Browne (1997) into constant negative proportional net cash flow rate[1] and borrows from Browne (1999) for the

[1]The negative constant proportional net cash flow rate is considered to depict the case of an investor who spends beyond her means.

analysis of the results obtained under the constrained case.

To reach our results, we first provide the background of the stochastic optimal control theory in Chapter 3. There, we first introduce the wealth process, then define the value function, admissibility and maximization principles of the value function along with an example whose result yields the growth optimal portfolio. Second, we introduce the dynamic programming principle that employs Hamilton-Jacobi-Bellman equation in order to solve the stochastic control problem. Finally, we present a verification theorem to show how the optimality of the results can be verified.

In Chapter 4, we present the problem formulation. First, we define the stochastic benchmark and benchmark adjusted wealth process, which is the controlled process in this thesis. Then, we provide details regarding the structure of the auxiliary market and fictitious parameters to deal with the incompleteness due to the constraints. Under the auxiliary market, we redefine the benchmarked wealth process by incorporating the fictitious parameters. Besides, we define a general value function covering the three problems and present the main theorem that provides general solutions for all three problems considered in this thesis.

In Chapter 5, we present the optimal solutions of all problems by using the main theorem introduced in Chapter 4. To this end, we first solve maximizing (minimizing) the expected time the benchmarked wealth process hits a lower (an upper) boundary. Then, we solve the survival probability maximization problem. Finally, we solve the expected discounted reward and penalty problem. As in Browne (1999), all strategies are constant proportional investment strategies [2]

[2] The strategies which are invariant against the changes in time and the state of underlying process.

that are related to the favourability condition of the markets. Differently, we show the effect of the constraints on the favourability condition and show how that effect changes the investment behaviour of the investors when borrowing is prohibited. Finally, we conclude by providing the summary of our results in Chapter 6.

Chapter 2

The Market

In this chapter we construct the market model we will use throughout this thesis. To this end, we first introduce the processes for the financial assets we consider when constructing the market. These are namely a risk-free bank account and N risky stocks[1].

We model the market under a complete filtered probability space denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbb{P})$ and assume infinite time horizon. We use the complete probability space to model the uncertainties surrounding the market. We assume that the market filtration is spanned by a N-dimensional standard Brownian motion $B(t) = (B_1(t), \ldots, B_N(t))^{'}, 0 \leq t < \infty$, which is defined on our complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbb{P})$. Here, the market filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$, formed by by finer partition of the sample space Ω, is used to describe the propagation of information in the market. On the other hand, the sigma algebra $\mathcal{F}_t$, a collection of the subsets of Ω, gives the status of the information at time t. We define $\mathcal{F} := \vee_{t\geq 0}\mathcal{F}_t$. In addition, we use the Brownian motions in order to model the

[1]In the following part of the thesis we also introduce a benchmark process that an investor is supposed to beat.

uncertainty in the market.

More clearly, Ω contains all the states of the economy, while $\mathcal{F}_t$ gives information about the status of the economy. In other words, since the sigma-algebra is a collection of the subsets of Ω, then we have an idea about the possible events in an economy. At time t the information is revealed and we know which event has happened. However, before time t, we may only know about the contents of the sigma-algebra and use probability functions to see the likelihood of events. As the time progresses, more information is revealed (i.e. for $t \leq s$, we obtain $\mathcal{F}_t \subseteq \mathcal{F}_s$) and by taking the collection the sigma-algebras created at each time we can form the filtration. In this way, we model the propagation of the information and use the Brownian motions to materialize the outcome of each random event. That is, once the information is revealed then its impact on the price processes of assets must appear as a number.

Given the model of the uncertainty, we assume that there is an investor who trades in a Black-Scholes type market. That is, in this market:

- There are no arbitrage opportunities;
- There are no transaction costs;
- There is no spread between buy and sell price;
- There are no dividends;
- Trading is done continuously;
- It is possible to borrow and lend at a constant risk-free rate;

- The risky asset price process is modeled as a geometric Brownian motion (GBM, hereafter) with constant drift and volatility.

The price processes of the traded assets are given by

$$\mathrm{d}V_0(t) = rV_0(t)\mathrm{d}t, ; \tag{2.1}$$

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i \mathrm{d}t + \sum_{j=1}^{N} \sigma_{ij}\mathrm{d}B_j(t)\right] \quad \text{for} \quad i = 1, \dots, N, \tag{2.2}$$

where Equation (2.1) is the risk-free money market account process with a constant riskless rate $r \geq 0$ and Equation (2.2) is the risky assets' price processes. We call risky assets stocks and, as mentioned previously, we model them as GBM with constant drift μ_i and volatility σ_{ij}, for $i, j = 1, \dots, N$. Since the price process of the stocks is GBM, then the closed form solution for $S_i(\cdot)$ is log-normal. As a result, for $t < \infty$, $S_i(t)$ never hits zero.

When trading in the market, the investor invests some proportion of her wealth in stocks and the remainder in the money market account. The proportions of wealth invested in the risky assets at time t is denoted by a vector control process $\boldsymbol{w}(t) := (w_1(t), \dots, w_N(t))'$. Mainly, $\boldsymbol{w}(\cdot)$ is called an investment strategy, and when admissible for an initial capital x_o, we use $\boldsymbol{w}(\cdot) \in \mathcal{A}(x_o)$. More clearly, $\mathcal{A}(x_o)$ is the set of admissible strategies. We say that $\boldsymbol{w}(t) \in \mathcal{A}(x_o)$ if $\boldsymbol{w}(t)$ is $\{\mathcal{F}_t\}$-progressively measurable, satisfies $\int_0^t \|\boldsymbol{w}(s)\|^2 ds < \infty$ almost surely[2] for $t < \infty$.

Note that the more general form of the investment strategy is of the feedback form $\boldsymbol{w}(t, X^{\boldsymbol{w}}(t))$. That is, it is a function of time and level of wealth process at time t. Therefore, we have $\boldsymbol{w} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^N$. That is, the control process $\boldsymbol{w}(t, X^{\boldsymbol{w}}(t))$

[2] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If a set $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, we say that the event A occurs almost surely (See **?**).

is a path dependent function which is renewed depending on history of the wealth process and its time t value is determined based on the value of the wealth level at time t. In the sequel, we will use $\boldsymbol{w}(t)$ and $\boldsymbol{w}(t, X^{\boldsymbol{w}}(t))$ interchangeably when necessary. Given the aforementioned form we then proceed to give the definition of the progressively measurable processes.

Definition 2.1. (See Protter (2004)). A progressively measurable process is a process $\boldsymbol{w} := \{\boldsymbol{w}(t), t \geq 0\}$ on $\mathbb{R}_+ \times \Omega$ such that for each $t \in \mathbb{R}_+$ the mapping $(s, \omega) \to \boldsymbol{w}(s, \omega)$ of $[0, t] \times \Omega$ into $\mathbb{R}$ is measurable with respect to $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

In the above definition $\mathcal{B}([0, t])$ represents the Borel sigma-algebra of subsets of $[0, t]$. This is the sigma-algebra obtained by beginning with closed intervals and adding everything else necessary (See Karatzas and Shreve (1998) for further details). More clearly, by using the Borel sigma-algebra we set the time interval and by taking its tensor product with $\mathcal{F}_t$, we define a random variable over the predetermined time interval. Therefore, by making $\boldsymbol{w}(t)$ $\{\mathcal{F}_t\}$-progressively measurable, we not only know the evolution of the investment process within the time interval $[0, t]$, but also establish the connection of the investment decision of an investor with the random occurrences in an economy. Second, if the investment strategy is square integrable, that is $\int_0^t \|\boldsymbol{w}(s)\|^2 ds < \infty$, almost surely for $t < \infty$ then the investment strategy, as mentioned above, becomes admissible. In this way, the *self-financing* wealth process associated to an admissible strategy is the solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}X^{\boldsymbol{w}}(t) &= X^{\boldsymbol{w}}(t)\left(\sum_{i=1}^{N}\boldsymbol{w}_i(t)\frac{\mathrm{d}S_i(t)}{S_i(t)}\right) + X^{\boldsymbol{w}}(t)\left(\sum_{i=1}^{N}(1-\boldsymbol{w}_i(t))\frac{\mathrm{d}V_0(t)}{V_0(t)}\right) \\ &= X^{\boldsymbol{w}}(t)\left(r+\sum_{i=1}^{N}\boldsymbol{w}_i(t)(\mu_i - r)\right)\mathrm{d}t + X^{\boldsymbol{w}}(t)\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{w}_i(t)\sigma_{ij}\mathrm{d}B_j(t). \end{aligned} \tag{2.3}$$

By self-financing, we mean that there is no external infusion or withdrawal of cash from the portfolio. We can then write the above portfolio dynamics in matrix form as

$$\mathrm{d}X^{\boldsymbol{w}}(t) = X^{\boldsymbol{w}}\left[r\mathrm{d}t + \boldsymbol{w}^{'}(t)(\mu - r\underset{\sim}{1})\mathrm{d}t + \boldsymbol{w}^{'}(t)\sigma\mathrm{d}B(t)\right], \tag{2.4}$$

where $\mu = (\mu_1, \ldots, \mu_N)^{'}$, $\underset{\sim}{1} = (1, \ldots, 1)^{'}$, and $\sigma = (\sigma_1, \ldots, \sigma_N)^{'}$ and $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{iN})$ for $i = 1, \ldots, N$. Notice from the equation (2.4) that the closed form solution always gives $X^{\boldsymbol{w}}(t) > 0$ for $t < \infty$. Therefore, under proportional investment strategies the portfolio process never hits zero in finite time.

In sum, we created a model of uncertainty to take into account the random occurrences in an economy. We then provided the asset price processes for the money market account and the stocks. We assume that an investor trading in a Black-Scholes type market invests in proportional amounts both in the money market account and the stocks. In this way, she forms a portfolio whose dynamics are given by the Equation (2.4). This equation will have a strong and unique solution if the investment strategy chosen by the investor takes into account the random occurrences in an economy and its accumulated value within a finite time horizon is finite. In other words, given the scarcity of the resources, an investor makes allocation based on the available information. In this way, the investment strategy becomes admissible.

Next, we proceed to the chapter where we introduce the technique that will be used when solving the problems considered in this thesis.

Chapter 3

Stochastic Optimal Control

The technique of stochastic optimal control, a subfield of control theory, helps its users to solve problems related to the maximization or minimization of some value function subject to a random underlying and possibly constraints. Metaphorically, use of this technique resembles steering a device in an uncertain environment to reach a predetermined goal in the space-time. The field is fathered by Bellman (1954) and has wide field of application. For its application, we refer the reader to other notable studies which include the work of Fleming and Rishel (1975) on the deterministic case, Fleming and Soner (2006) on the viscosity solutions of the controlled Markov processes and Merton (1971) on the application in the continuous time portfolio optimization literature. For the outline of the technique, we refer to Björk (2004) and Saß (2006) in addition to the aforementioned references.

The stochastic optimal control problems can be grouped in two cases; discrete and continuous time. The types of problems considered in this thesis are studied under continuous time. In this chapter we will briefly introduce the background of the technique and provide the definitions of the terms that are often used in this study.

Our aim is to find a unique control process that steers a stochastic process and satisfies the criteria defined (i.e. maximization or minimization) on an objective function. From now on we will refer $X^{\boldsymbol{w}}(\cdot)$ as the controlled stochastic process by the control process $\boldsymbol{w}(\cdot)$. This is because the portfolio process is controlled under the investment strategy that defines the control law. Note that during the exposition, we consider the stochastic dynamics that are relevant to our study. That is, the underlying stochastic process is the one defined in Equation (2.4). Given the underlying random process, we then define the value function $J(t, x, \boldsymbol{w})$ by

$$J(t, x, \boldsymbol{w}) = \mathbb{E}_{t,x}\left[\int_t^T F(s, X(s), \boldsymbol{w}(s))\mathrm{d}s + \Psi(T, X^{\boldsymbol{w}}(T))\right], \tag{3.1}$$

where $\mathbb{E}_{t,x}[\cdot\,] = \mathbb{E}[\cdot \mid X^{\boldsymbol{w}}(t) = x]$, F is an infinitesimal utility (or cost) function that depends on time, the level of controlled process and the control strategy. Ψ, on the other hand, is a legacy function that measures the terminal utility whose value is determined based on the level of $X^{\boldsymbol{w}}(T)$. More generally, Ψ is the expected terminal utility at time $T < \infty$.

Given the value function in (3.1), the aim is then to define the objective function. Depending on the aim of the controller the aim may either be maximizing (3.1) or minimizing it. More formally, if the problem involves maximizing the value function, we have

$$V(t, x) = \sup_{\boldsymbol{w}\in\tilde{\mathcal{A}}} J(t, x, \boldsymbol{w}), \tag{3.2}$$

subject to the process given in Equation (2.4) and $\tilde{\mathcal{A}}$ is given by

$$\tilde{\mathcal{A}} := \left\{ \boldsymbol{w} \in \mathcal{A} \,\middle|\, |J(t, x, \boldsymbol{w})| < \infty \right\}. \tag{3.3}$$

In sum, our aim is to maximize the expected utility function by steering the stochastic process $\{X^{\boldsymbol{w}}(t), t > 0\}$ and find the optimal control strategy $\boldsymbol{w}^*$ that allows us to maximize the expected utility function. Therefore we define the optimal value function and optimal control strategy as follows

$$V(t, x) = \sup_{\boldsymbol{w} \in \tilde{\mathcal{A}}} J(t, x, \boldsymbol{w}) \quad \text{and} \quad \boldsymbol{w}^* := \arg\sup_{\boldsymbol{w} \in \tilde{\mathcal{A}}} J(t, x, \boldsymbol{w}), \tag{3.4}$$

where V and J are

$$J : \ \mathbb{R}_+ \times \mathbb{R}_+ \times \tilde{\mathcal{A}} \to \mathbb{R}; \qquad V : \ \mathbb{R}_+ \times \mathbb{R}_+ \times \to \mathbb{R}.$$

Hence V can be written as $V(t, x) = J(t, x, \boldsymbol{w}^*)$. This is so because the supremum is attained by the unique control strategy. The existence of this strategy depends in turn on the form of the objective function. Particularly, there exists an optimal control strategy if the objective function is concave increasing or convex decreasing in a given domain.

3.1 An Example: Growth Optimal Portfolio

An influential study, Merton (1969) studied the stochastic optimal control for the problems due to portfolio optimization. In this section we present his findings to demonstrate a first example and to define a concept we use throughout this chapter.

To attain the optimal control strategy that allows us to calculate the growth optimal portfolio, we first present the solution to Equation (2.4) as

$$X^{\boldsymbol{w}}(T) = x_o \exp\left\{\int_0^T \left(r + \boldsymbol{w}'(t)(\mu - r\underset{\sim}{1}) - \frac{1}{2}\boldsymbol{w}'(t)\Sigma\boldsymbol{w}(t)\right)\mathrm{d}t + \int_0^T \boldsymbol{w}'(t)\sigma\mathrm{d}B(t)\right\}, \tag{3.5}$$

where $\Sigma = \sigma\sigma'$.

