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Compromise Rules Revisited

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Abstract

Decision makers often face a dilemma when they have to arbitrate between the quantity of support for a decision (i.e., the number of people who back it) and the quality of support (i.e., at which level to go down in voters' preferences to obtain sufficient level of support). The trade-off between the quality and quantity of support behind alternatives led to numerous suggestions in social choice theory: without being exhaustive we can mention Majoritarian Compromise, Fallback Bargaining, Set of Efficient Compromises, Condorcet Practical Method, Median Voting Rule, Majority Judgement. Our point is that all these concepts share a common feature which enables us to gather them in the same class, the class of compromise rules, which are all based upon elementary scoring rules described extensively by Saari. One can exploit his results to analyze the compromise rules with relative ease, which is a major point of our paper.

Keywords Compromise · Voting · Scoring rules · Borda · Condorcet · Saari

JEL Classification D71

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1 Introduction

There is a widespread interest in social choice theory for studying rules based on the trade-off between the quality and quantity of support behind alternatives. To be more concrete, the kth quality support of an alternative is the number of voters who rank that alternative among their first k best. So the trade-off is clear: It is possible to increase the number of voters (i.e., quantity of support) behind an alternative at the expense of going below in individual preferences by a higher choice of k—which we interpret as a support of a lower quality.

The plurality rule, which is one of the oldest ways of making decisions, has a clear bias in this trade-off. It selects the alternatives considered as best by the highest number of voters. In other words, it insists on a support of first and highest quality, disregarding the quantity of support this may lead to. Not surprisingly, this rule is well-known of being able to lead to outcomes with an unsatisfactory public support: For example, in a five candidate contest, a winner can be selected with only 21% of the ballots, while 79% of the voters rank him/her last in their preferences!

There is a vast literature on alternative social choice rule proposals aiming to recover this deficiency. This dates back to de Borda (1781) and de Caritat (1785) who are among the earliest criticizers of this aspect of plurality. The proposals based on the trade-off between the quality and quantity support are relatively more recent members of this literature.

To give a non-exhaustive list of these, we can start with the method suggested by James W. Bucklin. Several variants were used by a handful of US states during the first half of twentieth century. They all proposed to take into account voters' second best, or sometimes even third best alternatives, to reach a majority of votes. Sertel's Majoritarian Compromise (MC) (Sertel 1986), which is treated in details by Sertel and Yilmaz (1999), develops the same idea more precisely. MC picks alternatives receiving a majority support at the highest possible quality while ties are broken according to the quantity of support these receive. Notice that this rule gives up from the quality of support, in order to ensure a majority support behind the selected alternatives. Fortunately, this trade-off is bounded: With m alternatives, there will be always one which receives a majority support at quality $\lceil m/2 \rceil$ (where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x). Another example of a compromise type of rule is the Median Voting Rule (MVR) proposed by Bassett and Persky (1999) and further elaborated by Gehrlein and Lepelley (2003). MVR picks all alternatives receiving a majority support at the highest possible quality. It differs from MC, as it does not break ties according to the quantity of support. So MC is a refinement of MVR.

The concept of median has also been largely advocated by Balinski and Laraki (2007, 2011). In a context where a group of agents has to grade on a given common scale several alternatives, they propose to use the median rule to pick the median grade.[1] In their model, ties should be broken by removing one by one the median grades, until

[1] This method was implemented in December 2016 to select a candidate for the 2017 French presidential election. Via the website https://laprimaire.org/, 50.64% of the 32,685 participants gave the grade "very good" to Charlotte Marchandise, who was selected. Eventually, she could not participate in the 2017 French presidential elections, as she did not get the support of 500 elected officials, a necessary condition to register as a candidate.

one alternative gets a better new median grade than another. If we consider the m ranks of the alternatives as possible grades, with the condition that each grade is used only once, Majority Judgement is immediately transposable as a voting rule.

Fallback Bargaining (FB), introduced by Brams and Kilgour (2001), is a bargaining solution under which bargainers fall back, in lockstep, to less and less preferred alternatives until they reach a unanimous agreement. This approach, carried to a social choice context, leads to a social choice rule which picks alternatives receiving a unanimous support at the highest possible quality. Notice that in case every alternative is ranked last by at least one voter, obtaining a unanimous support will be at a cost of going down to a quality of support at rank m in which case all alternatives are chosen. FB also corresponds to the minimax rule, characterized by Congar and Merlin (2012), which picks the alternatives whose lowest rank among voters' preferences is highest.

Picking any number q of voters, Brams and Kilgour (2001) generalize FB into a procedure that picks the alternatives, called q-approval compromises, which receive the support of q voters at the highest possible quality—breaking ties according to the quantity of support. Note that MC and FB winners are particular cases of q-approval compromises, for q being respectively equal to majority and unanimity. Moreover for $q = 1$, q-approval compromises coincide with the plurality rule (PR) winners.

Another generalization of compromise rules is made through the efficiency in the degree of compromise axiom introduced by Özkal-Sanver and Sanver (2004): They call an alternative an efficient compromise of degree k if there exists no other alternative which receives an at least equal quantity of support of the same or a higher quality. The prime objective of this concept is, rather than determining a unique winner, to select a set of alternatives that can be viewed as admissible compromises. So there is no a priori level for the quality or quantity of support. Any alternative receiving the highest quantity of support at some quality can be considered as a compromise. Nevertheless, it makes sense to give up from the quality of support only if the quantity of support behind the outcome increases. Otherwise there is no gain to consider the lower quality which, in such a case, is classified as inefficient. Thus, at any preference profile, the efficient compromises are alternatives which receive the highest quantity of support at some efficient level of quality.[2] Interestingly, this set of efficient compromises (SEC) coincides with the set of q-approval compromises: Given any number of voters q, all q-approval compromises are efficient compromises. Moreover, every efficient compromise is a q-approval compromise for some appropriate choice of q (see Özkal-Sanver and Sanver 2004).

