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Uniqueness cases in odd-type groups of finite Morley rank

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Abstract

There is a longstanding conjecture, due to Gregory Cherlin and Boris Zilber, that all simple groups of finite Morley rank are simple algebraic groups. One of the major theorems in the area is Borovik's trichotomy theorem. The 'trichotomy' here is a case division of the generic minimal counterexamples within *odd type*, that is, groups with a large and divisible Sylow° 2-subgroup. The so-called 'uniqueness case' in the trichotomy theorem is the existence of a proper 2-generated core. It is our aim to drive the presence of a proper 2-generated core to a contradiction, and hence bind the complexity of the Sylow° 2-subgroup of a minimal counterexample to the Cherlin–Zilber conjecture. This paper shows that the group in question is a minimal connected simple group and has a strongly embedded subgroup, a far stronger uniqueness case. As a corollary, a tame counterexample to the Cherlin–Zilber conjecture has Prüfer rank at most two.

1. *Introduction*

This paper relates to the algebraicity conjecture for simple groups of finite Morley rank, also known as the Cherlin–Zilber conjecture, which states that all simple groups of finite Morley rank are simple algebraic groups over an algebraically closed field. As with most of the recent work on this conjecture, the present article seeks to transfer ideas from the classification of finite simple groups.

It is now a common practice to divide the Cherlin–Zilber conjecture into different cases depending on the nature of the connected component of the Sylow 2-subgroup, or Sylow° 2-subgroups (cf. § 2.1). We shall be working with groups with a divisible and non-trivial Sylow° 2-subgroup, or *odd-type* groups. Prior to [**9**], the main theorems in the area of odd-type groups were Borovik's trichotomy theorem [**5**] and the generic identification theorem [**4**]. Together, these two results prove the following.

Tame trichotomy theorem. *Let G be a simple* tame *K^*-group of finite Morley rank and of odd type. Then G is either a Chevalley group over an algebraically closed field of characteristic not 2, or has normal 2-rank at most 2, or has a proper 2-generated core.*

Here a group is said to be tame if it does not involve a field of finite Morley rank with a proper infinite definable subgroup of its multiplicative group. Such fields are presently believed to exist in characteristic zero [**16**]. Hence the tameness assumption must eventually be removed.

In this paper, we analyse groups with proper 2-generated cores (see § 3 for the definition), and drive them towards exceptional minimal connected simple configurations which should eventually turn out to be contradictory. In [**11**], Cherlin and Jaligot show that the Prüfer 2-rank of a tame minimal connected simple group is at most 2. In light of this result, and the tame trichotomy, the present paper shows the following.

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Tame generic case. *A tame minimal counterexample to the algebraicity conjecture has Prüfer 2-rank at most 2.*

It is our near-term aim to eliminate the need for tameness in the above theorem. In [**9**], tameness is removed from the tame trichotomy above, and the present paper will make no use of tameness either, and so all important applications of tameness now lie within [**11**]. For this reason, our results below will push beyond establishing that the group is minimal connected simple, and attempt to provide tools for the analysis of minimal connected simple groups, without tameness. In particular, we will show that the Sylow 2-subgroup is connected, and that G has a strongly embedded subgroup. Our results are summarized as follows.

Strong embedding theorem. *Let G be a simple K^*-group of finite Morley rank and of odd type with normal 2-rank at least 3 and Prüfer 2-rank at least 2. Let S be a Sylow 2-subgroup of G. Suppose that G has a proper 2-generated core $M = \Gamma_{S,2}(G) < G$. Then the following hold.*

(1) *G is a minimal connected simple group; that is, all proper definable connected subgroups are solvable.*
(2) *M is strongly embedded.*
(3) *$B := M^\circ$ is a Borel subgroup.*
(4) *S is connected.*
(5) *$N_G(B) = M$.*
(6) *$I(B \cap B^g) = \emptyset$ for any $g \notin M$.*
(7) *$\cup B^G$ is generic in G.*

Burdges, Cherlin, and Jaligot will eliminate this configuration in [**10**], thus replicating the main result of [**11**].

The notions of both a 2-generated core and a strongly embedded subgroup arise as the so-called *uniqueness cases* in finite group theory. These subgroups both exhibit a black-hole property reminiscent of a normal subgroup; and they seem similar when we compare Fact 3.1 below with Lemma 3.3 or Claim 5.3 of § 5. Strong embedding, however, is far more powerful and has global consequences (see Fact 3.2). Our proof of the fact that G is a minimal connected simple group will involve passing through strong embedding to obtain a contradiction under the assumption that B is non-solvable.

In bridging the gap between 2-generated cores and strong embedding, we employ the theory of Carter subgroups and make use of a result due to Olivier Frécon (Fact 2.6) in the final stage of the argument.

2. *Background*

We now recall essential facts about groups of finite Morley rank. The standard reference for our basic facts is [**6**]. Some of that material will be used without explicit mention.

A group of finite Morley rank is *connected* if it contains no proper definable subgroup of finite index. We will refer to maximal connected solvable subgroups of a group of finite Morley rank as *Borel* subgroups.