Next, we set infinitesimal utility function $F \equiv 0$ and $\Psi(T, x) = \log(x)$. Then, the value function becomes at time t

$$J(t, x, \boldsymbol{w}) = \mathbb{E}_{t,x}\left[\log(X^{\boldsymbol{w}}(T))\right].$$

We seek to maximize the above function overall control strategies in $\tilde{\mathcal{A}}$ defined by

$$\tilde{\mathcal{A}} := \left\{\boldsymbol{w} \in \mathcal{A} \,\middle|\, \mathbb{E}\left[\log(X^{\boldsymbol{w}}(T))^-\right] < \infty\right\}, \tag{3.6}$$

where $x^- = \max(0, -x)$. Note that if take the logarithm of the Equation (3.5) we obtain

$$\begin{aligned}\log(X^{\boldsymbol{w}}(T)) = \log(x_o) &+ \int_0^T \left(r + \boldsymbol{w}'(t)(\mu - r\underset{\sim}{1}) - \frac{1}{2}\boldsymbol{w}'(t)\Sigma\boldsymbol{w}(t)\right)\mathrm{d}t \\ &+ \int_0^T \boldsymbol{w}'(t)\sigma\mathrm{d}B(t).\end{aligned} \tag{3.7}$$

From the definition of admissibility in Equation (3.6), the stochastic integral is a martingale since for $t \leq T$

$$\mathbb{E}_{t,x}\left[\int_t^T \boldsymbol{w}^{'}(t)\sigma \mathrm{d}B(t)\right] = 0.$$

Then, we set $t = 0$ and the value function becomes for $\boldsymbol{w} \in \tilde{\mathcal{A}}$

$$J(0, x_o, \boldsymbol{w}) = \log(x_o) + \mathbb{E}\left[\int_0^T \left(r + \boldsymbol{w}^{'}(t)(\mu - r\underset{\sim}{1}) - \frac{1}{2}\boldsymbol{w}^{'}(t)\Sigma\boldsymbol{w}(t)\right)\mathrm{d}t\right]. \tag{3.8}$$

Our goal is to find the strategy that maximizes the terminal logarithmic utility. To achieve it, we take the derivatives of the integrand for the first and second order. Thus, we get

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{w}}\left(r + \boldsymbol{w}^{'}(\mu - r\underset{\sim}{1}) - \frac{1}{2}\boldsymbol{w}^{'}\Sigma\boldsymbol{w}\right) &= \mu - r\underset{\sim}{1} - \Sigma\boldsymbol{w};\\ \frac{\partial^2}{\partial \boldsymbol{w}^2}(r\mathrm{d}t + \boldsymbol{w}^{'}(\mu - r\underset{\sim}{1}) - \frac{1}{2}\boldsymbol{w}^{'}\Sigma\boldsymbol{w}) &= -\Sigma < 0.\end{aligned}$$

where the second line shows that the growth rate of the portfolio process is concave in $\boldsymbol{w}$. Then, a pointwise optimization of the first order condition by setting the first line equal to zero gives

$$\boldsymbol{w}^* := \Sigma^{-1}(\mu - r\underset{\sim}{1}). \tag{3.9}$$

Here $\boldsymbol{w}^*$ is known as *Merton strategy.* Substituting it in the value function gives

$$V(0, x_o) = J(0, x_o, \boldsymbol{w}^*) = \log(x_o) + \left(r + \frac{1}{2}(\mu - r\underset{\sim}{1})^{'}\Sigma^{-1}(\mu - r\underset{\sim}{1})\right)T. \tag{3.10}$$

3.2 Hamilton-Jacobi-Bellman Equation

We start by introducing the following definition that will be referred to in the main theorem of this section.

Definition 3.1. For every open set $\mathcal{O} \subset \mathbb{R}$ and for every function of the form $f(t,x) \in C^{1,2}([0,\infty) \times \mathcal{O})$ the second-order differential operator function $\mathcal{L}^{\boldsymbol{w}} f : [0,\infty) \times \mathcal{O} \to \mathbb{R}$ is given by for $\boldsymbol{w} \in \mathbb{R}^N$

$$\mathcal{L}^{\boldsymbol{w}} f(t,x) = f_t(t,x) + \left(r + \boldsymbol{w}^{'}(\mu - r\underset{\sim}{1})\right) x f_x(t,x) + \frac{1}{2}\|\sigma^{'}\boldsymbol{w}\|^2 x^2 f_{xx}(t,x). \quad (3.11)$$

where f_t is the first derivative with respect to time and f_x is the first derivative while f_{xx} is the second derivative with respect to x. Note that the arguments of the functions will be hidden when necessary to simplify the notation.

Remark 3.2.1. *By Itô's formula, we can write for* $\boldsymbol{w} \in \mathbb{R}^N$

$$\begin{aligned} & \mathrm{d}f(t, X^{\boldsymbol{w}}(t)) \\ = \; & f_t(t, X^{\boldsymbol{w}}(t))\mathrm{d}t + \left[\left(r + \boldsymbol{w}^{'}(\mu - r\underset{\sim}{1})\right)\right] X^{\boldsymbol{w}}(t) f_x(t, X^{\boldsymbol{w}}(t))\mathrm{d}t \\ & + \frac{1}{2}\|\sigma^{'}\boldsymbol{w}\|^2 X^{\boldsymbol{w}}(t)^2 f_{xx}(t, X^{\boldsymbol{w}}(t))\mathrm{d}t + \boldsymbol{w}^{'} X^{\boldsymbol{w}}(t) f_x(t, X^{\boldsymbol{w}}(t))\sigma \mathrm{d}B(t). \end{aligned}$$

Then, by using $\mathcal{L}^{\boldsymbol{w}}$ *we write the above as*

$$\mathrm{d}f(t, X^{\boldsymbol{w}}(t)) = \mathcal{L}^{\boldsymbol{w}} f(t, X^{\boldsymbol{w}}(t))\mathrm{d}t + \boldsymbol{w}^{'} X^{\boldsymbol{w}}(t) f_x(t, X^{\boldsymbol{w}}(t))\sigma \mathrm{d}B(t).$$

Solution to the stochastic optimal control problem in continuous case is provided by Bellman (1954) by extending the earlier work known as Hamilton-Jacobi equations. The following theorem presents his findings known as Hamilton-Jacobi-Bellman (HJB, hereafter) equation.

Theorem 3.1 (HJB Equation). *The optimal value function V is the solution of the PDE below.*

$$\sup_{\boldsymbol{w}}\{F(t,x,\boldsymbol{w}) + \mathcal{L}^{\boldsymbol{w}}V(t,x)\} = 0, \tag{3.12}$$

with the boundary condition

$$V(T,x) = \Psi(T,x), \quad \forall x \in \mathbb{R}^N, \tag{3.13}$$

and $\forall(t,x) \in [0,T] \times \mathbb{R}^N$ there is a unique $\boldsymbol{w}^*(t,x)$ for which the supremum of the above function is attained.

Assumption 3.1. The above theorem requires certain assumptions that are listed below.

1. There exists an optimal control strategy, $\boldsymbol{w}^*$;
2. The optimal value function is smooth enough, i.e. $V \in C^{1,2}$.

3.2.1 Proof of Hamilton-Jacobi-Bellman Equation

For the proof we follow two strategies outlined here below (See Björk (2004)):

1. We compute the value function with optimal control law $\boldsymbol{w}^*$ for the time interval $[t,T]$, hence we get $V(t,x) = J(t,x,\boldsymbol{w}^*)$.

2. We use an arbitrary strategy and then switch to the optimal control law. For this we divide the time interval $[t, T]$ into $[t, t')$ and $[t', T]$ provided that $t < t' \leq T$. Our new control law acts the same with the previous one in the latter time interval but different in the first one. That is, we write

$$\hat{\boldsymbol{w}}(s, x) = \begin{cases} \boldsymbol{w}(s, x) & \text{if } (s, x) \in [t, t'] \times \mathbb{R}_+; \\ \boldsymbol{w}^*(s, x) & \text{if } (s, x) \in (t', T] \times \mathbb{R}_+. \end{cases} \tag{3.14}$$

Note that the optimal control law is at least as good as the new control law. Therefore, by using this fact and letting $t' \downarrow t$ we obtain the HJB partial differential equation.

The **first step** is obvious since choosing the optimal control strategy for interval $[t, T]$ results in the optimal value function

$$V(t, x) = J(t, x, \boldsymbol{w}^*). \tag{3.15}$$

For the **second step**, we select the control law $\hat{\boldsymbol{w}}$ for $[t, t')$. Then, the conditional expected utility for $[t, t')$ given (t, x) is

$$\mathbb{E}_{t,x}\left[\int_t^{t'} F(s, X^{\hat{\boldsymbol{w}}}(s), \hat{\boldsymbol{w}}(s))\mathrm{d}s\right]. \tag{3.16}$$

For the interval $[t', T]$, we obtain $V(t', X^{\hat{\boldsymbol{w}}}(t')) = \sup_{\boldsymbol{w}} J(t', x, \boldsymbol{w})$. But we are interested in its value at time t instead of t'. Thus, the conditional expected utility given (t, x) is

$$\mathbb{E}_{t,x}[V(t', X^{\hat{\boldsymbol{w}}}(t'))]. \tag{3.17}$$

By combining equations (3.16) and (3.17), the total expected utility for interval $[t, T]$ is

$$\mathbb{E}_{t,x}\left[\int_t^{t'} F(s, X^{\hat{\boldsymbol{w}}}(s), \hat{\boldsymbol{w}}(s))\mathrm{d}s + V(t', X^{\hat{\boldsymbol{w}}}(t'))\right]. \tag{3.18}$$

From the first step, since $V(t,x) = J(t,x,\boldsymbol{w}^*)$, we obtain the inequality

$$V(t,x) \geq \mathbb{E}_{t,x}\left[\int_t^{t'} F(s, X^{\hat{\boldsymbol{w}}}(s), \hat{\boldsymbol{w}}(s))\mathrm{d}s + V(t', X^{\hat{\boldsymbol{w}}}(t'))\right]. \tag{3.19}$$

Next, we apply Itô's formula to $V(t', X^{\boldsymbol{w}}(t'))$, and write

$$\begin{aligned}
\mathrm{d}V(t', X^{\hat{\boldsymbol{w}}}(t')) &= V_t(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}t + \left[\left(r + \hat{\boldsymbol{w}}'(t')(\mu - r\underset{\sim}{1})\right)\right] X^{\hat{\boldsymbol{w}}}(t')V_x(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}t \\
&\quad + \frac{1}{2}\|\sigma' \hat{\boldsymbol{w}}(t')\|^2 X^{\hat{\boldsymbol{w}}}(t')^2 V_{xx}(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}t \\
&\quad + \hat{\boldsymbol{w}}'(t')\sigma X^{\hat{\boldsymbol{w}}}(t')V_x(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}B(t) \\
&= \mathcal{L}^{\hat{\boldsymbol{w}}}V(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}t + \hat{\boldsymbol{w}}'(t')\sigma X^{\hat{\boldsymbol{w}}}(t')V_x(t', X^{\hat{\boldsymbol{w}}}(t'))\mathrm{d}B(t).
\end{aligned} \tag{3.20}$$

The solution of (3.20) is

$$\begin{aligned}
V(t', X^{\hat{\boldsymbol{w}}}(t')) = V(t,x) &+ \int_t^{t'} \mathcal{L}^{\hat{\boldsymbol{w}}}V(s, X^{\hat{\boldsymbol{w}}}(s))\mathrm{d}s \\
&+ \int_t^{t'} \hat{\boldsymbol{w}}'(s)\sigma X^{\hat{\boldsymbol{w}}}(s)V_x(s, X^{\hat{\boldsymbol{w}}}(s))\mathrm{d}B(s).
\end{aligned} \tag{3.21}$$

After substituting (3.21) into (3.19), $V(t, X^{\hat{\boldsymbol{w}}})$ is cancelled out and we obtain

$$0 \geq \mathbb{E}_{t,x}\left[\int_t^{t'} \left(F(s, X^{\hat{\boldsymbol{w}}}(s), \hat{\boldsymbol{w}}(s)) + \mathcal{L}^{\hat{\boldsymbol{w}}} V(s, X^{\hat{\boldsymbol{w}}}(s)) \right) \mathrm{d}s + \int_t^{t'} \hat{\boldsymbol{w}}^{'}(s) \sigma X^{\hat{\boldsymbol{w}}}(s) V_x(s, X^{\hat{\boldsymbol{w}}}(s)) \mathrm{d}B(s) \right]. \tag{3.22}$$

If the stochastic part of the above equation is a martingale, we have

$$\mathbb{E}_{t,x}\left[\int_t^{t'} \hat{\boldsymbol{w}}^{'}(s) X^{\hat{\boldsymbol{w}}}(s) \sigma V_x(s, X^{\hat{\boldsymbol{w}}}(s)) \mathrm{d}B(s) \right] = 0,$$

and thus the following inequality follows

$$\mathbb{E}_{t,x}\left[\int_t^{t'} \left(F(s, X^{\hat{\boldsymbol{w}}}(s), \hat{\boldsymbol{w}}(s)) + \mathcal{L}^{\hat{\boldsymbol{w}}} V(s, X^{\hat{\boldsymbol{w}}}(s)) \right) \mathrm{d}s \right] \leq 0. \tag{3.23}$$

From the above we have the inequality

$$F(t, x, \hat{\boldsymbol{w}}) + \mathcal{L}^{\hat{\boldsymbol{w}}} V(t, x) \leq 0 \tag{3.24}$$

$\mathbb{P}$-almost surely, and holds with equality if and only if $\hat{\boldsymbol{w}}(t, x) = \boldsymbol{w}^*(t, x)$. Then, HJB partial differential equation follows:

$$\sup_{\boldsymbol{w}} \{ F(t, x, \boldsymbol{w}) + \mathcal{L}^{\boldsymbol{w}} V(t, x) \} = 0 \tag{3.25}$$

Given the HJB equation above the next question to tackle is to see when the solution of the HJB equation is the value function of the control problem. To this end, we proceed to give the method for solving the HJB equation. First, we write equation (3.25) as

$$\sup_{\boldsymbol{w}} \Big\{ F(t,x,\boldsymbol{w}) + V_t(t,x) + \Big[\Big(r + \boldsymbol{w}^{'}(\mu - r\underset{\sim}{1}) \Big) \Big] x V_x(t,x) \\ + \frac{1}{2} \|\sigma^{'} \boldsymbol{w}\|^2 x^2 V_{xx}(t,x) \Big\} = 0 \tag{3.26}$$

From the above we solve the maximizing control law $\boldsymbol{w} = \boldsymbol{w}^*(t,x)$. From the specification in (3.26), the control law will depend on t, x, V and various partial derivatives of V. By substituting the maximizing control law in (3.26), we obtain the partial differential equation that V solves. Next, we use the boundary condition $V(T,x) = \Psi(T,x)$ to obtain a candidate solution to the partial differential equation.

Once the maximizing control law and a candidate solution are obtained, we proceed to verify whether they are optimal. To this end, we provide a verification theorem in the next section.