However, not all rules based on the trade-off between the quality and quantity of support have to pick among efficient compromises. For example, Condorcet's Practical Method (CPM) described by Nurmi (1999), picks the alternative receiving a majority support of first quality, whenever it exists. If there is no such alternative, then those

[2] The term efficient compromise has also been used in the literature by Börgers and Postl (2009). When two persons with opposite preferences have to choose among three alternatives, it is the alternative that maximizes ex ante the weighted sum of von Neumann Morgenstern (vNM) utilities of the voters. Their primary objective is to understand whether this solution is implementable when the vNM utilities are privately observed. Our model is not as precise, as we ignore the utilities of the agents in this paper, and just focus on their ranking; hence, Börgers and Postl's "utilitarian" efficient compromise is not defined in our context.

receiving the highest support of second quality are chosen. We show in the sequel that CPM is not an efficient compromise.

All these social choice rules which seem a priori different share the common feature of searching a social compromise through the trade-off between the quantity and quality of support behind alternatives. We introduce two classes of compromise rules in which all these rules of the literature can be expressed. Moreover, we observe that compromise rules are based upon specific scoring rules, namely elementary scoring rules, which have been extensively described by Saari (1989, 1992, 1994) in a series of masterpiece papers on the properties of scoring rules. In fact, one can exploit the powerful techniques used in these papers to analyze compromise rules with relative ease, which is one of the major points of our analysis.

The rest of this paper is organized as follows. Section 2 presents the basic setting and introduces various classes of compromise rules. Section 3 relates the compromise rules to Saari's framework and presents our main results showing to which extent SEC can differ from the Borda count and the Condorcet criterion. More precisely, the winner of any strict scoring rule (e.g. the Borda count) may not be an efficient compromise whenever $m \geq 4$. Moreover, this is true for any number $p = 2, \ldots, m-2$ of alternatives that SEC may contain! However, if SEC contains a unique element, all the scoring rules will pick it. Furthermore, if the scoring rule is strict (e.g. the Borda count), it picks the unique efficient compromise only. Moreover, for $m \geq 4$, the Condorcet winner may not be an efficient compromise and this is even true if SEC contains p alternatives, $p = 1, \ldots, m-2$. We conclude in Sect. 4.

2 Definition

2.1 Basic Setting

Consider a set of voters $N = \{1, \ldots, n\}$ confronting a non empty finite set of candidates A, with $\#A = m \geq 3$. The *preference* of voter i is a linear order $p_i \in P$ where P is the set of linear orders on A. A *profile of individual preferences* (or simply a *profile*) is denoted by $p \in P^N$. A *social choice correspondence* (SCC) F assigns to each profile p some non-empty subset $F(p)$ of A.

Let $r(x, p_i) = \#\{y \in A \mid y\ p_i\ x\} + 1$ be the rank of x at p_i. A *score vector* is an m-tuple of real numbers $w = (w_1, w_2, \ldots, w_m)$ with $w_1 = 1$, $w_m = 0$ and $w_j \geq w_{j+1}$ for all $j = 1, \ldots, m-1$. A score vector is strict if $w_j > w_{j+1}$ for all $j = 1, \ldots, m-1$. Denoting $n_r(x, p)$ for the number of voters who rank candidate x at rank r in their preferences at profile p, the score of alternative x under the score vector w is defined as

$$S_w(x, p) = \sum_{r=1,\ldots,m} n_r(x, p) \times w_r \tag{1}$$

A *scoring rule* F_w is a SCC that selects at each p the alternatives whose scores are maximal. Axiomatic characterizations of scoring rules have been proposed by Smith

(1973) and Young (1974, 1975). A well-known scoring rule which we use throughout the paper is the Borda count where $w = (1, \frac{m-2}{m-1}, \ldots, \frac{1}{m-1}, 0)$.

The class of *elementary scoring rules* is defined by the family of m-dimensional score vectors $\{w^r\}_{r=1,\ldots,m-1}$ with

$$w^r = (\underbrace{1, 1, 1, \ldots, 1}_{r\ times}, 0, \ldots, 0)$$

The most famous elementary scoring rules are the plurality rule $w^1 = (1, 0, \ldots, 0)$ and the anti-plurality rule[3] $w^{m-1} = (1, \ldots, 1, 0)$. Notice that the case $r = m$ gives the null rule which selects all alternatives whatever the profile is. As $w_1^m = w_m^m$, the null rule does not match our definition of a scoring rule. Nevertheless, it will be used in the forthcoming analysis.

Another tradition, that dates back to the works of Condorcet, is to consider pairwise comparisons to establish a social choice. Let us denote by $N_{a,b}(p)$ the number of voters who prefer a to b in profile p. Alternative a is a Condorcet winner at profile p if $N_{a,b}(p) > N_{b,a}(p)$ for all $b \in A, b \neq a$.

2.2 Compromise Rules and Set of Efficient Compromises

We now give precise definitions of the two different classes of compromise rules we will work with. For any strictly positive integer $r \leq m$ and any $x \in A$, we write $S_r(x, p) = \#\{i \in N \mid r(x; p) \leq r\}$ for the rth quality support of x at p, i.e., the number of voters who rank x among their r best alternatives at p. Note that $S_r(x, p)$ is equivalent to the score $S_w(x, p)$ obtained by x under the scoring vector w^r. For any $q \in \{1, \ldots, n\}$, we denote $F_{q,r}(p) = \{x \in A : S_r(x, p) \geq q\}$ for the alternatives whose rth quality support at p is at least q and $F_r(p) = \{x \in A : S_r(x, p) \geq S_r(y, p)\ \forall y \in A\}$ for the alternatives who receive the maximal rth quality support at p. Note that $F_r(p)$ is the set of winners under the elementary scoring rule w^r.