We define the 2-*rank* $m_2(G)$ of a group G to be the maximum rank of its elementary abelian 2-subgroups. Also, the *Prüfer* 2-*rank* $\mathrm{pr}_2(G)$ is the maximum rank of its Prüfer 2-subgroups $\mathbb{Z}(2^\infty)^k$, and the *normal* 2-*rank* $n_2(G)$ is the maximum rank of a normal elementary abelian 2-subgroup of Sylow 2-subgroup of G. These ranks must all be finite for subgroups of an odd-type group of finite Morley rank.

We define the *odd part* $O(G)$ of a group G of finite Morley rank to be the maximal definable connected normal $2^{\perp}$-subgroup of G. The subgroup $O(G)$ is well defined by the following exercise from [**6**].

FACT 2.1 [**6**, Exercise 11, p. 93]. *Let G be a group of finite Morley rank, and let $H \lhd G$ be a definable subgroup. Let $x \in G$ be an element such that $\bar{x} \in G/N$ is a p-element. Then xH contains a p-element.*

2.1. *Sylow and Carter subgroups*

We provide a basic notion of 'characteristic' for groups of finite Morley rank as follows.

Let S be a Sylow 2-subgroup of a group G of finite Morley rank. By [**7**] (see also [**6**, Lemma 10.8]), $S^\circ = B * T$ is a central product of a definable connected nilpotent subgroup B of bounded exponent and of a 2-torus T; that is, T is a divisible abelian 2-group.

The group G is said to be of *odd type* if $B = 1$ and $T \neq 1$. This notion is well defined because the Sylow 2-subgroups of a group of finite Morley rank are conjugate by [**7**, **18**] (see also [**6**, Theorem 10.11]). The following two corollaries of conjugacy, known as a 'Frattini argument' and a 'fusion control lemma,' respectively, will be useful.

FACT 2.2 [**6**, Corollary 10.12]. *Let G be a group of finite Morley rank, let $N \lhd G$ be a definable subgroup, and let S be a characteristic subgroup of the Sylow 2-subgroup of N. Then $G = N_G(S)N$.*

FACT 2.3 [**6**, § 10.6.1]. *Let G be a group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G. Then $N_G(S^\circ)$ controls fusion in $C_S(S^\circ)$; that is, two elements of $C_S(S^\circ)$ which are G-conjugate are in fact $N_G(S^\circ)$-conjugate.*

A useful property of Sylow 2-subgroups is that they can be lifted.

FACT 2.4 [**19**; **23**, Corollary 1.5.5]. *Let G be a group of finite Morley rank, and let N be a normal subgroup of G. Then the Sylow 2-subgroups of G/N are the images of the Sylow 2-subgroups of G.*

FACT 2.5 [**6**, Theorem 9.29] (see also [**12**, Corollary 7.15]). *Let G be a connected solvable group of finite Morley rank. Then the Sylow p-subgroups of G are connected.*

Let G be a group of finite Morley rank. A definable subgroup $C \leqslant G$ which is nilpotent and self-normalizing in G is called a *Carter subgroup* of G.

The following result is a summary, in order, of [**12**, Proposition 3.2, Corollary 4.8], [**22**; **23**, Theorem 5.5.12], and [**12**, Corollary 7.15].

FACT 2.6. *Let H be a connected solvable group of finite Morley rank. Then the following hold.*

(1) *The group H has a Carter subgroup.*

(2) *The Carter subgroups of H are the definable nilpotent subgroups of H with $N_H^\circ(C) = C$. In particular, Carter subgroups of H are connected.*

(3) *The Carter subgroups of H are H-conjugate. As a corollary, we have the Frattini argument: if H is a definable connected normal solvable subgroup of a group G of finite Morley rank, and C is a Carter subgroup of H, then $G = N_G(C)H$.*

(4) *Let R be a Sylow p-subgroup of H. Then $N_H(R)$ contains a Carter subgroup of H.*

2.2. *Algebraic groups and K-groups*

A group G will be called *quasi-simple* if $G = G'$ and $G/Z(G)$ is simple. The group G will be called *semi-simple* if $G = G'$ and $G/Z(G)$ is *completely reducible*, that is, $G/Z(G)$ is a direct sum of finitely many simple subgroups. Therefore quasi-simple groups are semi-simple.

We will need the following results from the classification of quasi-simple algebraic groups.

FACT 2.7. *The only quasi-simple algebraic groups over an algebraically closed field F without definable proper quasi-simple subgroups are $\mathrm{SL}_2(F)$ and $\mathrm{PSL}_2(F)$.*

FACT 2.8 [**6**, Theorem 8.4]. *Let $G \rtimes H$ be a group of finite Morley rank, where G and H are definable, G is an infinite quasi-simple algebraic group over an algebraically closed field, and $C_H(G)$ is trivial. Then, viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leqslant \mathrm{Inn}(G)\Gamma$, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G, relative to a fixed choice of Borel subgroup B and maximal torus T contained in B.*

A group G of finite Morley rank is called a *K-group* if every connected definable simple section of G is a Chevalley group over an algebraically closed field. We shall also call a group G of finite Morley rank a *K^*-group* if every proper definable section is a K-group. Clearly, a *minimal* non-algebraic connected simple group of finite Morley rank will be a K^*-group. We also observe that $O(H)$ is solvable if H is a K-group, since simple algebraic groups contain involutions.