3.3 The Verification Theorem

The theorem proclaims that the HJB equation is necessary and sufficient condition for optimality. That is, the optimality of the maximizing control law and the candidate value function is verified via this theorem.

Theorem 3.2 (Verification). *Suppose that* $|H(t, X^{\boldsymbol{w}}(t))| \leq \mathcal{C} \int_0^t (1 + (X^{\boldsymbol{w}}(s))^p)\mathrm{d}s$ *for constants* $\mathcal{C} > 0$, $p \geq 2$, *and that* $H \in C^{1,2}$ *with* $H^{'} > 0$, $H^{''} < 0$ *satisfies the*

HJB equation

$$\sup_{\boldsymbol{w}} \{F(t,x,\boldsymbol{w}) + \mathcal{L}^{\boldsymbol{w}} H(t,x)\} = 0 \tag{3.27}$$

with the boundary condition $H(T,x) = \Psi(T,x)$*, and suppose for each* (t,x) *the supremum in the expression (3.27) is attained by the choice* $\boldsymbol{w} = g(t,x)$*. Then,*

(i) The optimal value function to the control problem is given by $H(t,X(t)) = V(t,X(t))$*, and*

(ii) the optimal control strategy is given by $\boldsymbol{w}^*(t,x) = g(t,x)$*.*

3.3.1 Proof of The Verification Theorem

In order to prove the theorem, we first introduce the function

$$M(t,X^{\boldsymbol{w}}(t)) = \int_0^t F(s,X^{\boldsymbol{w}}(s),\boldsymbol{w}(s))\mathrm{d}s + H(t,X^{\boldsymbol{w}}(t)) \tag{3.28}$$

For the proof we will refer to a localization argument. To this end, we fix $(t,x) \in [0,T] \times \mathbb{R}_+$ and define the stopping time

$$\tau_n = T \wedge \inf\{u > t \mid |X^{\boldsymbol{w}}(u) - X^{\boldsymbol{w}}(t)| \geq n\}, \quad n \in \mathbb{N}.$$

Then, for an admissible strategy $\boldsymbol{w}(\cdot) \in \tilde{\mathcal{A}}(x)$ and for $0 \leq t \leq \tau_n^w$, we write by

Itô's formula

$$M(\tau_n, X^{\boldsymbol{w}}(\tau_n)) = M(t, X^{\boldsymbol{w}}(t)) + \int_t^{\tau_n} \left(F(s, X^{\boldsymbol{w}}(s), \boldsymbol{w}(s)) + \mathcal{L}^{\boldsymbol{w}} H(s, X^{\boldsymbol{w}}(s))\right) \mathrm{d}s + \int_t^{\tau_n} \boldsymbol{w}^{'}(s)\sigma X^{\boldsymbol{w}}(s) H_x(s, X^{\boldsymbol{w}}(s)) \mathrm{d}B(s) \tag{3.29}$$

In the above specification the expectation of the stochastic integral is zero, because from the continuity of H, the admissibility of $\boldsymbol{w}$, and the boundedness of $X^{\boldsymbol{w}}$ in $[t, \tau_n]$ we have

$$\mathbb{E}_{t,x}\left[\int_t^{\tau_n} \|\boldsymbol{w}^{'}(s)\sigma X^{\boldsymbol{w}}(s) H_x(s, X^{\boldsymbol{w}}(s))\|^2 \mathrm{d}s\right] < \infty. \tag{3.30}$$

Therefore, the conditional expectation of the stochastic integral is equal to 0. Furthermore, for every $\boldsymbol{w} \in \mathbb{R}^N$ and for each s, the inequality

$$F(s, X^{\boldsymbol{w}}(s), \boldsymbol{w}(s)) + \mathcal{L}^{\boldsymbol{w}} H(s, X^{\boldsymbol{w}}(s))) \mathrm{d}s \leq 0 \tag{3.31}$$

holds $\mathbb{P}$-almost surely. Then, it follows from (3.29)-(3.31) that

$$M(t, X^{\boldsymbol{w}}(t)) \geq \mathbb{E}_{t,x}[M(\tau_n, X^{\boldsymbol{w}}(\tau_n))] \tag{3.32}$$

On the other hand, by using (3.28) we obtain

$$H(t, X^{\boldsymbol{w}}(t)) = M(t, X^{\boldsymbol{w}}(t)) - \int_0^t F(s, X^{\boldsymbol{w}}(s), \boldsymbol{w}(s)) \mathrm{d}s$$

$$M(\tau_n, X^{\boldsymbol{w}}(\tau_n)) = \int_0^{\tau_n} F(s, X^{\boldsymbol{w}}(s), \boldsymbol{w}(s)) \mathrm{d}s + H(\tau_n, X^{\boldsymbol{w}}(\tau_n))) \tag{3.33}$$

By substituting (3.33) into (3.32) we get

$$H(t, X^{\boldsymbol{w}}(t)) \geq \mathbb{E}_{t,x}\left[\int_t^{\tau_n} F(s, X^{\boldsymbol{w}}(s)\boldsymbol{w}(s))\mathrm{d}s + H(\tau_n, X^{\boldsymbol{w}}(\tau_n))\right]. \tag{3.34}$$

Note that as $n \to \infty$, $\tau_n \to T$. Furthermore, since $|H(t, X^{\boldsymbol{w}}(t))| \leq \mathcal{C}\int_0^t (1 + (X^{\boldsymbol{w}}(s))^p)\mathrm{d}s$ for constants $\mathcal{C} > 0,\ p \geq 2$, then it follows from dominated convergence theorem that

$$\mathbb{E}_{t,x}\left[\int_t^{\tau_n} F(s, X^{\boldsymbol{w}}(s)\boldsymbol{w}(s))\mathrm{d}s + H(\tau_n, X^{\boldsymbol{w}}(\tau_n))\right] \to J(t, x, \boldsymbol{w}) \quad \text{as} \quad n \to \infty$$

Then, we have from above and (3.34)

$$H(t, X^{\boldsymbol{w}}(t)) \geq J(t, X^{\boldsymbol{w}}(t), \boldsymbol{w})$$

By taking the supremum of the right-hand side over all $\boldsymbol{w} \in \tilde{\mathcal{A}}$

$$H(t, X^{\boldsymbol{w}^*}(t)) \geq V(t, X^{\boldsymbol{w}^*}(t))$$

On the other hand, by selecting the maximizing control law $\boldsymbol{w} = g(t, X^{\boldsymbol{w}}(t))$, the inequality in (3.31) becomes equality giving us

$$H(t, X^{\boldsymbol{g}}(t)) = J(t, X^{\boldsymbol{g}}(t), \boldsymbol{g}) \leq V(t, X^{\boldsymbol{g}}(t))$$

Besides we know that

$$J(t, X^{\boldsymbol{g}}(t), \boldsymbol{g}) \leq V(t, X^{\boldsymbol{w}^*}(t)) \quad \text{and} \quad J(t, X^{\boldsymbol{w}^*}(t), \boldsymbol{w}^*) = V(t, X^{\boldsymbol{w}^*}(t))$$

which concludes in $g(t,x) = \boldsymbol{w}^*$. It follows that $H(t,x) = V(t,x)$ and therefore, $H(t,x)$ is the optimal value function, $\boldsymbol{w}^*(X^{\boldsymbol{w}}(t))$ is the optimal control process and $X^{\boldsymbol{w}^*}(t)$ is the optimal control process.

Chapter 4

Problem Formulation

In this chapter, we define the problems we solve in the thesis, introduce the benchmarked wealth process, and write the main theorem for the generalized version of the problem. We consider a portfolio manager whose performance is measured relative to a benchmark such as an index, inflation or exchange rate. Her aim is to outperform the benchmark via the objectives defined here below:

P1. Minimizing the expected time of beating the benchmark or maximizing the expected time until ruin;

P2. Maximizing the probability of reaching a predetermined higher wealth level before incurring a shortfall;

P3. Maximizing or minimizing the expected discounted reward.

Given the above problems, we then proceed to introduce an exogenously given benchmark process and the benchmarked wealth process.

4.1 The Benchmarked Wealth Process

For measuring the performance of portfolio we define an exogenously given stochastic benchmark process $Y(\cdot)$, which is the solution to the stochastic differential equation

$$\mathrm{d}Y(t) = Y(t)[\alpha \mathrm{d}t + b^{'}\mathrm{d}B(t) + \beta \mathrm{d}B^{N+1}(t)], \tag{4.1}$$

where α and β are constants and b denotes a constant column vector; $b = (b_1, b_2, \ldots, b_N)^{'}$. Furthermore, $B^{N+1}(\cdot)$ is the additional standard Brownian motion.

The specification in (4.1) is the benchmark process studied by Browne (1999). We observe that the benchmark process is partially correlated with the wealth process $X^{\boldsymbol{w}}(\cdot)$ for $\beta \neq 0$. This inequality allows for further generalization and makes the benchmark process be interpreted in various forms such as inflation or exchange rate,the price process of a non-traded asset and such. However, when $\beta = 0$, then the benchmark process can be interpreted as a benchmark portfolio process (For the case when $\beta = 0$ and the maturity is $T < \infty$ we refer the reader to Browne (1997)). Note that, with additional Brownian motion the market becomes incomplete as there are more risk factors than the number of liquidly traded financial assets.

Next, we call for a new controlled stochastic process $Z^{\boldsymbol{w}}(t) := X^{\boldsymbol{w}}(t)/Y(t)$. We observe that $Z^{\boldsymbol{w}}(\cdot)$ is ratio process which is equivalent to the traded portfolio expressed in units of the benchmark portfolio. In other words, it is the benchmarked portfolio process. Via straightforward application of the Itô's formula, we obtain

$$\mathrm{d}Z^{\boldsymbol{w}}(t) = Z^{\boldsymbol{w}}(t)\left[\left(\hat{r} + \boldsymbol{w}'(t)(\hat{\mu} - r\underset{\sim}{1})\right)\mathrm{d}t + (\boldsymbol{w}'(t)\sigma - b')\mathrm{d}B(t) - \beta\mathrm{d}B^{(N+1)}(t)\right], \tag{4.2}$$

where $\hat{r} = r + b'b - \alpha + \beta^2$ and $\hat{\mu} = \mu - \sigma b$.

4.2 The Auxiliary Market

The market defined in the first chapter is complete as it involves N risky assets and one risk free asset. We already mentioned that the market becomes incomplete with the additional Brownian motion in the benchmark process. Furthermore, under the investment constraints the market becomes incomplete as well. In this section, to address the problem, we construct an auxiliary market endowed with fictitious assets under the context of Cvitanic and Karatzas (1992) (see also Karatzas and Shreve (1998)). The substance of the auxiliary market is to provide a mathematical ground that allows us to trade freely as if there are no constraints.

Auxiliary market is an augmentation of the unconstrained market by the use of fictitious parameters, specifically dual processes so-called Lagrange multipliers. To provide a background, we first define a closed convex set $\mathcal{K} \neq \emptyset$ of $\mathbb{R}^N$. This is the constraint set that contains proportional investment strategies in N risky assets. For example, under borrowing prohibition $\mathcal{K}$ is given by

$$\mathcal{K} := \{\boldsymbol{w} \in \mathbb{R}^N \mid \sum_{i=1}^{N} w_i \leq 1\}.$$

Next, for a given $\mathcal{K}$, we define the *support function* of the convex set $-\mathcal{K}$ by

$$\delta(\nu) := \sup_{\boldsymbol{w}\in\mathcal{K}}(-\boldsymbol{w}'\nu), \quad \nu \in \mathbb{R}^N.$$

The support function is finite on its effective domain defined by

$$\tilde{\mathcal{K}} := \{\nu \in \mathbb{R}^N \mid \delta(\nu) < \infty\}.$$

Here, $\tilde{\mathcal{K}}$ is the barrier cone of $-\mathcal{K}$ and we assume that $\tilde{\mathcal{K}}$ contains the origin on $\mathbb{R}^N$. This assumption arises from the fact that $\nu = 0$ when the constraints are not binding. Then, from the specification of the support function it follows that $\delta(0) = 0$. In fact, $\delta(\nu) \geq 0 \ \forall \nu \in \mathbb{R}^N$ and $\delta(\nu) + \boldsymbol{w}'\nu \geq 0$, $\forall \nu \in \tilde{\mathcal{K}}$ if and only if $\boldsymbol{w} \in \mathcal{K}$. Then, when borrowing is prohibited $\delta(\nu) = -\nu_1$ on $\tilde{\mathcal{K}}$ for some scalar $\nu_1 \leq 0$, and the corresponding barrier cone is

$$\tilde{\mathcal{K}} = \{\nu \in \mathbb{R}^N \mid \nu_1 = \ldots = \nu_N \leq 0\}.$$

Furthermore, we let $\nu := \{\nu(t) \mid 0 \leq t < \infty\}$ be the vector of $\{\mathcal{F}_t\}$-progressively measurable Markovian fictitious processes in the space $\mathcal{D}$ of fictitious processes. Here, $\mathcal{D}$ is the space of fictitious processes taking values in $\tilde{\mathcal{K}}$.

Via the use of fictitious processes, we construct an auxiliary market by relaxing the constraints and allowing investment to be done as if there are no constraints at all. Once the optimal investment strategy in the auxiliary market is found, we then find the optimal results under the constrained market by optimizing over $\nu(\cdot) \in \mathcal{D}$. In this way, we find a particular value $\nu^*(t)$ that makes $\boldsymbol{w}(t) \in \mathcal{K}$ for $t < \infty$. More clearly, via $\nu^*(\cdot)$ the unconstrained investment strategy in the auxiliary market is the same as the investment strategy in the constrained market. Therefore, $\nu^*(\cdot)$ is equivalent to the value the proportional investment amount that violates the constraints.