Given a pair (q, r) where $q \in \{1, \ldots, n\}$ reflects a quantity of support and $r \in \{1, \ldots, m\}$ reflects a rank, we say that r is non-binding for q iff $F_{q,r}(p) \neq \varnothing\ \forall p \in P^N$. Thus, r is binding for q if there are preference profiles where the level of support q cannot be reached at rank r. For example, $r = 1$ is non-binding for $q = 1$, as we are sure that there is always an alternative which is top ranked by one voter while $r = 1$ is binding for $q = \lceil \frac{m}{2} \rceil$, as we may have no alternative with a majority support at level 1. Proposition 1 below immediately follows from Theorem 1 of Brams and Kilgour (2001).

Proposition 1 *The rank r is non-binding for q iff*

$$r \geq \left\lfloor \frac{m(q-1)+n}{n} \right\rfloor. \tag{2}$$

For example, if $m = 12$ and $n = 3$, we are sure that the quota $q = 2$ will be reached latest at rank 5, while this is not the case for any rank lower than 5.

[3] Also called the Veto Rule, or the Negative Plurality Rule in the literature.

For each $p \in P^N$, let $h_q(p) \in \{1, \ldots, r\}$ be such that $F_{q,h_q(p)}(p) \neq \varnothing$ and $F_{q,h'}(p) = \varnothing$ for all $h' < h_q(p)$. So $h_q(p)$ is the smallest rank at which q voters supports some candidate.

We are now ready to define two classes of compromise rules.

A (q, r)*-compromise* is a social choice rule $C_{q,r}$ which is defined at every $p \in P^N$ as

$$C_{q,r}(p) = \begin{cases} F_{q,h_q(p)}(p) & \text{when } F_{q,r}(p) \neq \varnothing \\ F_r(p) & \text{otherwise} \end{cases}$$

A (q, r)-compromise seeks for the first rank $h_q(p) \leq r$ where the threshold q is met and returns all alternatives that meet this threshold. If the threshold cannot be met at r, it just returns the winner under the elementary scoring rule w^r.

Note that when r is non-binding for q, we have $C_{q,r} = F_{q,h_q(p)}$ which we call the q*-partisan compromise rule* that selects all alternatives that reach a support of q for the highest quality. Examples of q-partisan compromise rules include the union of tops ($q = 1$), the Median Voting Rule ($q = \lceil \frac{n}{2} \rceil, r = \lceil \frac{m}{2} \rceil$), and Fallback Bargaining ($q = n, r = m$).

A *refined* (q, r)*-compromise* is a social choice rule $C^*_{q,r}$ which is defined at every $p \in P^N$ as

$$C^*_{q,r}(p) = \begin{cases} F_{h_q}(p) & \text{when } F_{q,r}(p) \neq \varnothing \\ F_r(p) & \text{otherwise} \end{cases}$$

A refined (q, r)-compromise seeks for the first rank $h_q(p) \leq r$ where the threshold q is met and returns all the alternatives that meet this threshold while ties are broken according to the quantity of support these winners receive. If the threshold cannot be met at r, it just returns the winner under the elementary scoring rule w^r.

Again when r is non-binding for q, we have $C^*_{q,r}(p) = F_{h_q}(p)$ which we call the *refined* q*-partisan compromise rule* that selects all the alternatives that reach a support of q for the maximal quality by breaking ties according to the quantity of support. Note that our refined q-partisan compromise is what Brams and Kilgour (2001) call q-approval compromise. Remark that by this tie-breaking, and as the names suggest, the refined q-partisan compromise rule is a subcorrespondence of the q-partisan compromise rule. Examples of refined q-partisan compromise rules include plurality ($q = 1$), the Majoritarian Compromise ($q = \lceil \frac{n}{2} \rceil, r = \lceil \frac{m}{2} \rceil$), and Fallback Bargaining ($q = n, r = m$).

The refined q-partisan compromise rule selects at each profile p, the winners of the elementary scoring rule w^r where $r = h_q(p)$ is the earliest rank where at least one candidate gets a support of q voters. Thus, the choice of w^r depends on p and it is not possible to state a priori at which rank the fallback process will stop. However, for each q, there is a maximal value of r as expressed by Proposition 1. While the same reasoning holds for the q-partisan compromise rule, the winners are the alternatives whose scores exceed a given threshold under the score vector w^r with $r = h_q(p)$.[4]

[4] For more on scoring rules using thresholds, see Saari (1994).

Table 1 A profile where all the ranks are efficient

4 :	a	b	c	d
3 :	b	c	d	a
2 :	c	d	a	b
1 :	d	a	b	c

We now define the set of efficient compromises. At each $p \in P^N$, we define the set of efficient degree of compromises as $ED(p) = \{r \in \{1, \ldots, m\}$: there exists no $r' < r$ with $S_{r'}(x, p) = S_r(x, p)$ for all $x \in F_r(p)\}$. Said differently, r is not efficient if the maximal score has been obtained before at a lower rank r'. We qualify an elementary scoring rule w^r as being efficient at p if $r \in ED(p)$.

The set of efficient compromises (Özkal-Sanver and Sanver 2004) is defined as:

$$SEC(p) = \cup_{r \in ED(p)} F_r(p)$$

So an alternative is an efficient compromise at p iff it is selected by an elementary scoring rule which is efficient at p. Remark that $SEC(p)$ is always non-empty, enabling to introduce the efficiency in the degree of compromise (EC) axiom which requires from a SCC to pick among the efficient compromises. Thus a SCC F satisfies EC iff $F(p) \subseteq SEC(p)$ at every $p \in P^N$.

Below, we present two examples to clarify ED and SEC. Example 1 illustrates a profile where all ranks are efficient whereas at the profile in Example 2 some ranks are inefficient.

Example 1 Let $n = 10$ and $A = \{a, b, c, d\}$ and take $p \in P^N$ as in Table 1.

We read the first line as follows: Each of the 4 voters ranks a as the first, b as the second, c as the third and d as the fourth best.

If we look for a first degree compromise, we have $F_1(p) = \{a\}$ where a receives the highest support of the first quality, namely the support of 4 voters. Note that rank $r = 1$ is efficient at any profile. If the society compromises so as to agree on voters' first or second best alternatives, i.e., $r = 2$; we have $F_2(p) = \{b\}$ where b receives the highest support of the second quality, which is 7. Since the support level increases from 4 to 7, as the degree of compromise increases from 1 to 2, rank $r = 2$ is efficient. One can check that $F_3(p) = \{c\}$ with support of third quality by 9 voters and $F_4(p) = \{a, b, c, d\}$ with support of fourth quality by 10 voters. As the support level always increases by compromising to a lower rank, ranks 3 and 4 are also efficient. Thus, $ED(p) = \{1, 2, 3, 4\}$. Furthermore, we have $SEC(p) = \{a, b, c, d\}$.

Example 2 Let $n = 6$ and $A = \{a, b, c, d\}$ and take $p \in P^N$ as described in Table 2.

One can check that $F_1(p) = \{a\}$, $F_2(p) = \{a, b, c, d\}$, $F_3(p) = \{c\}$ and $F_4(p) = \{a, b, c, d\}$. As increasing the degree of compromise from 1 to 2, leaves the quantity of support intact at 3, rank 2 is inefficient, because one can obtain the same support by compromising less. Nevertheless, increasing the degree of compromise from 2 to 3, increases the quantity of support from 3 to 6 , hence rank 3 is efficient. Finally,

Table 2 A profile with inefficient degrees of compromise

1:	a	b	c	d
1:	a	c	b	d
1:	a	d	c	b
1:	d	c	a	b
1:	c	b	a	d
1:	d	b	c	a

Table 3 Interpretation

r	(q, r)-compromise	Refined (q, r)-compromise
Non-binding	q-partisan compromise (e.g. Union of the tops, Median Voting Rule, Fallback Bargaining)	Refined q-partisan compromise (e.g. Plurality, Majoritarian Compromise, Fallback Bargaining)
Binding	"Coarse" CPM for $q = \lceil \frac{n}{2} \rceil$ and $r = 2$	Condorcet's Practical Method, for $q = \lceil \frac{n}{2} \rceil$ and $r = 2$

increasing the degree of compromise from 3 to 4, leaves the quantity support intact at 6, hence rank 4 is also inefficient. Thus, we have $ED(p) = \{1, 3\}$ which implies $SEC(p) = \{a, c\}$.

In the introduction we briefly mentioned that the set of efficient compromises coincides with the set of refined q-partisan compromises.[5] Now we are ready to state this result in the following proposition more formally.

Proposition 2 (Özkal-Sanver and Sanver 2004) *The refined q-partisan compromise rule satisfies EC at any $q \in \{1, \ldots, n\}$. Moreover, for any $p \in P^N$ and any $x \in A$, if x is an efficient compromise, then x is a refined q-partisan compromise for some $q \in \{1, \ldots, n\}$.*

Interestingly, not only the refined q-partisan compromise rule satisfies EC but also every alternative which is an efficient compromise has to be a refined q-partisan compromise for some $q \in \{1, \ldots, n\}$. Note also that, as the refined q-partisan compromise rule is equivalent to pick refined (q, r)-compromises for some appropriate choice of r so that r is non-binding for q, we can rewrite Proposition 2 by replacing the refined q-partisan compromise rule with refined (q, r)-compromises where r is non-binding for q.

Table 3 suggests a classification of the rules discussed in the introduction in terms of (q, r)-compromises or refined (q, r)-compromises. In this table, CPM, which selects the w^1 winner if it gets the support of a majority and the w^2 winner otherwise, stands as a refined (q, r)-compromise with $q = \lceil \frac{n}{2} \rceil$ and $r = 2$ where r is binding for q. The coarser counterpart of CPM picks the two majoritarian alternatives (when they exist)

[5] Recall that our refined q-partisan compromise is what Brams and Kilgour (2001) call q-approval compromise.

when we have to move up to rank 2. Note that while CPM and its coarser version are examples within the class to which they belong, the q-partisan compromise rule and the refined q-partisan compromise rule fully characterize the class they represent.

3 Discrepancies Among Voting Rules

In this section, we elaborate on the analysis of the rules we have presented, by showing to which extent compromise rules and the set of efficient compromises can differ from the Borda count and the Condorcet criterion. By using examples, it has already been shown in Özkal-Sanver and Sanver (2004) that several compromise rules fail to satisfy efficiency in the degree of compromise. Here, we go a step further and describe how one can design examples where a given rule selects a set B of alternatives while another one selects another set C, with almost no constraint on the size of these sets. Our proofs rely on Saari's results on scoring rules (Saari 1989, 1992, 1994) and can be viewed as illustrations of the power of his techniques. First, we recall some of his results, and next, we apply them to the set of efficient compromises.