Next, for every process $\nu(\cdot) \in \mathcal{D}$, we express the assets of the auxiliary market by

$$\mathrm{d}V_0(t) = V_0(t)(r + \delta(\nu(t)))\mathrm{d}t; \tag{4.3}$$

$$\mathrm{d}S_i^{\nu}(t) = S_i^{\nu}(t)\left[(\mu_i + \nu_i(t) + \delta(\nu(t)))\mathrm{d}t + \sum_{j=1}^{N} \sigma_{ij}\mathrm{d}B_j(t)\right] \text{ for } i = 1, \dots, N. \tag{4.4}$$

Then, the wealth process is the solution of the stochastic differential equation

$$\mathrm{d}X_{\nu}^{\boldsymbol{w}}(t) = X_{\nu}^{\boldsymbol{w}}(t)\left[(r + \delta(\nu(t)))\mathrm{d}t + \boldsymbol{w}^{'}(t)(\mu + \nu(t) - r\underset{\sim}{1})\mathrm{d}t + \boldsymbol{w}^{'}(t)\sigma\mathrm{d}B(t)\right]. \tag{4.5}$$

Given the wealth process dynamics in (4.5), the benchmarked portfolio process in the auxiliary market is the solution of

$$\begin{aligned} \mathrm{d}Z_{\nu}^{\boldsymbol{w}}(t) &= Z_{\nu}^{\boldsymbol{w}}(t)\left((\hat{r} + \delta(\nu(t)))\mathrm{d}t + \boldsymbol{w}^{'}(t)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\right)\mathrm{d}t \\ &\quad + Z_{\nu}^{\boldsymbol{w}}(t)\left((\boldsymbol{w}^{'}(t)\sigma - b^{'})\mathrm{d}B(t) - \beta\mathrm{d}B^{(N+1)}(t)\right) \end{aligned} \tag{4.6}$$

Moreover, given the above specification we redefine the second-order differential operator $\mathcal{L}_{\nu}^{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \mathbb{R}^N$ and $\nu \in \mathbb{R}^N$ as in the following way: For every open set $\mathcal{O} \subset \mathbb{R}$ and for every $\Upsilon_{\nu}(x) \in C^2(\mathcal{O})$ the function $\mathcal{L}_{\nu}^{\boldsymbol{w}}\Upsilon_{\nu} : \mathcal{O} \to \mathbb{R}$ is given by

$$\mathcal{L}_\nu^{\boldsymbol{w}}\Upsilon_\nu(z) = \left((\hat{r}+\delta(\nu))\mathrm{d}t + \boldsymbol{w}'(\hat{\mu}+\nu-r\underset{\sim}{1})\right) z\Upsilon_\nu'(z) + \frac{1}{2}\left(\|\boldsymbol{w}'\sigma - b'\|^2+\beta^2\right) z^2\Upsilon_\nu''(z). \quad (4.7)$$

where Υ_ν' is the first derivative and Υ_ν'' is the second derivative with respect to z. Note that the sign $'$ on the coefficients and $\boldsymbol{w}$ denote the transpose.

Finally, the market price of risk under the constrained market is

$$\begin{aligned}\zeta_\nu(t) &= \sigma^{-1}(\mu+\nu(t)-r\underset{\sim}{1}) \\ &= \zeta + \sigma^{-1}\nu(t),\end{aligned}$$

where $\zeta = \sigma^{-1}(\mu - r\underset{\sim}{1})$.

4.3 The Main Theorem

After representing HJB equations and defining the auxiliary market, we proceed to formulate the general form [1] of the three problems considered in this thesis. With this in mind, we let

$$\tau_L^{\boldsymbol{w}} := \inf\{t>0 | Z_\nu^{\boldsymbol{w}}(t) \le L\},$$

be the first time the benchmarked auxiliary portfolio process $Z_\nu^{\boldsymbol{w}}(\cdot)$ crosses the

[1]Please see Browne (1999) for the unconstrained case.

lower wealth level L, where $0 < L < x$, and

$$\tau_U^{\boldsymbol{w}} := \inf\{t > 0 | Z^{\boldsymbol{w}}(t) \geq U\},$$

be the first time the benchmarked auxiliary portfolio process $Z_\nu^{\boldsymbol{w}}(\cdot)$ crosses the upper wealth level U, where $x < U$. Given $\tau_L^{\boldsymbol{w}}$ and $\tau_U^{\boldsymbol{w}}$, we also let $\tau^{\boldsymbol{w}} := \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}$. Mainly, $\tau^{\boldsymbol{w}}$ denotes the first escape time of $Z_\nu^{\boldsymbol{w}}(\cdot)$ from the interval (L, U) under an admissible control process $\{\boldsymbol{w}(t), t \geq 0\}$.

Now we introduce the general form by

$$\begin{aligned} J_\nu^{\boldsymbol{w}} = \mathbb{E}_z \Bigg[\int_0^{\tau^{\boldsymbol{w}}} & \exp\left\{ -\int_0^s \rho(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s \right\} q(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s \\ & + \exp\left\{ -\int_0^{\tau^{\boldsymbol{w}}} \rho(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s \right\} H(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}})) \Bigg], \end{aligned} \tag{4.8}$$

where $\rho(z) \geq 0$ is a real-valued function, $q(z)$ is a real-valued bounded and continuous function and $H(z)$ is defined on a domain set $z \in \{L, U\}$. Here we use $\mathbb{E}_z[\,\cdot\,] = \mathbb{E}[\,\cdot\,|Z(0) = z]$ for shorthand in notation.

Remark 4.3.1. *By using the specification in (4.8), we obtain the value functions of the problems we are solving in this thesis*

- ***P1.*** *We obtain the objective function of the* ***expected time of hitting upper or lower level*** *by taking $\rho(\cdot) = 0$, $q(\cdot) = 1$ and $H(U) = H(L) = 0$. In this case*

$$J_\nu^{\boldsymbol{w}}(z) = \mathbb{E}_z \left[\int_0^{\tau^{\boldsymbol{w}}} e^0 \mathrm{d}t \right] = E_z[\tau^{\boldsymbol{w}}].$$

P2. *Similarly, we get the* ***probability of hitting upper level before lower level*** *with letting* $\rho(\cdot\,) = q(\cdot\,) = 0$, $H(U) = 1$ *and* $H(L) = 0$ *and thus*

$$\begin{aligned} J_\nu^{\boldsymbol{w}}(z) &= \mathbb{E}_z\left[H\left(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}})\right)\right] \\ &= \mathbb{P}_z\left(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}}) = U\right) = \mathbb{P}(\tau_U < \tau_L). \end{aligned}$$

P3. *Finally, for the last problem we take* $\rho(\cdot\,) = \rho$, $q(\cdot\,) = 0$, $H(U) = 1$ *and* $H(L) = 0$ *thus*

$$\begin{aligned} J_\nu^{\boldsymbol{w}}(z) &= \mathbb{E}_z\left[H(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}}))\exp\left\{-\int_0^{\tau^{\boldsymbol{w}}} \rho \mathrm{d}s\right\}\right] \\ &= \mathbb{E}_z\left[H(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}}))e^{-\rho\tau^{\boldsymbol{w}}}\right] \\ &= \mathbb{E}_z\left[e^{-\rho\tau_U^{\boldsymbol{w}}}\right], \end{aligned}$$

gives us ***the expected discounted reward****.*

Depending on the problems we are solving in the thesis, the objective, in turn involves, either the maximization or minimization of (4.8). Thus,

$$V_\nu = \sup_{\boldsymbol{w}\in\mathcal{A}_\nu} J_\nu^{\boldsymbol{w}} \quad \text{or} \quad V_\nu = \inf_{\boldsymbol{w}\in\mathcal{A}_\nu} J_\nu^{\boldsymbol{w}}, \tag{4.9}$$

where $\mathcal{A}_\nu(z)$ is the set of admissible strategies in the auxiliary market. As mentioned in section (4.2), when constraints are binding we have $\delta(\nu) + \boldsymbol{w}'\nu \geq 0\ \forall\nu \in \tilde{\mathcal{K}}$ if and only if $\boldsymbol{w} \in \mathcal{K}$. Therefore once we find a fictitious parameter $\nu^* \in \tilde{\mathcal{K}}$ and establish that the maximizer $\boldsymbol{w}^* \in \mathcal{K}$ then we obtain $\delta(\nu) + \boldsymbol{w}'\nu = 0$. Furthermore, once the optimality is shown, we have for the maximization problem

$$V(z) = V_{\nu^*}(z) = \inf_{\nu \in \mathcal{D}} V_\nu(z),$$

and for the minimization problem

$$V(z) = V_{\nu^*}(z) = \sup_{\nu \in \mathcal{D}} V_\nu(z).$$

Then, the main theorem follows.

Theorem 4.1. *Suppose $G_{\nu^*} : [L, U] \to \mathbb{R}$ is a concave increasing function, smooth enough in the sense $G_{\nu^*} \in C^2((L, U))$. Furthermore, assume for $c = (c_1, c_2, c_3)$ and $c \geq 0$ that $|G_{\nu^*}(z)| < c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$ and that G_{ν^*} satisfies the non-linear partial differential equations*

$$\begin{cases} -\rho(z)G_{\nu^*} + q(z) + \left((\hat{r} + b'\zeta)zG'_{\nu^*} - \frac{1}{2}\|\zeta\|^2 \frac{(G'_{\nu^*})^2}{G''_{\nu^*}} + \frac{1}{2}\beta^2 z^2 G''_{\nu^*}\right) = 0 & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1 \\ -\rho(z)G_{\nu^*} + q(z) + \Bigg(\left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right) zG'_{\nu^*} & \\ \qquad + \frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right) z^2 G''_{\nu^*} - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(G'_{\nu^*})^2}{G''_{\nu^*}}\Bigg) = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1 \end{cases} \tag{4.10}$$

where $D = \zeta'\sigma^{-1}\underset{\sim}{1}$, $K = \underset{\sim}{1}'\Sigma^{-1}\underset{\sim}{1}$ and $Q = b'\sigma^{-1}\underset{\sim}{1}$ along with boundary conditions

$$G_{\nu^*}(L) = H(L) \quad \textit{and} \quad G_{\nu^*}(U) = H(U),$$

and where $\nu^*(z)$ is given by

$$\nu^*(z) = \begin{cases} \underset{\sim}{0} & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1 \\ \frac{1}{K}\left((-1+Q)\frac{zG''_{\nu^*}}{G'_{\nu^*}} - D\right)\underset{\sim}{1} & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1 \end{cases} \tag{4.11}$$

with $\underset{\sim}{0} = (0, 0, \ldots, 0)'$ and $\boldsymbol{w}^*(z)$ is

$$\boldsymbol{w}^*(z) = (\sigma^{-1})'b - (\sigma^{-1})'\zeta\frac{G'_{\nu^*}}{zG''_{\nu^*}}. \tag{4.12}$$

If there exists a maximizer such that $-\rho(z)G_{\nu^*}(z) + q(z) + \mathcal{L}^{\boldsymbol{w}^*}_{\nu^*}G_{\nu^*}(z) = 0$, $\nu^*(z) \in \tilde{\mathfrak{K}}$ and $\boldsymbol{w}^*(Z^*_{\nu^*}(t))$ is admissible, then, $G_{\nu^*}(z)$ is the optimal value function (That is, $G_{\nu^*}(z) = V_{\nu^*}(z)$), $\nu^*(Z^*_{\nu^*}(t))$ is the optimal fictitious process and $\boldsymbol{w}^*(Z^*_{\nu^*}(t))$ is the optimal investment strategy, where $Z^*_{\nu^*}(t)$ is the optimal benchmarked wealth process for $t < \tau^w$.

Proof. We will first find the maximizing control strategy and maximum value function and then check for optimality. For the **first step** we write the HJB equation that G_ν satisfies for all $\nu \in \mathbb{R}^N$

$$\begin{aligned} -\rho(z)G_\nu + q(z) + \sup_{\boldsymbol{w}}\Big\{ &\left((\hat{r} + \delta(\nu)) + \boldsymbol{w}'(\hat{\mu} + \nu - r\underset{\sim}{1})\right) z\frac{\partial}{\partial z}G_\nu \\ &+ \frac{1}{2}\left((\boldsymbol{w}'\Sigma\boldsymbol{w} - 2\boldsymbol{w}'\sigma b + b'b + \beta^2\right) z^2\frac{\partial^2}{\partial z^2}G_\nu\Big\} = 0. \end{aligned} \tag{4.13}$$

From the first-order condition, we obtain the maximizing control strategy

$$\boldsymbol{w}^*(z) = (\sigma^{-1})'b - (\sigma^{-1})'\zeta_\nu\frac{G'_\nu}{zG''_\nu}. \tag{4.14}$$

Next, we proceed to calculate the vector of fictitious parameters. To this end, we substitute the above into (4.13) and rearrange the terms to write

$$-\rho(z)G_\nu + q(z) + \left((\hat{r} + \delta(\nu) + b'\zeta_\nu)zG'_\nu - \frac{1}{2}\|\zeta_\nu\|^2\frac{(G'_\nu)^2}{G''_\nu} + \frac{1}{2}\beta^2 z^2 G''_\nu\right) = 0. \tag{4.15}$$

Using the fact that $\delta(\nu) = \sup_{\boldsymbol{w}}(-\boldsymbol{w}'\nu) = -\nu_1$, $\nu = \nu_1 \underset{\sim}{1}$ and $\zeta_\nu = \zeta + \sigma^{-1}\nu_1\underset{\sim}{1}$, we rewrite (4.15) as

$$-\rho(z)G_\nu + q(z) + \Bigg((\hat{r} - \nu_1 + b'(\zeta + \sigma^{-1}\nu_1\underset{\sim}{1}))zG'_\nu \\ - \frac{1}{2}(\|\zeta\|^2 + 2\nu_1 D + \nu_1^2 K)\frac{(G'_\nu)^2}{G''_\nu} + \frac{1}{2}\beta^2 z^2 G''_\nu\Bigg) = 0. \tag{4.16}$$

By setting the first-order derivative of the above with respect to ν_1 to 0, we find optimal fictitious parameter as

$$\nu_1^* = \frac{1}{K}\left((-1+Q)\frac{zG''_\nu}{G'_\nu} - D\right). \tag{4.17}$$

And by replacing above parts into respective places we obtain the PDE that G_{ν^*} solves

$$-\rho(z)G_{\nu^*} + q(z) + \Bigg(\left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right)zG'_{\nu^*} \\ + \frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2 z^2 G''_{\nu^*}\right)z^2 G''_{\nu^*} - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(G'_{\nu^*})^2}{G''_{\nu^*}}\Bigg) = 0. \tag{4.18}$$

Next, we show that the above results are optimal. First, we check if the maximizer $\boldsymbol{w}^*(z) \in \mathcal{K}$. For the case when the constraints are not binding, we have

$\nu_1 = 0$ and $\boldsymbol{w}^*(z) \in \mathcal{K}$ in this case. On other hand, when the constraints are binding we have

$$\begin{aligned}
\underset{\sim}{1}'\boldsymbol{w}^*(z) &= \underset{\sim}{1}'((\sigma^{-1})'b - (\sigma^{-1})'\zeta_\nu \frac{G'_\nu}{zG''_\nu}) \\
&= Q - (D + (-1+Q)\frac{zG''_\nu}{G'_\nu} - D)\frac{G'_\nu}{zG''_\nu} \\
&= Q - (-1+Q) \\
&= 1. \qquad (4.19)
\end{aligned}$$

Therefore, we have $\boldsymbol{w}^*(z) \in \mathcal{K}$ in each case and it follows that $\delta(\nu^*) + \underset{\sim}{1}'\boldsymbol{w}^*\nu_1^* = 0$, giving $\nu^*(z) \in \tilde{\mathcal{K}}$.