3.1 Set of Profiles as a Simplex

The key point in Saari's approach is the following: Under some mild conditions, the set of profiles is identified with the unit simplex of dimension $m!$, $Si(m!)$. Then, with an appropriate normalization of the scoring vectors (all the coordinates of a vector are multiplied by the same scalar so that they add up to one) all scoring rules can be considered as linear mapping from the simplex $Si(m!)$ into the unit simplex of dimension m, $Si(m)$.

More precisely, when $|A| = m$, there are $m!$ possible preference types on A . As scoring rules are anonymous (a permutation of the names of the voters does not affect the final outcome) and homogeneous (a replication of the preferences of each voter k times, $k \in N$, to create a population of kn voters, does not affect the result of the voting process),[6] we can directly consider the vectors $v = (v_1, \dots, v_{m!}) \in R^{m!}$, where v_t is the fraction of voters whose preference is of type t, as the input of the SCCs. The set of all the profiles is now identified with the set of rational points v in the unit simplex of $R^{m!}$, and is denoted by $Si(m!)$.

It is easy to see from Eq. (1) that the multiplication of the coordinates of a score vector w by the same scalar $\theta > 0$ also multiplies by θ the total scores of any alternative:

$$S_{\theta w}(x, v) = \sum_{r=1,\dots,m} n_r(x, v) \times \theta w_r = \theta S_w(x, v) \tag{3}$$

Thus, the ranking of the candidates by the scoring rule F_w coincides with the ranking obtained by the rule $F_{\theta w}$. Hence, it is possible to associate to each score vector w its normalized version $\hat{w}$ by dividing all its coordinates by $\sum_{j=1}^{m} w_j = 1$. In particular,

[6] See Young (1974, 1975)

for any $r = \{1, \ldots, m-1\}$, the elementary score vector w^r can be associated to the score vector

$$\hat{w}^r = \frac{1}{r}(\underbrace{1, \ldots, 1}_{r\ times}, 0, \ldots, 0).$$

It is now easy to see that any normalized score vector can be expressed as a convex combination of the elementary score vectors. So all the normalized score vectors lie in the convex hull defined by $\hat{w}^r$. Thus, we have

$$\hat{w} = \sum_{r=1}^{m-1} \lambda^r \hat{w}^r,\ \lambda^r \geq 0\ \forall r = 1, \ldots, m-1,\ \sum_{r=1}^{m-1} \lambda^r = 1. \tag{4}$$

Note that $\lambda^r > 0\ \forall r = 1, \ldots, m-1$ corresponds to the case where the scoring rule in question is strict. For example, the normalized Borda vector is strict and $\hat{w}_B = \frac{2}{m(m-1)}(m-1, m-2, \ldots, 1, 0)$ can be written as a convex combination of all the $\hat{w}^r$'s:

$$w_B = \sum_{r=1}^{m-1} \frac{2(m-r)\hat{w}^j}{m(m-1)} \tag{5}$$

This normalization also implies that the scores $S_{\hat{w}}(x, v)$ add up to one for every profile in $Si(m!)$. Thus, the vector of the scores for $F_{\hat{w}}$, $S_{\hat{w}}(v) = (S_{\hat{w}}(a_1, v), S_{\hat{w}}(a_2, v), \ldots, S_{\hat{w}}(a_m, v))$, lies in the unit simplex $Si(m)$.

Saari hence proves that the possible relationships among the winners of the different scoring rules are governed by the properties of the elementary scoring rules. Theorem 1 states that at a given profile, the $m-1$ elementary scoring rules can lead to the selection of $m-1$ different winners. In fact, we build Table 4 on this basis.

Theorem 1 (Saari 1992) *In the unit simplex $Si(m)$, there exists a ball $B(i_m, \varepsilon)$ with radius $\varepsilon > 0$ centered on the barycenter point $i_m = (\frac{1}{m}, \ldots, \frac{1}{m})$ with the following property:*

Choose $m-1$ points E_r in $B(i_m, \varepsilon)$, $r = 1, \ldots, m-1$. There exists a profile $v \in Si(m!)$ such that the scores obtained with the elementary scoring rule $\hat{w}^r$ are respectively the ones given by the point E^r, for all $r = 1, \ldots, m-1$.

This result is a consequence of another theorem by Saari (1989), which asserts that the rankings of the alternatives by $m-1$ scoring rules can be completely different for a given profile as long as the $m-1$ scoring vectors are linearly independent. In fact, all Saari type results we use throughout this paper are proved in the same way. To be more precise, consider the profile $i_{m!}$ where each preference type is equally represented:

$$i_{m!} = \left(\frac{1}{m!}, \ldots, \frac{1}{m!}\right)$$

Table 4 A profile which illustrates the discrepancies among elementary scoring rules and (q, r)-compromise rules

1:	a d c f b e	8:	a b f e c d
10:	a b f e d c	1:	c b d e f a
13:	b c d f e a	3:	b c d f a e
1:	b c e d f a	17:	c d e a f b
1:	d f c e b a	14:	d a c b f e
1:	d a c e f b	12:	e f a c b d
3:	e f a c d b	13:	f e b d c a
1:	f e d b c a	1:	f e d c b a

Table 5 The different winners according to the quality of support

Alternatives	Quality of support						Borda	Majority
	1st	2nd	3rd	4th	5th	6th	Scores	$N_{cx}(p)$
a	**19**	34	49	66	69	100	237	51
b	17	**36**	49	64	79	100	246	51
c	18	35	**52**	68	90	100	263	–
d	16	34	**53**	67	80	100	250	58
e	15	30	48	**69**	82	100	244	52
f	15	31	49	66	**100**	100	261	51

For this profile, any neutral SCC selects all alternatives in A. But Saari managed to show that it is possible to slightly modify the preferences in this profile, staying at a distance ε from $i_{m!}$, in order to obtain the desired result for $\hat{w}^k$, while the scores for the other elementary scoring rules are unchanged.[7] Theorem 1 can be proved by performing the operation repeatedly. Thus, there exists a ball of profiles centered around $i_{m!}$, where all the preferences types are almost equally represented, and for which we can obtain the rankings given by the $m - 1$ elementary scoring rules as desired. We will call such profiles Saari's profiles.

Provided that we can derive a compromise rule from elementary scoring rules, we can immediately obtain new results on their relationships and properties. For example, we can immediately infer that many (q, r)-compromises and refined (q, r)-compromises presented in Table 3 (PR, CPM, MC, MVR and FB) can lead to different results simultaneously as long as they are based on different vectors w^r. This is illustrated by the example displayed on Table 4. Consider a profile for $n = 100$ voters and $m = 6$ candidates, labeled by a, b, c, d, e and f. Table 4 reads as follows: There are 10 voters whose preferences are given by the transitive ordering a is preferred to b, b is preferred to f, f is preferred to e, e is preferred to d and d is preferred to c. Table 5 displays the quantity of support at each quality for all alternatives. It can be seen that a is the unique Plurality winner while b is the unique CPM winner. The MVR selects the set $\{c, d\}$, and d is picked by MC . Candidate e would be the winner with $q = 66$ under the refined q-partisan compromise rule. f is the unique FB winner. At last, SEC

[7] This is precisely described for the three candidate case in Saari (1999).

considers $\{a, b, d, e, f\}$ as equally good. Thus, the choice for a best alternative is very sensitive to the trade off between the quality and the quantity of support. Moreover, as already noticed by Özkal-Sanver and Sanver (2004), SEC fails to satisfy the Condorcet criterion or to select the Borda winner. From Table 5, we can check that candidate c, who is not an efficient compromise, is the Borda winner. Moreover, candidate c is also the Condorcet winner (it beats a by 51:49, b by 51:49, d by 58:42, e by 52:48 and f by 51:49). By using Saari's techniques, we will see that this situation can be almost as sophisticated as desired.

3.2 Set of Efficient Compromises and Scoring Rules

First, we show that even when SEC is very large, it may fail to pick the winner of almost any given scoring rule.

Theorem 2 *Consider $m \geq 4$. The winner of any strict scoring rule (e.g. the Borda count) may not be an efficient compromise. In particular, this is true for any number $p = 2, \ldots, m-2$ of alternatives that SEC may contain.*

If SEC contains a unique element, all scoring rules will pick it. Furthermore, if the scoring rule is strict (e.g. the Borda count), it will pick the unique efficient compromise only.

Proof First, consider the case where the set of efficient compromises is a singleton. It means that all the efficient elementary scoring rules select the same winner. Hence, it is also selected by the inefficient elementary scoring rules. As the elementary scoring rules $\hat{w}^k$ form a base for the space of scoring rules, any scoring rule can be expressed as a convex combination of the tallies of the elementary scoring rules (for example, see Eq. (2) for the Borda count case). Hence, all scoring rules will pick this unique efficient compromise. Moreover, if the scoring rule is strict, it will select this alternative as a unique winner.

Even if SEC is a singleton, say $\{a\}$, it may be the case that for some inefficient $\hat{w}^k$, the corresponding scoring rule picks additional alternatives, b, c, etc. Hence, it might be that non strict scoring rules could also pick other alternatives on the top of a.

Next, consider the case where SEC contains p alternatives, $p = 2, \ldots, m-2$. When SEC contains at least two different elements, again it is possible to use the same approach as in Theorem 1 in order to show that it can differ from the result of any given strict scoring rule. We illustrate the general proof by focusing on the case $m = 4$, $A = \{a_1, a_2, a_3, a_4\}$ and the Borda count. First, assume that a_4 is always ranked last, and consider only $\{a_1, a_2, a_3\}$ with the scoring rule $\hat{w}_s = (1-s, s, 0)$. With this normalization, note that $s \in [0, \frac{1}{2}]$, $s = 0$ defines the plurality rule, $s = 1/3$ defines the Borda count, and $s = 1/2$ defines the anti-plurality rule. The image of a profile v by a scoring rule $F_{\hat{w}_s}$ is the point $S_s(v) = (S_s(a_1, v), S_s(a_2, v), S_s(a_3, v))$, where $S_s(a_t, v)$ is the score of alternative a_t with the scoring vector $\hat{w}_s$ at profile v. The point $S_s(v)$ lies in the unit simplex $Si(3)$. With this normalization of the scores, we know from Saari (see for example Geometry of Voting, Saari 1994, p. 56) that:

$$S_s(v) = (1-2s)S_0(v) + 2s\, S_{\frac{1}{2}}(v),\ \ \forall s \in \left[0, \frac{1}{2}\right]. \tag{6}$$