To continue the proof of optimality, we will follow the steps of the Section 3.3.1 similarly. However, notice that the problem considered here is under infinite horizon. Therefore, we make the necessary changes during the proof. To this end, we start by defining the stopping time

$$\tau_n^{\boldsymbol{w}} = \tau^{\boldsymbol{w}} \wedge \inf\{u > t | Z_{\nu^*}^{\boldsymbol{w}}(u) - Z_{\nu^*}^{\boldsymbol{w}}(t)| \geq n\} \quad \text{for } n \in \mathbb{N}, \qquad (4.20)$$

and fixing $z \in (L, U)$. We also introduce

$$\begin{aligned}
M(t, Z_{\nu^*}^{\boldsymbol{w}}(t)) = \int_0^t &\exp\left\{-\int_0^s \rho(Z_{\nu^*}^{\boldsymbol{w}}(v))\mathrm{d}v\right\} q(Z_{\nu^*}^{\boldsymbol{w}}(s))\mathrm{d}s \\
&+ \exp\left\{-\int_0^t \rho(Z_{\nu^*}^{\boldsymbol{w}}(s))\mathrm{d}s\right\} G_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(t)). \qquad (4.21)
\end{aligned}$$

Then, for an admissible strategy $\boldsymbol{w}(\cdot)$ and for $0 \leq t \leq \tau_n^{\boldsymbol{w}}$, we can write by Itô's formula

$$
\begin{aligned}
&M(\tau_n^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}})) \\
&= M(t, Z_\nu^{\boldsymbol{w}}(t)) \\
&\quad + \int_t^{\tau_n^{\boldsymbol{w}}} e^{-\int_t^s \rho(Z_{\nu^*}^{\boldsymbol{w}}(v))\mathrm{d}v}(-\rho(Z_{\nu^*}^{\boldsymbol{w}}(s))G_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(s)) + q(Z_{\nu^*}^{\boldsymbol{w}}(s)) + \mathcal{L}_{\nu^*}^{\boldsymbol{w}} G_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(s)))\mathrm{d}s \\
&\qquad + \int_t^{\tau_n} e^{-\int_t^s \rho(Z_{\nu^*}^{\boldsymbol{w}}(v))\mathrm{d}v} Z_{\nu^*}^{\boldsymbol{w}}(s) G'_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(s))(\boldsymbol{w}'(Z_{\nu^*}^{\boldsymbol{w}}(s))\sigma - b)\mathrm{d}B(s) - \beta \mathrm{d}B^{(N+1)}(s)).
\end{aligned}
\tag{4.22}
$$

Because of the continuity of G_{ν^*}, admissibility of $\boldsymbol{w}$ and boundedness of $Z_{\nu^*}^{\boldsymbol{w}}$ in $[t, \tau_n^{\boldsymbol{w}}]$ we have

$$
\begin{aligned}
\mathbb{E}_{t,z}\Big[\int_t^{\tau_n^{\boldsymbol{w}}} &\Big[e^{-\int_t^s \rho(Z_{\nu^*}^{\boldsymbol{w}}(v))\mathrm{d}v} Z_{\nu^*}^{\boldsymbol{w}}(s) G'_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(s))\Big]^2 \\
&\times \left(\|(\boldsymbol{w}'(Z_{\nu^*}^{\boldsymbol{w}}(s))\sigma - b)\|^2 + \beta^2\right) \mathrm{d}s\Big] < \infty,
\end{aligned}
\tag{4.23}
$$

and the expectation of the stochastic integral in (4.22) is zero. Furthermore, for every $\boldsymbol{w} \in \mathbb{R}^N$ and for each s we have the inequality

$$
-\rho(Z_{\nu^*}^{\boldsymbol{w}}(s))G_{\nu^*}(Z_{\nu^*}^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}})) + q(Z_{\nu^*}^{\boldsymbol{w}}(s)) + \mathcal{L}_{\nu^*}^{\boldsymbol{w}} G_{\nu^*}(z) \le 0, \tag{4.24}
$$

which holds $\mathbb{P}$-almost surely. Then from (4.22) - (4.24) it follows that

$$
M(t, Z_{\nu^*}^{\boldsymbol{w}}(t)) \ge \mathbb{E}_{t,z}[M(\tau_n^{\boldsymbol{w}}, Z_{\nu^*}^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}}))] \quad \forall n. \tag{4.25}
$$

Next, by using (4.21), we write the above specification as

$$\begin{aligned}&\int_0^t \exp\left\{-\int_0^s \rho(Z^{\boldsymbol{w}}_{\nu^*}(v))\mathrm{d}v\right\} q(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s \\ &\qquad + \quad \exp\left\{-\int_0^t \rho(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s\right\} G_{\nu^*}(Z^{\boldsymbol{w}}_{\nu^*}(t)) \\ &\geq \mathbb{E}_{t,z}\Bigg[\int_0^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_0^s \rho(Z^{\boldsymbol{w}}_{\nu^*}(v))\mathrm{d}v\right\} q(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s \\ &\qquad + \quad \exp\left\{-\int_0^{\tau_n^{\boldsymbol{w}}} \rho(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s\right\} G_{\nu^*}(Z^{\boldsymbol{w}}_{\nu^*}(\tau_n^{\boldsymbol{w}}))\Bigg], \end{aligned} \tag{4.26}$$

By setting $t = 0$, we rewrite the above as

$$\begin{aligned} G_{\nu^*}(z) \geq \mathbb{E}_z\Bigg[&\int_0^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_0^s \rho(Z^{\boldsymbol{w}}_{\nu^*}(v))\mathrm{d}v\right\} q(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s \\ &+ \exp\left\{-\int_0^{\tau_n^{\boldsymbol{w}}} \rho(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s\right\} G_{\nu^*}(Z^{\boldsymbol{w}}_{\nu^*}(\tau_n^{\boldsymbol{w}}))\Bigg]. \end{aligned} \tag{4.27}$$

Since we $|G_{\nu^*}(z)| < c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$, and $\tau_n^{\boldsymbol{w}} \to \tau^{\boldsymbol{w}}$ as $n \to \infty$, from the dominated convergence theorem we have

$$\begin{aligned} \mathbb{E}_z\Bigg[&\int_0^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_0^s \rho(Z^{\boldsymbol{w}}_{\nu^*}(v))\mathrm{d}v\right\} q(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s \\ &+ \exp\left\{-\int_0^{\tau_n^{\boldsymbol{w}}} \rho(Z^{\boldsymbol{w}}_{\nu^*}(s))\mathrm{d}s\right\} G_{\nu^*}(Z^{\boldsymbol{w}}_{\nu^*}(\tau_n^{\boldsymbol{w}}))\Bigg] \to J_{\nu^*}(z, \boldsymbol{w}). \end{aligned} \tag{4.28}$$

Then we have from above and (4.27)

$$G_{\nu^*}(z) \geq J_{\nu^*}(z, \boldsymbol{w}). \tag{4.29}$$

Taking the supremum of the right-hand side over all admissible control laws gives

$$G_{\nu^*}(z) \geq V_{\nu^*}(z). \tag{4.30}$$

On the other hand, if we take the maximizer such that the inequality in (4.24) becomes equality, we then have

$$G_{\nu^*}(z) = J_{\nu^*}(z, \boldsymbol{w}^*) \leq V_{\nu^*}(z).$$

It then follows that

$$G_{\nu^*}(z) = J_{\nu^*}(z, \boldsymbol{w}^*) \leq V_{\nu^*}(z) \leq G_{\nu^*}(z),$$

implying that $G_{\nu^*}(z) = V_{\nu^*}(z)$. Therefore, $G_{\nu^*}^{\boldsymbol{w}^*}(z)$ is the optimal value function, $\boldsymbol{w}^*$ is the optimal investment strategy. □

Chapter 5

Application

5.1 Maximizing/Minimizing Expected Time

In this subsection we will introduce first problem. We consider an investor who is interested in minimizing (maximizing) the expected time to reach (stay above) an upper (a lower) level. When investing the investor takes into consideration the condition of the markets so that she can decide whether her portfolio may attain the desired objective. To model the objective, in turn, we denote the condition via a *market favourability* parameter whose value depends on the direction of the trend of the investor's portfolio. As we will see in what follows, our investor can minimize the expected time until her portfolio reaches an upper level if and only if the market is *favourable*. On the other hand, if the markets are *unfavourable*, she can only invest to maximize the expected time to stay above a predetermined wealth level.

For more clarity, we start by defining the generalized version of the *favourability parameter* via

$$\theta_\nu = \theta - (1 - Q - D)\nu_1 + \frac{1}{2}K\nu_1^2. \tag{5.1}$$

Note that for the unconstrained case, we have $\nu_1 = 0$. Then, the parameter becomes

$$\theta = \hat{r} + b'\zeta + \frac{1}{2}\|\zeta\|^2 - \frac{1}{2}\beta^2. \tag{5.2}$$

Then, it follows that, for $\theta_\nu > 0$, the market is said to be favourable, and for $\theta_\nu < 0$, it is said to be unfavourable. Consequently, we end up with two different objectives which are given by

- for $\theta_\nu > 0$,

$$\bar{E}_\nu(z) = \inf_{\boldsymbol{w} \in \mathcal{A}_\nu} \mathbb{E}[\tau_U^{\boldsymbol{w}}],$$

- and for $\theta_\nu < 0$

$$\underline{E}_\nu(z) = \sup_{\boldsymbol{w} \in \mathcal{A}_\nu} \mathbb{E}[\tau_L^{\boldsymbol{w}}],$$

respectively. We will see in the next theorem that two different problems have the same investment strategy.

Theorem 5.1. *Let $Z_\nu^{\boldsymbol{w}}$ be solution to (4.6) and $Z_\nu^{\boldsymbol{w}}(0) = z \in [L, U]$. The vector of* *optimal fictitious parameters is*

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \textit{if } Q + D < 1; \\ \frac{1-Q-D}{K}\underset{\sim}{1}, & \textit{if } Q + D \geq 1, \end{cases} \tag{5.3}$$

and the favourability parameter $\theta^*_\nu \in \mathbb{R} \setminus \{0\}$ *is thereby*

$$\theta_{\nu^*} = \begin{cases} \theta, & \textit{for } Q + D < 1; \\ \theta - \frac{(1-Q-D)^2}{2K}, & \textit{for } Q + D \geq 1. \end{cases} \tag{5.4}$$

Then, the optimal value function when $\theta_{\nu^*} > 0$ *is*

$$\bar{E}_{\nu^*}(z) = \frac{1}{\theta_{\nu^*}} \log\left(\frac{U}{z}\right) \quad \textit{for} \quad z \leq U, \tag{5.5}$$

and for $\theta_{\nu^*} < 0$

$$\underline{E}_{\nu^*}(z) = \frac{1}{|\theta_{\nu^*}|} \log\left(\frac{z}{L}\right) \quad \textit{for} \quad z \geq L. \tag{5.6}$$

In both cases, the optimal investment strategy is

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'\zeta + (\sigma^{-1})'b, & \textit{if } Q + D < 1; \\ (\sigma^{-1})'\left(\zeta + (\sigma^{-1})\frac{1-Q-D}{K}\underset{\sim}{1}\right) + (\sigma^{-1})'b, & \textit{if } Q + D \geq 1. \end{cases} \tag{5.7}$$

Therefore, the optimal portfolio process relative to benchmark for $t \in [0, \tau_L \wedge \ \tau_U]$

is

$$Z^{\boldsymbol{w}^*}_{\nu^*}(t) = \begin{cases} Z^{\boldsymbol{w}^*}_{\nu^*}(0)\exp\left\{\theta t + \zeta' B(t) - \beta B^{(N+1)}(t)\right\}, & \text{if } Q + D < 1; \\ Z^{\boldsymbol{w}^*}_{\nu^*}(0)\exp\left\{\theta_{\nu^*} t + \zeta'_{\nu^*} B(t) - \beta B^{(N+1)}(t)\right\}, & \text{if } Q + D \geq 1, \end{cases} \tag{5.8}$$

where $\zeta_{\nu^*} = \zeta + (\sigma^{-1})\frac{1-Q-D}{K}\underset{\sim}{1}$.

Proof. For this problem, we take $\rho(\cdot) = 0,\ q(\cdot) = 1,\ H(\cdot) = 0$ in Equation 4.8. As a result, from Equation 4.10 of the *Main Theorem*, the partial differential equation that E_{ν^*} is a solution to is given by

$$\begin{cases} 1 + \left((\hat{r} + b'\zeta) z E'_{\nu^*} - \frac{1}{2}\|\zeta\|^2 \frac{(E'_{\nu^*})^2}{E''_{\nu^*}} + \frac{1}{2}\beta^2 z^2 E''_{\nu^*}\right) = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1; \\ 1 + \Bigg(\left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right) z E'_{\nu^*} & \\ \quad + \frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right) z^2 E''_{\nu^*} - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(E'_{\nu^*})^2}{E''_{\nu^*}}\Bigg) = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1. \end{cases} \tag{5.9}$$

Since we have two market conditions, we have two solutions for the above equation. Note that both equations

$$\bar{E}_{\nu^*}(z) = \frac{1}{\theta_{\nu^*}} \log\left(\frac{U}{z}\right) \quad \text{for } z \leq U, \tag{5.10}$$

where $\bar{E}'_{\nu^*} > 0$ with $\bar{E}''_{\nu^*} < 0$ and

$$\underline{E}_{\nu^*}(z) = \frac{1}{|\theta_{\nu^*}|} \log\left(\frac{z}{L}\right) \quad \text{for } z \geq L, \tag{5.11}$$

where $\underline{E}'_{\nu^*} < 0$ with $\underline{E}''_{\nu^*} > 0$ satisfy the equality in (5.9). Furthermore, both equations satisfy their boundary conditions, i.e. $\bar{E}_{\nu^*}(U) = 0$ and $\underline{E}_{\nu^*}(L) = 0$. Then, the minimizing fictitious parameters are given by

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1; \\ \frac{1-Q-D}{K}\underset{\sim}{1}, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1. \end{cases} \tag{5.12}$$

Note that using both value functions give the same result for the maximizer and the minimizer. As a result, we obtain

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'\zeta + (\sigma^{-1})'b, & \text{if } Q + D < 1; \\ (\sigma^{-1})'\left(\zeta + (\sigma^{-1})\frac{1-Q-D}{K}\underset{\sim}{1}\right) + (\sigma^{-1})'b, & \text{if } Q + D \geq 1. \end{cases} \tag{5.13}$$

We prefer to write $Q + D$ instead of $\underset{\sim}{1}'\boldsymbol{w}^*$ to indicate the cases since by definition $Q + D = 1'[(\sigma^{-1})'\zeta + (\sigma^{-1})'b]$. Last, by substituting (5.12) into (5.1) we obtain

$$\theta_{\nu^*} = \begin{cases} \theta, & \text{for } Q + D < 1; \\ \theta - \frac{(1-Q-D)^2}{2K}, & \text{for } Q + D \geq 1. \end{cases} \tag{5.14}$$

Then, the benchmarked portfolio process is given by

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\{\theta t + \zeta' B(t) - \beta B^{N+1}(t)\}, & \text{if } Q + D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\{\theta_{\nu^*} t + \zeta'_{\nu^*} B(t) - \beta B^{N+1}(t)\}, & \text{if } Q + D \geq 1. \end{cases} \tag{5.15}$$

Now, we verify that the above results are optimal. To this end, we first check if $\boldsymbol{w}^* \in \mathcal{K}$ with $\nu^* \in \tilde{\mathcal{K}}$. We first consider the case when $Q + D \geq 1$:

$$\begin{aligned}
\underset{\sim}{1}'\boldsymbol{w}^*(z) &= \underset{\sim}{1}'(\sigma^{-1})'b + \underset{\sim}{1}'(\sigma^{-1})'(\zeta + \frac{1-Q-D}{K}\sigma^{-1}\underset{\sim}{1}) \\
&= \underset{\sim}{1}'(\sigma^{-1})'b + \underset{\sim}{1}'(\sigma^{-1})'\zeta + \underset{\sim}{1}'(\sigma^{-1})'\sigma^{-1}\underset{\sim}{1}\frac{1-Q-D}{K}) \\
&= Q + D + \frac{1-Q-D}{K}K = 1,
\end{aligned} \tag{5.16}$$

which implies $\boldsymbol{w}^* \in \mathcal{K}$, and thus $\delta(\nu^*) + \boldsymbol{w}^*(z)'\nu_1^*\underset{\sim}{1} = 0$ gives $\nu^* \in \tilde{\mathcal{K}}$. When $Q + D < 1$, $\underset{\sim}{1}'\boldsymbol{w}^* < 1$ which by definition gives us also $\boldsymbol{w}^* \in \mathcal{K}$ and $\nu^* \in \tilde{\mathcal{K}}$.