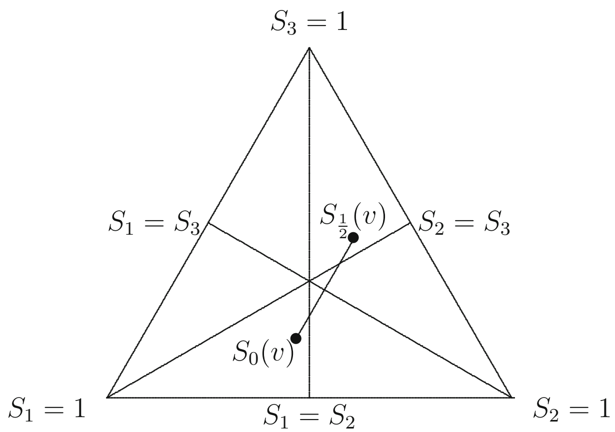

Fig. 1 A possible position of the procedure line in $Si(3)$

Saari calls the line depicted by Eq. 6 the procedure line. Around the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ where all the candidates get the same scores, the points $S_0(v)$ and $S_{\frac{1}{2}}(v)$ can be chosen as desired: There will be always a profile that gives these results (see Geometry of Voting, Saari 1994, Theorem 2.4.2, p. 56). In particular, it is possible to choose $S_0(v)$ and $S_{\frac{1}{2}}(v)$ such that a_1 is chosen by plurality, a_3 is chosen by anti-plurality, and the line connecting both points enters the region where a_2 is the winner for some other given s. Without loss of generality, this can be done for Borda ($s = 1/3$), as seen on Fig. 1. Thus, we have a_2, for example, as a Borda winner, while a_1 and a_3 are respectively the plurality and anti-plurality winners. However, as a_4 is always ranked last, SEC is $\{a_1, a_2, a_3\}$ for a profile v which corresponds to the situation depicted on Fig. 1, as a_1 is the $\hat{w}^1$ winner by more than one third of the votes, a_3 is the $\hat{w}^2$ winner by more than 2/3 of the votes, a_1, a_2 and a_3 are the $\hat{w}^3$ winners by unanimity. But v can correspond to a population as large as desired (say, one million voters), where the changes in preferences of a handful of voters will not have any impact on the final strict rankings. Thus, one voter with a_2 next to the last can switch the positions of a_2 and a_4 without modifying the plurality, Borda and anti-plurality orderings on the set $\{a_1, a_2, a_3\}$. Thus, the fallback process will stop with $\hat{w}^3$, but selecting only $\{a_1, a_3\}$ as now a_2, the Borda winner, fails to reach unanimity by one vote. QED.

Theorem 2 calls for some comments. First,the method we described can be generalized as long as SEC contains $p = 2, \ldots, m-2$ alternatives. We just use an initial profile where $a_1, \ldots, a_p$ are respectively chosen by the corresponding p elementary scoring rules, while a_{p+1} is the Borda winner, and alternative a_t, $t > p+1$ is always positioned at rank t. The trick we used to build these examples does not cover the case where SEC contains exactly $m-1$ alternatives. Secondly, the technique we used can be generalized for any rule which can be expressed as an interior point in the convex hull of elementary scoring rules. To realize it, consider Fig. 1 and $s \in]0, \frac{1}{2}[$. For any such s, by Theorem 1, it is always possible to position the procedure line such that the results for rule $\hat{w}_s$ lie in the a_2 zone, while $S_1(v)$ (resp, $S_3(v)$) lies in the a_1 zone (resp., a_3 zone). Thirdly, although the proof has been developed in the case we have a unique winner for any strict scoring rule (e.g., the Borda count), it can be generalized to the case with the several winners. Using the same reasoning, the reader can show that the set of Borda winners $B \subset A$ and SEC $= C \subset A$ are disjoint as long as $\#B + \#C \leq m - 1$.

3.3 Set of Efficient Compromises and the Condorcet Winner

We now show that, even when SEC is large, it may fail to pick the Condorcet winner.

Theorem 3 *For $m \geq 4$, the Condorcet winner may not be an efficient compromise. In particular, this is true for any number $p = 1, \ldots, m-2$ of alternatives that SEC may contain.*

Proof Our proof is an application of another result by Saari which we quote below:

Theorem 4 (Saari 1989, Theorem 5) *Let $m \geq 4$, and F be a family of subsets of candidates that consists of all $m(m-1)/2$ pairs of candidates and the set of all candidates. Choose $m-2$ linearly independent normalized score vectors whose convex hull does not include the normalized Borda vector. Choose a ranking for each pair of alternatives, and $(m-2)$ rankings for the set of m alternatives. There is a profile of voters so that, for each pair of alternatives, their majority ranking is the selected one, and when their ballots are tallied with the jth vector, the outcome is the jth ranking.*

Although it is not explicitly mentioned in this statement, the proof considers again Saari's profiles, that is profiles v around $I_{m!}$, the point in $Si(m!)$ where all the preference types on $\{a_1, \ldots, a_m\}$ are equally represented. Let us now adapt this result to our case. Consider the set $A' = \{a_1, \ldots, a_{m-1}\}$. We know that there exists a profile v around $I_{(m-1)!}$ such that the rankings for the first elementary scoring rules $\hat{w}^k$, $k = 1, \ldots, m-3$ and for the pairwise comparisons are as desired. Thus, it can be the case that each alternative, a_k is selected with the elementary scoring rule $\hat{w}^k$, $k = 1, \ldots, m-3$, while a_{m-1} is the Condorcet winner. For this profile, we do not know which alternative wins with $\hat{w}^{m-2}$ but we know that each alternative has a normalized score close to $\frac{1}{m-1}$ by definition of the Saari's profiles. Create a profile v' by switching a_{m-2} and a_k, $k = 1, \ldots, m-3$ when a_{m-2} is the last and a_k next to the last. The scores of these alternatives become significantly lower than $\frac{1}{m-1}$, the score of a_{m-1} remains unchanged, and the score of a_{m-2} jumps as he remains last for a fraction of the votes close to $\frac{1}{(m-1)(m-2)}$. Hence, a_{m-2} is the unique winner with w^{m-2}; we have a profile v' where each alternative a_k is the unique winner with $\hat{w}^k$ for $k = \{1, \ldots, m-2\}$ while a_{m-1} is the Condorcet winner.

Now, consider the profile v'' on A, by adding a_m as the bottom candidate for each preference in v'. At this stage, a_{m-1} remains the Condorcet winner, but $SEC = \{a_1, \ldots, a_{m-1}\}$ as all alternatives are selected with $\hat{w}^{m-1}$. Now, as v'' can represent a population as large as we want, we can imagine that this population is large enough for a single switch not to change any scoring or pairwise rankings. Consider $v^\star$, obtained from v'' by switching just for one voter a_m and a_{m-1} when a_{m-1} is next to the last. With this single switch, a_{m-1} is expelled from $SEC(v^\star) = \{a_1, \ldots, a_{m-2}\}$ while a_{m-1} remains the Condorcet winner.

Notice that the same reasoning remains valid even if one of the alternatives in $\{a_1, \ldots, a_{m-2}\}$ is chosen by several elementary scoring rules. In the worst case, a_1 is the unique winner for all elementary scoring rules, $\hat{w}^k$, $k = 1, \ldots, m-1$. It is the unique member of SEC, but fails to be the Condorcet winner. QED.

Although, the proof has been developed in the case we have a (unique) Condorcet winner, it can be generalized to the case with several weak Condorcet winners.[8] Using the same reasoning, the reader can show that the set of Condorcet winners $B \subset A$ and $\mathrm{SEC} = C \subset A$ are disjoint as long as $\#B + \#C \leq m - 1$.

4 Conclusion

Our paper aims to achieve three main objectives. First, it suggests that several rules proposed in the literature over the years and which are based on the trade-off between the quantity and quality of support, can be classified through the concept of compromise rules. As a result, they can be considered on a common ground without the need of being analyzed separately. Second, observing that compromise rules are based on elementary scoring rules, we show that the powerful techniques and results of Saari on scoring rules can be wisely adapted to obtain result on compromise rules. Third, by this adaptation, we show in our Theorems 2 and 3 that compromise rules strongly diverge from both scoring rules and the Condorcet criterion. In that sense, compromise rules can be seen to stand apart from the well-known Borda-Condorcet dichotomy in social choice theory.

One path we have not explored yet is the link between the compromise rules and the concept of Ordered Weighted Average (OWA) which is recently introduced to the literature by Goldsmith et al. (2016). OWAs are rank-dependent scoring rules which associate with each candidate the ordered vector of its ranks in individual preferences. For example, when there are seven voters and six alternatives, associating the vector $(1, 1, 1, 2, 4, 6, 6)$ with candidate a indicates that candidate a is ranked first by three voters, second by one voter, fourth by one voter and last by two voters. The OWA operator attributes a (positive) weight to each coordinate and considers the weighted sum in order to pick the best candidate. Notice that some of the rules we discussed throughout the paper can be expressed as an OWA. For instance, MVR is defined by the weight vector $(0, 0, 0, 1, 0, 0, 0)$, as it only takes into account the median rank of each alternative. In a similar vein, FB is defined by the weight vector $(0, 0, 0, 0, 0, 0, 1)$. Nevertheless, the plurality rule cannot be expressed as an OWA while this is possible for the union of the tops via the weight vector $(1, 0, 0, 0, 0, 0, 0)$. Characterization of q-partisan compromise rules that can be expressed as an OWA is a further open area worth to study.

An obvious open question is whether Saari's work, which contains many more results, can be further elaborated towards a better understanding of compromise rules. As a case in point, one can ask whether the results of Saari on the stability of voting rules when candidates are added or dropped can be used to obtain a complete characterization of the stability of compromise rules.

[8] Alternative a is a weak Condorcet winner at profile p if $N_{a,b}(p) \geq N_{b,a}(p)$ for all $b \in A$, $b \neq a$.

References

Balinski M, Laraki R (2007) A theory of measuring, electing and ranking. Proc Natl Acad Sci USA 104(21):8720–8725

Balinski M, Laraki R (2011) Majority judgment: measuring ranking and electing. MIT Press, Cambridge

Bassett GW, Persky J (1999) Robust voting. Public Choice 99:299–310

Börgers T, Postl P (2009) Efficient compromising. J Econ Theory 144:2057–2076

Brams S, Kilgour DM (2001) Fallback bargaining. Group Decis Negot 10:287–316

Congar R, Merlin V (2012) A characterization of the maximin rule in the context of voting. Theory Decis 72–1:131–147

de Borda J-C (1781) Mémoire sur les Élections au Scrutin. Histoire de l'Académie Royale des Sciences, Paris

de Caritat MJAN (1785) Marquis de Condorcet, Éssai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix, L'Imprimerie Royale, Paris

Gehrlein WV, Lepelley D (2003) On some limitations of the median voting rule. Public Choice 117–2:177–190

Goldsmith J, Lang J, Mattei N, Perny P (2016) Voting with rank dependant scoring rules. AAAI, Menlo Park

Nurmi H (1999) Voting paradoxes and how to deal with them. Springer, Berlin

Özkal-Sanver İ, Sanver MR (2004) Efficiency in the degree of compromise: a new axiom for social choice theory. Group Decis Negot 13:375–380

Saari DG (1989) A dictionary for voting paradoxes. J Econ Theory 48:443–475

Saari DG (1992) Millions of election outcomes from a single profile. Social Choice Welf 9:227–306

Saari DG (1994) Geometry of voting. Springer, Berlin

Saari DG (1999) Explaining all three-alternative voting outcomes. J Econ Theory 87:313–355

Sertel MR (1986) Lectures notes in microeconomics. Bogazici University, unpublished

Sertel MR, Yilmaz B (1999) The majoritarian compromise is majoritarian optimal and subgame perfect implementable. Social Choice Welf 16:615–627

Smith JH (1973) Aggregation of preferences with variable electorate. Econometrica 41:1027–1041

Young HP (1974) A note on preference aggregation. Econometrica 42:1129–1131

Young HP (1975) Social choice scoring functions. SIAM J Appl Math 28:824–838