Next, we will verify that $\boldsymbol{w}^*$ is admissible. To this end, we check if the stopping time $\tau^{\boldsymbol{w}^*} = \tau_U^{\boldsymbol{w}^*} \wedge \tau_L^{\boldsymbol{w}^*}$ is finite. Since $\boldsymbol{w}^*$ is a constant vector, $\int_0^{\tau^{\boldsymbol{w}^*}} \|\boldsymbol{w}^*\|^2 \mathrm{d}s < \infty$ almost surely if and only if $\tau^{\boldsymbol{w}^*} = \tau_U^{\boldsymbol{w}^*} \wedge \tau_L^{\boldsymbol{w}^*} < \infty$ almost surely. From (5.15), we write

$$\frac{1}{t}\,\mathbb{E}\left[\log(Z_{\nu^*}^{\boldsymbol{w}^*}(t))\right] = \begin{cases} \theta, & \text{if } Q + D < 1; \\ \theta_{\nu^*}, & \text{if } Q + D \geq 1. \end{cases} \tag{5.17}$$

Note that for $\theta, \theta_{\nu^*} > 0$, $Z_{\nu^*}^{\boldsymbol{w}^*}(t) \to \infty$ and for $\theta, \theta_{\nu^*} < 0$, $Z_{\nu^*}^{\boldsymbol{w}^*}(t) \to 0$ as $t \to \infty$. Then, for $\theta, \theta_{\nu^*} > 0$, $\tau_U^{\boldsymbol{w}^*} < \infty$ almost surely and for $\theta, \theta_{\nu^*} < 0$, $\tau_L^{\boldsymbol{w}^*} < \infty$ almost surely. Then, it follows that $\tau_L^{\boldsymbol{w}^*} \wedge \tau_U^{\boldsymbol{w}^*} < \infty$, and $\boldsymbol{w} \in \mathcal{A}_\nu$.

Finally, to verify that $\underline{E}_{\nu^*}$, $\bar{E}_{\nu^*}$ are the optimal value functions and $\boldsymbol{w}^*$ is the optimal investment strategy, we will first show that value functions are dominated by some function. Because of the assumption that $|G_\nu^*|$ is dominated by $c_1 + c_2\log(\frac{z}{L}) + c_3\log(\frac{U}{z})$, taking $(c_1, \frac{1}{|\theta_{\nu^*}|}, 0)$ and $(c_1, 0, \frac{1}{\theta_{\nu^*}})$ with some $c_1 > 0$ for relevant problem is enough to show that $\underline{E}_{\nu^*}$ and $\bar{E}_{\nu^*}$ are dominated by such function. For the remainder of the verification, the same steps in Proof of Theorem 4.1 can be followed. □

For the analysis of the results, we first refer to the specifications in (5.17). Note that, the case for θ (i.e. the unconstrained case) is equivalent to the case for θ_{ν^*} with $\nu^* = 0$ and is already shown in Browne (1999). Therefore, we only repeat the analysis of Browne (1999) for the constrained case (see also Yener (2014)). That is, when $\theta_{\nu^*} < 0$, $\inf_{\boldsymbol{w}\in\mathcal{A}_{\nu^*}} \mathbb{E}[\tau_U^{\boldsymbol{w}}] = \infty$. In other words, if the markets are unfavourable there is no admissible strategy that will make the traded portfolio process beat the benchmark, and if they are favourable, then, there will always be an admissible strategy that make the time to ruin infinite.

However, for the constrained case, we have

$$\theta_{\nu^*} = \theta - \frac{(1-Q-D)^2}{2K}.$$

Therefore, $\theta_{\nu^*} > 0$ gives $\theta > (1-Q-D)^2/2K$ showing that the favourability parameter of the unconstrained case must be larger than zero by a factor of $\theta > (1-Q-D)^2/2K$ under the constrained case so that the investor can find an optimal investment strategy to make the time to ruin infinite. On the other hand, if $\theta_{\nu^*} < 0$, $\theta < (1-Q-D)^2/2K$ showing that the expected time to beat the benchmark might be infinite under the constrained case even when $\theta \in [0, \theta > (1-Q-D)^2/2K]$, i.e. θ is positive.

5.2 Maximizing the Probability of Beating the Benchmark Before Being Beaten

In this section, we consider a portfolio manager who is trading under borrowing prohibitions and whose performance is measured by a benchmark process. In particular, the performance of the manager is evaluated according to hitting above or below a certain percentage of the benchmark process. In this respect, the second problem is the probability of beating the benchmark before being beaten by it. Therefore, we express the value function mathematically by

$$P_\nu(z) = \sup_{\boldsymbol{w} \in \mathcal{A}_\nu} \mathbb{P}_z(\tau_U^{\boldsymbol{w}} > \tau_L^{\boldsymbol{w}}). \tag{5.18}$$

Furthermore, as we previously mentioned (see *Remark 4.3.1*), this value function is a special case that can be obtained by letting $\rho = q = 0,\ H(U) = 1,\ H(L) = 0$ in equation (4.10). We represent the solution of such problem in the following theorem.

Theorem 5.2. *Let $Z_\nu^{\boldsymbol{w}}(t)$ be solution to (4.6) and set $Z_\nu^{\boldsymbol{w}}(0) = z$ where $z \in [L, U]$. The vector of optimal fictitious parameters is*

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \text{if } Q - \frac{1}{\gamma^-} D < 1; \\ \frac{1}{K}\left(\gamma^-(-1+Q) - D\right) \underset{\sim}{1}, & \text{if } Q - \frac{1}{\gamma^-} D \geq 1, \end{cases} \tag{5.19}$$

where $\gamma^- \in \ (-\infty, 0) \setminus \{-1\}$ *for* $\|\zeta\|^2 - \frac{D^2}{K} \geq 0$ *is*

$$\gamma^- = \begin{cases} \frac{1}{\beta^2}\left(-(\hat{r} + b'\zeta) - \sqrt{(\hat{r} + b'\zeta)^2 + \beta^2\|\zeta\|^2}\right), & \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ \frac{-\left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right) - \sqrt{\Xi}}{\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)}, & \text{if } Q - \frac{1}{\gamma^-}D \geq 1, \end{cases} \tag{5.20}$$

and that Ξ *is*

$$\Xi = \left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right)^2 + \left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)\left(\|\zeta\|^2 - \frac{D^2}{K}\right). \tag{5.21}$$

Then, the optimal value function is, $P_{\nu^*} \in [0, 1]$*, is given by*

$$P_{\nu^*}(z) = \frac{L^{1+\gamma^-} - z^{1+\gamma^-}}{L^{1+\gamma^-} - U^{1+\gamma^-}}, \tag{5.22}$$

and the constant proportional optimal investment strategy is

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'b - (\sigma^{-1})'\zeta\frac{1}{\gamma^-}, & \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ (\sigma^{-1})'b - \frac{1}{\hat{\gamma}^-}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\frac{\hat{\gamma}^-(-1+Q)-D}{K}\underset{\sim}{1}\right), & \text{if } Q - \frac{1}{\hat{\gamma}^-}D \geq 1. \end{cases} \tag{5.23}$$

Therefore, the optimal portfolio process relative to the benchmark is

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta - \frac{1}{2}\left(1+\frac{1}{\gamma^-}\right)^2\|\zeta\|^2\right)t - \frac{1}{\gamma^-}\zeta' B(t) - \beta B^{(N+1)}(t)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\gamma^-}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t - \frac{1}{\gamma^-}\zeta'_{\nu^*}B(t) \right.\\ \qquad\qquad \left. -\beta B^{(N+1)}(t)\right], \qquad \text{if } Q - \frac{1}{\gamma^-}D \geq 1, \end{cases} \tag{5.24}$$

where $\zeta_{\nu^*} = \zeta + (\sigma^{-1})\left(\frac{1}{K}(\gamma^-(-1+Q) - D\right)\underset{\sim}{1}$.

Proof. As mentioned in Remark 4.3.1 the value function for Probability Maximization problem is obtained by taking $\rho(\cdot) = q(\cdot) = 0,\ H(L) = 0$ and $H(U) = 1$ in Equation 4.8. Hence by applying the same replacements in Equation 4.10 in the *Main Theorem*, we can write P_ν as a solution to the following PDE

$$\begin{cases} \left((\hat{r} + b'\zeta)zP'_{\nu^*} - \frac{1}{2}\|\zeta\|^2\frac{(P'_{\nu^*})^2}{P''_{\nu^*}} + \frac{1}{2}\beta^2 z^2 P''_{\nu^*}\right) = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1; \\ \left(\left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right)zP'_{\nu^*} + \frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)z^2 P''_{\nu^*} \right. \\ \qquad \left. -\frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(P'_{\nu^*})^2}{P''_{\nu^*}}\right) = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1. \end{cases} \tag{5.25}$$

To find the maximizing strategy and the minimizing fictitious parameters, we guess a solution to P_{ν^*} in form of $A_1 - A_2 z^{1+\gamma}$ where A_1 and A_2 are constants. Solving by using the boundary conditions $P_{\nu^*}(L) = H(L) = 0$ and $P_{\nu^*}(H) = H(U) = 1$, we get

$$A_1 - A_2 z^{1+\gamma} = \frac{L^{1+\gamma} - z^{1+\gamma}}{L^{1+\gamma} - U^{1+\gamma}}. \tag{5.26}$$

We substitute the above function and its first and second order derivatives into (5.25) and solve for γ. It follows that

$$\gamma^{\pm} = \begin{cases} \frac{1}{\beta^2}\left(-(\hat{r} + b'\zeta) \pm \sqrt{(\hat{r} + b'\zeta)^2 + \beta^2\|\zeta\|^2}\right), & \text{for } \underset{\sim}{1}\boldsymbol{w}^*(z) < 1; \\ \frac{-\left(\hat{r}+b'\zeta-\frac{D}{K}(-1+Q)\right)\pm\sqrt{\Xi}}{\left(\frac{1}{K}(-1+Q)^2+\beta^2\right)}, & \text{for } \underset{\sim}{1}\boldsymbol{w}^*(z) \geq 1, \end{cases} \tag{5.27}$$

where

$$\Xi = \left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right)^2 + \left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)\left(\|\zeta\|^2 - \frac{D^2}{K}\right).$$

For $\|\zeta\|^2 - \frac{D^2}{K} \geq 0$, (5.27) has two roots: $\gamma^- < 0 < \gamma^+$. We see that $P_{\nu^*}(z)$ is concave increasing in z only for $\gamma < 0$. Therefore, the candidate value function is

$$P_{\nu^*} = \frac{L^{1+\gamma^-} - z^{1+\gamma^-}}{L^{1+\gamma^-} - U^{1+\gamma^-}}. \tag{5.28}$$

Next, from the *Main Theorem*, we can find the optimal fictitious parameters by replacing P_{ν^*} and its first and second order derivatives into (4.11). Thus, we obtain

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) < 1; \\ \left(\frac{\gamma^-}{K}(-1+Q) - \frac{D}{K}\right)\underset{\sim}{1}, & \text{if } \underset{\sim}{1}'\boldsymbol{w}^*(z) \geq 1. \end{cases} \tag{5.29}$$

Likewise, by replacing (5.29) and first and second order derivations of (5.28) into

(4.12), we find the maximizer as

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'b - (\sigma^{-1})'\zeta\frac{1}{\gamma^-}, & \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ (\sigma^{-1})'b - \frac{1}{\gamma^-}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\frac{\gamma^-(-1+Q)-D}{K}\underset{\sim}{1}\right), & \text{if } Q - \frac{1}{\gamma^-}D \geq 1, \end{cases} \tag{5.30}$$

where $Q - \frac{1}{\gamma^-}D = \underset{\sim}{1}'\boldsymbol{w}^*(z)$ states the indebtedness. Notice that the maximizer is a constant proportion strategy and independent of the initial state of the portfolio process, $Z_\nu^{\boldsymbol{w}}(0) = z$. Last, with the minimizing fictitious parameters and the maximizer, we can write the portfolio process as

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta - \frac{1}{2}\left(1+\frac{1}{\gamma^-}\right)^2\|\zeta\|^2\right)t - \frac{1}{\gamma^-}\zeta' B(t) - \beta B^{(N+1)}(t)\right], \\ \qquad \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\gamma^-}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t - \frac{1}{\gamma^-}\zeta'_{\nu^*}B(t) \right. \\ \qquad \left. -\beta B^{(N+1)}(t)\right], \quad \text{if } Q - \frac{1}{\gamma^-}D \geq 1, \end{cases} \tag{5.31}$$

where $\zeta_{\nu^*} = \zeta + (\sigma^{-1})\left(\frac{1}{K}(\gamma^-(-1+Q) - D\right)\underset{\sim}{1}$.

Now, we need to verify that the above results are optimal. To this end, we first check if $\boldsymbol{w}^* \in \mathcal{K}$ with $\nu^* \in \tilde{\mathcal{K}}$. Consider the case when $Q - \frac{1}{\gamma^-}D \geq 1$. It follows that

$$
\begin{aligned}
\underset{\sim}{1}'\boldsymbol{w}^* &= \underset{\sim}{1}'(\sigma^{-1})'b - \frac{1}{\gamma^-}\underset{\sim}{1}'(\sigma^{-1})'\left(\zeta + \frac{1}{K}\left((-1+Q)\hat{\gamma^-} - D\right)\sigma^{-1}\underset{\sim}{1}\right) \\
&= Q - \frac{1}{\hat{\gamma}^-}D - K\frac{1}{K}(-1+Q-\frac{1}{\hat{\gamma}^-}D) \\
&= 1.
\end{aligned}
$$

This implies $\boldsymbol{w}^* \in \mathcal{K}$ and it follows that $\nu^* \in \tilde{\mathcal{K}}$. On the other hand, when constraints are not binding, $\nu_1^* = 0$ and $\boldsymbol{w} \in \mathcal{K}$ by definition. For both cases $\delta(\nu^*) + \underset{\sim}{1}'\boldsymbol{w}^*\nu^* = 0$.

Next, we will verify $\boldsymbol{w}^*$ is admissible. To this end, we check if the stopping time $\tau^{\boldsymbol{w}^*} = \tau_U^{\boldsymbol{w}} \wedge \tau_L^{\boldsymbol{w}}$ is finite. To see that is true, we call the probabilities of hitting L and U which are respectively given by

$$
N\left(\frac{\log\left(\frac{L}{z}\right) - \Theta(\gamma^-)t}{\sqrt{\frac{1}{(\gamma^-)^2}(\|\zeta_{\nu^*}\|^2 + \beta^2)t}}\right), \quad \text{and} \quad N\left(\frac{\log\left(\frac{z}{U}\right) + \Theta(\gamma^-)t}{\sqrt{\frac{1}{(\gamma^-)^2}(\|\zeta_{\nu^*}\|^2 + \beta^2)t}}\right),
$$

where N is standard normal c.d.f. and

$$
\Theta(\gamma^-) = \begin{cases} \theta - \frac{1}{2}\left(1 + \frac{1}{\gamma^-}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right), & \text{if } Q - \frac{1}{\gamma^-}D < 1; \\ \theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\gamma^-}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right), & \text{if } Q - \frac{1}{\gamma^-}D \geq 1. \end{cases}
$$

It is clear that when $\Theta(\gamma^-) > 0$, as t increases, probability of hitting L converges to 0 while probability of hitting U converges to 1, and vice versa when $\Theta(\gamma^-) < 0$, implying that $\tau^{\boldsymbol{w}^*} < \infty$. Therefore, we have $\int_0^{\tau^{\boldsymbol{w}^*}} \|\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(s))\|^2 \,\mathrm{d}s < \infty$ $\mathbb{P}$-a.s. As a result, $\boldsymbol{w}(\cdot) \in \mathcal{A}_\nu$.

Finally, the optimality of the maximizer and the value function follows from the *Main Theorem.* Note that $|P_{\nu^*}(z)| < 1$. Since we assumed that the general form of the value functions is dominated; $|G_{\nu^*}| \leq c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$, choosing $(c_1, c_2, c_3) = (1, 0, 0)$ shows that P_{ν^*} is dominated by such a function. The remainder of the verification follows the same steps in the proof of Theorem 4.1, which implies that $P_{\nu^*}(z)$ is the optimal value function. Furthermore, for $t < \tau^{\boldsymbol{w}^*}$, $\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(t))$ is the optimal investment strategy and $Z_{\nu^*}^{\boldsymbol{w}^*}(t)$ is the optimal benchmarked wealth process. □

From the above results, we next show by borrowing the arguments in Browne (1999) how the change in market favourability affects the investment behaviour. In this respect, we form a relationship between risk and favourability via presenting the link between γ^- and the favourability parameter.

Corollary 5.1. *We obtain via direct manipulation of the terms of* γ^- *specified in the previous problem:*

$$\begin{cases} \gamma^- < -1, & \textit{when } \theta_{\nu^*} > 0; \\ \gamma^- = -1, & \textit{when } \theta_{\nu^*} = 0; \\ \gamma^- > -1, & \textit{when } \theta_{\nu^*} < 0. \end{cases}$$

From the above corollary, we observe that when the markets are favourable, investor takes more risk to minimize the expected time to beat a benchmark than she would when maximizing the probability of beating the benchmark. However, the investor needs to wait more when the constraints bind as θ must exceed the penalty arising due to constraints. On the other hand, under the unfavourable markets, the investor pursues more aggressive strategy for maximizing the probability of beating the benchmark than she would for maximizing the survival time. Note that when the constraints are binding she would still pursue such strategy even if θ may be positive.

In sum, under binding constraints, the favourability parameter θ of the unconstrained markets must be larger than the penalty value determined via the optimal fictitious parameter. If this is so, then, as explained in Browne (1999), *bold play* maximizes the probability of beating the benchmark while *timid play* minimizes the expected time to beat the benchmark. However, if θ is below the penalty value then timid play maximizes the probability of beating the benchmark and bold play maximizes the time to ruin.

5.3 Maximizing/Minimizing Expected Discounted Rewards

In the first two sections we solved problems related to maximizing the probability of beating a benchmark and minimizing (maximizing) the time to beat (to stay above) a benchmark. We also showed the impact of the market favourability to the investment decision. In this section, we will consider expected discounted reward and penalty problems. We define the problems via two different value functions. We will see that each problem has different solution and the choice of the value function depends on the favourability of the market. More clearly, the investor may choose to maximize the expected discounted reward (goal reaching problem) or minimize the expected discounted penalty (survival problem) given the favourability of the market. To express the reward problem mathematically we write

$$\bar{R}_\nu = \sup_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[e^{-\rho\tau_U}], \tag{5.32}$$

and for the penalty problem we write

$$\underline{R}_\nu = \inf_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[e^{-\rho\tau_L}], \tag{5.33}$$

where $\rho > 0$ is the subjective discount rate. We present the solutions of the above problems in the theorems that follow. We first consider the maximization problem then give the results of the minimization problem.

Theorem 5.3. *Let $Z_\nu^{\boldsymbol{w}}$ be solution to (4.6) and set $Z_\nu^{\boldsymbol{w}}(0) = z \in [L, U]$. If η_2 and $\hat{\eta}_2$, defined in (5.44) and (5.46) respectively, are such that $-1 < \eta_2, \hat{\eta}_2 < 0$, and the portfolio is traded in a favourable market in sense that $\theta > \frac{1}{2}\left(1+\frac{1}{\eta}\right)^2 \|\zeta\|^2$ and $\theta_\nu > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}}\right)^2 \|\zeta\|^2$, then the vector of optimal fictitious parameters is*

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \text{if } Q - \frac{1}{\eta_2}D < 1; \\ \frac{1}{K}\left((-1+Q)\hat{\eta}_2 - D\right)\underset{\sim}{1}, & \text{if } Q - \frac{1}{\eta_2}D \geq 1, \end{cases} \tag{5.34}$$

where η_2 and $\hat{\eta}_2$ are defined in (5.44) and (5.46) respectively. Furthermore the optimal value function $\bar{R}_{\nu^*} \in [0, 1]$ is

$$\bar{R}_{\nu^*} = \begin{cases} \left(\frac{z}{U}\right)^{\eta_2+1}, \text{if } Q - \frac{1}{\eta_2}D < 1; \\ \left(\frac{z}{U}\right)^{\hat{\eta}_2+1}, \text{if } Q - \frac{1}{\eta_2}D \geq 1, \end{cases} \tag{5.35}$$

and the optimal investment strategy is given by

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'b - \frac{1}{\eta_2}(\sigma^{-1})'\zeta, & \text{if } Q - \frac{1}{\eta_2}D < 1; \\ (\sigma^{-1})'b - \frac{1}{\hat{\eta}_2}(\sigma^{-1})'\left(\zeta + \frac{1}{K}\left((-1+Q)\hat{\eta}_2 - D\right)\sigma^{-1}\underset{\sim}{1}\right), & \text{if } Q - \frac{1}{\eta_2}D \geq 1. \end{cases} \tag{5.36}$$

Under this strategy the optimal portfolio process becomes

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta - \frac{1}{2}\left(1+\frac{1}{\eta_2}\right)^2\|\zeta\|^2\right)t - \frac{1}{\eta_2}\zeta' B(t) - \beta B^{(N+1)}(t)\right], \\ \qquad\qquad \text{if } Q - \frac{1}{\eta_2}D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_2}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t - \frac{1}{\hat{\eta}_2}\zeta'_{\nu^*}B(t) \right. \\ \qquad \left. -\beta B^{(N+1)}(t)\right], \qquad \text{if } Q - \frac{1}{\eta_2}D \geq 1, \end{cases} \tag{5.37}$$

where $\zeta_{\nu^*} = \zeta + \sigma^{-1}\frac{1}{K}\left((-1+Q)\hat{\eta}_2 - D\right)\underset{\sim}{1}$.

As we observe from the above theorem, the optimal investment is again a constant proportional strategy that is independent from the boundary level U. Here, because $-1 < \eta_2, \hat{\eta}_2 < 0$, the optimal strategy is riskier than the growth optimal strategy. In other words, when maximizing the discounted reward, an investor benefits from the favourable market by taking more risk than she would with the growth maximizing strategy; we observe that the optimal strategy becomes the growth maximizing strategy when $\eta_2 = -1$ (or $\hat{\eta_2} = -1$).

Furthermore, from the above theorem we see that the possibility of following a riskier strategy is possible when when $\theta > \frac{1}{2}\left(1+\frac{1}{\eta}\right)^2\|\zeta\|^2$ and $\theta_\nu > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}}\right)^2\|\zeta\|^2$.

Especially, under the binding constraints, when $\theta_{\nu^*} > 0$, the property

$$\theta > \frac{(1-Q-D)^2}{2K} + \frac{1}{2}\left(1 + \frac{1}{\hat{\eta}_2}\right),$$

indicating an even larger shift in the positive direction for the unconstrained favourability parameter. Therefore, the markets must be favourable enough so that the investor can maximize the value of her reward under a highly risky strategy.

Theorem 5.4. *Let $Z_\nu^{\boldsymbol{w}}$ be solution to (4.6) and set $Z_\nu^{\boldsymbol{w}}(0) = z \in [L, U]$. If η_1 and $\hat{\eta}_1$, defined in (5.44) and (5.46) respectively, are such that $\eta_1, \hat{\eta}_1 < -1$, and the portfolio is traded in an unfavourable market, then the vector of optimal fictitious parameters is*

$$\nu^* = \begin{cases} \underset{\sim}{0}, & \text{if } Q - \frac{1}{\eta_1}D < 1; \\ \frac{1}{K}\left((-1+Q)\hat{\eta}_1 - D\right), & \text{if } Q - \frac{1}{\eta_1}D \geq 1. \end{cases} \tag{5.38}$$

Then, the optimal value function $\underline{R}_{\nu^*} \in [0, 1]$ is given by

$$\underline{R}_{\nu^*} = \begin{cases} \left(\frac{L}{z}\right)^{\eta_1+1}, \text{if } Q - \frac{1}{\eta_1}D < 1; \\ \left(\frac{L}{z}\right)^{\hat{\eta}_1+1}, \text{if } Q - \frac{1}{\eta_1}D \geq 1, \end{cases} \tag{5.39}$$

along with the optimal investment strategy

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'b - \frac{1}{\eta_1}(\sigma^{-1})'\zeta, & \textit{if } Q - \frac{1}{\eta_1}D < 1; \\ (\sigma^{-1})'b - \frac{1}{\hat{\eta}_1}(\sigma^{-1})'\left(\zeta + \frac{1}{K}\left((-1+Q)\hat{\eta}_1 - D\right)\sigma^{-1}\underset{\sim}{1}\right), & \textit{if } Q - \frac{1}{\eta_1}D \geq 1. \end{cases} \tag{5.40}$$

Under this strategy the optimal portfolio process becomes

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta - \frac{1}{2}\left(1+\frac{1}{\eta_1}\right)^2\|\zeta\|^2\right)t - \frac{1}{\eta_1}\zeta'B(t) - \beta B^{(N+1)}(t)\right], \\ \qquad\qquad \textit{if } Q - \dfrac{1}{\eta_1}D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_1}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t - \frac{1}{\hat{\eta}_1}\zeta_{\nu^*}'B(t) \right. \\ \qquad\qquad \left. -\beta B^{(N+1)}(t)\right], \qquad \textit{if } Q - \dfrac{1}{\eta_1}D \geq 1, \end{cases} \tag{5.41}$$

where $\zeta_{\nu^*} = \zeta + \sigma^{-1}\frac{1}{K}\left((-1+Q)\hat{\eta}_1 - D\right)\underset{\sim}{1}$.

The constant proportional strategy for minimizing the discounted penalty is also equal to the growth optimal strategy for asymptotic values of η_1 and $\hat{\eta}_1$ or i.e. -1. However, since $\eta_1, \hat{\eta}_1 < -1$, an investor takes less risk than she would with the growth optimal strategy. In other words, along with the results of the maximization problem, we observe that when the markets are unfavourable *timid play* minimizes the expected discounted penalty and when the markets are favourable, *bold play* maximizes the expected discounted reward.

Proof. The problem in consideration can be obtained by taking $\rho(\cdot) = \rho,\ q(\cdot) = 0$ and $H(U) = 1$ (or $H(L) = 1$ for minimization) in (4.8). Since we used (4.8) to derive the *Main Theorem*, applying the same replacements in Equation 4.10 gives us the PDE for which R_{ν^*} is solution to

$$\begin{cases} -\rho R_{\nu^*} + (\hat{r} + b'\zeta) z R'_{\nu^*} - \frac{1}{2}\|\zeta\|^2 \frac{(R'_{\nu^*})^2}{R''_{\nu^*}} + \frac{1}{2}\beta^2 z^2 R''_{\nu^*}, & \text{if } \underset{\sim}{1}'\boldsymbol{w}(z) < 1; \\ -\rho R_{\nu^*} + \left(\hat{r} + b'\zeta - \frac{D}{K}(-1+Q)\right) z R'_{\nu^*} & \\ \qquad + \frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right) z^2 R''_{\nu^*} - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(R'_{\nu^*})^2}{R''_{\nu^*}} = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}(z) \geq 1. \end{cases} \tag{5.42}$$

To proceed one step ahead we guess a solution to both $\bar{R}_{\nu^*}$ and $\underline{R}_{\nu^*}$ in form of $Cz^{\eta+1}$ where C and η are constants. After the substitution, we obtain the following cubic equations

$$\begin{cases} \eta^3 + \psi_1 \eta^2 + \psi_2 \eta + \psi_3 = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}(z) < 1; \\ \hat{\eta}^3 + \Psi_1 \hat{\eta}^2 + \Psi_2 \hat{\eta} + \Psi_3 = 0, & \text{if } \underset{\sim}{1}'\boldsymbol{w}(z) \geq 1, \end{cases} \tag{5.43}$$

where

$$\psi_1 = \frac{\frac{1}{2}\beta^2 + \hat{r} + b'\zeta}{\frac{1}{2}\beta^2}; \quad \psi_2 = \frac{\hat{r} + b'\zeta - \rho - \frac{1}{2}\|\zeta\|^2}{\frac{1}{2}\beta^2}; \quad \psi_3 = -\frac{\|\zeta\|^2}{\beta^2},$$

and

$$\begin{aligned} \Psi_1 &= \frac{\frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right) + \hat{r} + b'\zeta - \frac{D}{K}(-1+Q)}{\frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)}; \\ \Psi_2 &= \frac{\hat{r} + b'\zeta - \frac{D}{K}(-1+Q) - \rho - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)}{\frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)}; \\ \Psi_3 &= -\frac{\frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)}{\frac{1}{2}\left(\frac{1}{K}(-1+Q)^2 + \beta^2\right)}. \end{aligned}$$

To solve the equations we will follow the steps outlined in Yener (2014) (see also Weisstein (2003)). We define discriminants of the two cases by

$$\Delta_1 = \frac{1}{3}(3\psi_2 - \psi_1^2); \qquad \Lambda_1 = \frac{1}{3}(3\Psi_2 - \Psi_1^2);$$
$$\Delta_2 = \frac{1}{27}(2\psi_1^3 - 9\psi_1\psi_2 + 27\psi_3); \qquad \Lambda_2 = \frac{1}{27}(2\Psi_1^3 - 9\Psi_1\Psi_2 + 27\Psi_3),$$

where Δ_1, Δ_2 are discriminants of the first case of (5.43) and Λ_1, Λ_2 are of the second case. For both cases, there exists three unequal real roots if and only if $\frac{\Delta_2^2}{4} + \frac{\Delta_1^3}{27} < 0$, and $\frac{\Lambda_2^2}{4} + \frac{\Lambda_1^3}{27} < 0$, respectively. Then the roots η_i and $\hat{\eta}_i$ are

$$\eta_i = 2\sqrt{-\frac{\Delta_1}{3}} \cos\left(\frac{\phi}{3} + \frac{2k\pi}{3}\right) - \frac{\psi_1}{3}, \tag{5.44}$$

where

$$\phi = \begin{cases} \arccos\left(-\sqrt{\frac{\Delta_2^2/4}{-\Delta_1^3/27}}\right), & \text{if } \Delta_2 > 0; \\ \arccos\left(\sqrt{\frac{\Delta_2^2/4}{-\Delta_1^3/27}}\right), & \text{if } \Delta_2 < 0, \end{cases} \tag{5.45}$$

and

$$\hat{\eta}_i = 2\sqrt{-\frac{\Lambda_1}{3}} \cos\left(\frac{\phi}{3} + \frac{2k\pi}{3}\right) - \frac{\Psi_1}{3}, \tag{5.46}$$

where

$$\phi = \begin{cases} \arccos\left(-\sqrt{\frac{\Lambda_2^2/4}{-\Lambda_1^3/27}}\right), & \text{if } \Lambda_2 > 0; \\ \arccos\left(\sqrt{\frac{\Lambda_2^2/4}{-\Lambda_1^3/27}}\right), & \text{if } \Lambda_2 < 0. \end{cases} \tag{5.47}$$

Say the roots of two equations are such that

$$\eta_1 < -1 < \eta_2 < 0 < \eta_3, \qquad \hat{\eta}_1 < -1 < \hat{\eta}_2 < 0 < \hat{\eta}_3.$$

Since for η_2 and $\hat{\eta}_2$, R_{ν^*} is concave increasing and for η_1 and $\hat{\eta}_1$ it is convex decreasing, then the value functions are given in the following form:

$$\underline{R}_{\nu^*} = C_1 z^{\eta_1+1}, \quad \text{and} \quad \bar{R}_{\nu^*} = C_2 z^{\eta_2+1}.$$

Next, we guess a solution for both value functions that satisfies the boundary conditions which are

$$\underline{R}_{\nu^*}(z) = \left(\frac{L}{z}\right)^{\eta_1}; \quad \bar{R}_{\nu^*}(z) = \left(\frac{z}{U}\right)^{\eta_2}. \tag{5.48}$$

Now we can find the vector of optimal fictitious parameters by replacing above equations and their first and second order derivatives into (4.11). Thus, we obtain for both problems

$$\boldsymbol{w}^* = \begin{cases} (\sigma^{-1})'b - \frac{1}{\eta_1}(\sigma^{-1})'\zeta, & \text{if } Q - \frac{1}{\eta_1}D < 1; \\ (\sigma^{-1})'b - \frac{1}{\hat{\eta}_1}(\sigma^{-1})'\left(\zeta + \frac{1}{K}\left((-1+Q)\hat{\eta}_1 - D\right)\sigma^{-1}\underset{\sim}{1}\right), & \text{if } Q - \frac{1}{\eta_1}D \geq 1. \end{cases} \tag{5.49}$$

Under this strategy the optimal portfolio process becomes

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = \begin{cases} Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta - \frac{1}{2}\left(1+\frac{1}{\eta}\right)^2\|\zeta\|^2\right)t - \frac{1}{\eta}\zeta' B(t) - \beta B^{(N+1)}(t)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } Q - \frac{1}{\eta}D < 1; \\ Z_{\nu^*}^{\boldsymbol{w}^*}(0)\exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\hat{\eta}}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t - \frac{1}{\hat{\eta}}\zeta'_{\nu^*}B(t) \\ \qquad\qquad -\beta B^{(N+1)}(t)\right], \qquad \text{if } Q - \frac{1}{\eta}D \geq 1, \end{cases} \tag{5.50}$$

where

$$\theta_{\nu^*} = \theta - (1-Q-D)\frac{\hat{\eta}}{K}\left(-1+Q-\frac{D}{\hat{\eta}}\right) + \frac{\hat{\eta}^2}{2K}\left(-1+Q-\frac{D}{\hat{\eta}}\right)^2 \tag{5.51}$$

To verify that the above results are optimal, we will follow the same steps as in previous sections. When $Q - \frac{1}{\eta}D \geq 1$

$$\begin{aligned} \underset{\sim}{1}'\boldsymbol{w}^* &= \underset{\sim}{1}'(\sigma^{-1})'b - \frac{1}{\hat{\eta}}\underset{\sim}{1}'(\sigma^{-1})'\left(\zeta + \frac{1}{K}\left((-1+Q)\hat{\eta} - D\right)\sigma^{-1}\underset{\sim}{1}\right) \\ &= Q - \frac{1}{\hat{\eta}}D - K\frac{1}{K}(-1+Q-\frac{1}{\hat{\eta}}D) \\ &= 1. \end{aligned}$$

This implies that $\boldsymbol{w}^* \in \mathcal{K}$, and thus $\delta(\nu^*) + \boldsymbol{w}^*(z)'\nu_1^*\underset{\sim}{1} = 0$ implies $\nu^* \in \tilde{\mathcal{K}}$. On the other hand for $Q - \frac{1}{\eta}D = \underset{\sim}{1}'\boldsymbol{w}^* < 1$, by definition, we have $\boldsymbol{w}^* \in \mathcal{K}$ and $\nu \in \tilde{\mathcal{K}}$.

We have both $\boldsymbol{w}(\cdot)$ and $\nu(\cdot)$, $\mathcal{F}_t$-progressively measurable processes. Therefore, for admissibility we will only check if the variables are square integrable. As in the previous problems ,since $\boldsymbol{w}^*$ and ν^* are constant vectors, $\int_0^{\tau^{w^*}} \|\boldsymbol{w}^*\|^2 \mathrm{d}t < \infty$

and $\int_0^{\tau^{\boldsymbol{w}^*}} \|\nu^*\|^2 \mathrm{d}t < \infty$ if and only if $\tau^{\boldsymbol{w}^*}$ is finite. For a maximization problem, it is meaningless to aim upper goal when the drift of $Z_{\nu^*}^{\boldsymbol{w}^*}(t)$; $\Theta_{\nu^*}(\eta) = \theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\eta}\right)^2 \|\zeta_{\nu^*}\|^2 < 0$. In such a market, $\boldsymbol{w}^*$ and ν^* are admissible if $\tau_L^{\boldsymbol{w}^*} < \infty$. Likewise, when $\Theta_{\nu^*}(\eta) > 0$ we should check if $\tau_U^{\boldsymbol{w}^*} < \infty$. Then to see if they are admissible, we call the probabilities

$$\mathbb{P}_z(Z_{\nu^*}^{\boldsymbol{w}^*} \leq L) = N\left(\frac{\log\left(\frac{L}{z}\right) - \Theta_{\nu^*}(\eta)t}{\sqrt{\frac{1}{\eta^2}(\|\zeta_\nu\|^2 + \beta^2)t}}\right), \tag{5.52}$$

and

$$\mathbb{P}(Z_{\nu^*}^{\boldsymbol{w}^*}(t) \geq U) = 1 - N\left(\frac{\log\left(\frac{U}{z}\right) - \Theta_{\nu^*}(\eta)t}{\sqrt{\frac{1}{\eta^2}(\|\zeta_\nu\|^2 + \beta^2)t}}\right). \tag{5.53}$$

It is clear that when $\Theta < 0$, as $t \to \infty$, former converges to 1, then $\boldsymbol{w}^*$ and ν^* are admissible for a minimization problem. On the other hand when the drift is positive, as $t \to \infty$, $\mathbb{P}(Z_{\nu^*}^{\boldsymbol{w}^*} \geq U) \to 1$ which again implies admissibility.

Finally, to verify that the above results are optimal, we have to show that both value functions are dominated by some other functions. Recall from Theorem 4.1 that G_{ν^*} is bounded above with $c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$. Hence by choosing $c_1 = 1$ and $c_2 = c_3 = 0$ will be enough to show boundedness. The rest of the verification theorem follows the same steps in Theorem 4.1. □

Chapter 6

Conclusion

We applied the techniques of stochastic optimal control on three problems considering a portfolio manager who is evaluated by her performance against a benchmark and has no chance for borrowing. To address the borrowing constraint, we constructed an auxiliary market by utilizing fictitious parameters, acting as Lagrange multipliers, which allows the portfolio to be traded as if there are no constraints and is identical with the actual market when constraints are not binding. Then, we minimized the solutions found under the auxiliary market over all fictitious parameters in order to obtain the optimal results under the constrained markets.

Our solutions show that the optimal investment laws are constant proportional strategies that share the same structure (i.e. independent from the wealth level and barriers U and L). However, the portfolio manager's goal changes in accordance with an indicator, favourability parameter, that shows the favourability of the market. For instance, if the markets are unfavourable, the manager cannot aim to minimize the expected hitting time or maximize expected reward.

Furthermore, we saw that the behaviour of the manager against three objectives are related. The manager follows growth optimal portfolio to minimize/maximize the expected time for each case of the market favourability. On the other hand, if the manager is rewarded as in the third problem, she follows a strategy similar to the fractional (multiple of) growth optimal strategy, and plays safer (riskier) if markets are unfavourable (favourable). On the contrary, to maximize the probability, she would follow a risky strategy if the markets are unfavourable, but invests safely otherwise.

The major contributing result of the paper is to show that the constrained case has its own exclusive parameter. For instance, when the constraints bind, the portfolio manager faces a different circumstance designated by another favourability parameter which is less than the unconstrained one by a positive constant, $(1 - Q - D)^2/2K$. Therefore, there may be some cases when the manager cannot afford to aim the upper goal while she could if borrowing was allowed.

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Appendix A

Dominated Convergence Theorem

We present Dominated Convergence Theorem as outlined in Shreve (2004). Let $f_1, f_2, \dots f_n$ be a sequence of Lebesgue-integrable functions converges to a Lebesgue-integrable function f almost everywhere. If f_n are **dominated** by a Lebesgue-integrable function g, i.e. $|f_n| < g$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

The above property infers the following implication: Let $X_1, X_2, \dots, X_N$ be a sequence of random variables converging to a random variable X almost surely and that X_n dominated by Y such as $\mathbb{E}[Y] < \infty$, i.e. $|X_n| \leq Y, \ \forall n$. Then

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$$

Appendix B

Addendum to Auxiliary Market

We will show why $Z_{\nu}^{\boldsymbol{w}}(t) \geq Z^{\boldsymbol{w}}(t)$ $\forall \nu \in \tilde{\mathcal{K}}$ Lebesgue almost everywhere for $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$ by following the outline from Karatzas and Shreve (1998). We start by defining

$$\mathcal{Z}(t) = \tilde{Z}_{\nu}^{\boldsymbol{w}}(t) - \tilde{Z}^{\boldsymbol{w}}(t), \tag{B.1}$$

where $\tilde{Z}_{\nu}^{\boldsymbol{w}}(t)$ and $\tilde{Z}^{\boldsymbol{w}}(t)$ are discounted values of $Z_{\nu}^{\boldsymbol{w}}(t)$ and $Z^{\boldsymbol{w}}(t)$. We can rewrite $\mathcal{Z}(t)$ as

$$\begin{aligned}\mathcal{Z}(t) := &\int_0^t \mathcal{Z}(s)(\hat{r} + \delta(\nu))\mathrm{d}s + \int_0^t \mathcal{Z}(s)\boldsymbol{w}'(s)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\mathrm{d}s \\ &+ \int_0^t \mathcal{Z}(s)(\boldsymbol{w}'(s)\sigma - b')\mathrm{d}B(s) - \int_0^t \mathcal{Z}(s)\,\beta\,\mathrm{d}B^{(N+1)}(s) \\ &+ \int_0^t \tilde{Z}^{\boldsymbol{w}}(s)\left(\delta(\nu) + \boldsymbol{w}'(s)\nu\right)\mathrm{d}s.\end{aligned} \tag{B.2}$$

Next, we define

$$\begin{aligned}\mathcal{H}(t) := \exp\Bigg\{ \int_0^t -(\hat{r} + \delta(\nu))\mathrm{d}s - \int_0^t \boldsymbol{w}'(s)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\mathrm{d}s \\ - \int_0^t (\boldsymbol{w}'(s)\sigma - b')\mathrm{d}B(s) + \int_0^t \beta\,\mathrm{d}B^{(N+1)}(s) \\ - \frac{1}{2}\int_0^t (\|\boldsymbol{w}'(t)\sigma - b'\|^2 + \beta^2)\mathrm{d}s \Bigg\}.\end{aligned} \tag{B.3}$$

Thus, differential of the product is

$$\mathrm{d}\left(\mathcal{Z}(t)\mathcal{H}(t)\right) = \mathcal{H}(t)\tilde{Z}^{\boldsymbol{w}}(t)\left(\delta(\nu) + \boldsymbol{w}'(t)\nu\right)\mathrm{d}t. \tag{B.4}$$

The above equation is non-negative since $\mathcal{Z}(0) = 0$ and, by definition,$\delta(\nu) + \boldsymbol{w}'\nu \geq 0\ \forall \nu \in \tilde{\mathcal{K}}$ Lebesgue almost every $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$. This induces to $Z_\nu^{\boldsymbol{w}}(t) \geq Z^{\boldsymbol{w}}(t)$. On the other hand, the equality is attained only for a unique vector of minimizing fictitious parameters that minimizes the above equation.

This implies that $\mathcal{A}_c \subseteq \mathcal{A}_\nu$ and hence $V_\nu(z) \geq V(z)\ \forall \nu \in \tilde{\mathcal{K}}$. Therefore, equality can only be obtained by the optimal fictitious parameter, ν^* for which $\delta(\nu^*) + \boldsymbol{w}'\nu = 0\ \forall \nu \in \tilde{\mathcal{K}}$ and $Z_{\nu^*}^{\boldsymbol{w}}(t) = Z^{\boldsymbol{w}}(t)$ Lebesgue almost every for $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$. Thus, for maximization problem we can obtain ν^* by

$$\nu^* = \arg \inf_{\nu \in \mathcal{D}} V_\nu(z),$$

and for the minimization problem

$$\nu^* = \arg \sup_{\nu \in \mathcal{D}} V_\nu(z).$$