A quasi-simple subnormal subgroup of a group G is referred to as a *component* of G.

FACT 2.9 [**3**, **15**] (see also [**6**, § 7.4]). *Let G be a group of finite Morley rank. Then the components of G are definable subgroups, and there are only finitely many of them. Furthermore, G acts by conjugation on the set of components* (*see* [**6**, *Lemma* 7.12(ii)]).

The subgroup $L(G)$ generated by the components of G is now definable, being the setwise product of the components. We will refer to $L(G)$ as the *layer* of G and define $E(G) = L^\circ(G)$.

FACT 2.10 [**2**]. *A group of finite Morley rank which is a perfect central extension of a quasi-simple algebraic group over an algebraically closed field is an algebraic group and has a finite center.*

We define the *Fitting subgroup* $F(G)$ of G to be the subgroup generated by all the normal nilpotent subgroups of G. The Fitting subgroup is nilpotent and definable [**3**, **15**] (see also [**6**, Theorem 7.3]).

FACT 2.11. *Let G be a connected K-group of odd type. Then $G/O(G)$ is isomorphic to a central product of quasi-simple algebraic groups over algebraically closed fields of characteristic* *not 2 and of a definable connected abelian group. In particular, if $\overline{G} = G/O(G)$ then $\overline{G} = F(\overline{G})E(\overline{G})$ and $F(\overline{G})$ is an abelian group.*

Proof. The 'in particular' part of the statement is [**5**, Theorem 5.9]. By definition, $E(G) = L_1 * \ldots * L_k$ is a central product of connected quasi-simple groups. Since G is a K-group, each L_i is a perfect central extension of a Chevalley group over an algebraically closed field. Now the result follows from Fact 2.10. □

A *Klein four-group*, or just *four-group* for short, is a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We will use the notation $H^{\#} = H\backslash\{1\}$ to denote the set of non-identity elements of a group H.

The following generation principle for K-groups will be used frequently.

FACT 2.12 [**5**, Theorem 5.14]. *Let G be a connected K-group of finite Morley rank and of odd type. Let V be a four-subgroup acting definably on G. Then*

$$G = \langle C_G^{\circ}(v) \mid v \in V^{\#} \rangle.$$

3. *Uniqueness subgroups*

We first discuss the notions of '2-generated core' and 'strongly embedded subgroup'.

A proper definable subgroup M of a group G of finite Morley rank is said to be *strongly embedded* if $I(M) \neq \emptyset$ and $I(M \cap M^g) = \emptyset$ for any $g \in G\backslash M$. Here $I(H)$ denotes the set of involutions of H. We will apply the usual criteria for strong embedding.

FACT 3.1 [**13**, Theorem 9.2.1] (see also [**6**, Theorem 10.20]). *Let G be a group of finite Morley rank with a proper definable subgroup M. Then the following are equivalent.*

(1) *The subgroup M is a strongly embedded subgroup.*

(2) *$I(M) \neq \emptyset$, $C_G(i) \leqslant M$ for every $i \in I(S)$, and $N_G(S) \leqslant M$ for some Sylow 2-subgroup S of M.*

(3) *$I(M) \neq \emptyset$ and $N_G(S) \leqslant M$ for every non-trivial 2-subgroup S of M.*

The following is one of the major applications of strong embedding.

FACT 3.2 [**6**, Theorem 10.19] (see also [**1**, Fact 3.3]). *Let G be a group of finite Morley rank with a proper definable strongly embedded subgroup M. Then the following hold.*

(1) *A Sylow 2-subgroup of M is a Sylow 2-subgroup of G.*

(2) *The groups G and M each have only one conjugacy class of involutions.*

Let G be a group of finite Morley rank, and let S be a Sylow 2-subgroup of G. We define the *2-generated core* $\Gamma_{S,2}(G)$ of G to be the definable hull of the group generated by all normalizers $N_G(U)$ of all elementary abelian 2-subgroups $U \leqslant S$ with $m_2(U) \geqslant 2$. As it is this last rank condition to which the '2-generated' is referring, a strongly embedded subgroup would be a proper 1-generated core, by Fact 3.1.

A priori, merely possessing a proper 2-generated core need not entail the global consequences of Fact 3.2. However, the following easy consequence of Fact 2.12 indicates that 2-generated cores are not far from being strongly embedded.

LEMMA 3.3. *Let G be a simple K^*-group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G, and let $M = \Gamma_{S,2}(G)$ be the 2-generated core associated with S. Let A be an elementary abelian 2-subgroup of M with $m_2(A) \geqslant 3$. Then $C_G^\circ(a) \leqslant M$ for any $a \in A^\#$.*

Proof. Let $K = C_G^\circ(a)$. Let A_1 be a four-subgroup of A disjoint from $\langle a \rangle$. Consider the K-group K of odd type, which contains A_1. By Fact 2.12, we see that $K = \langle C_K^\circ(x) \mid x \in A_1^\# \rangle$. Now $C_K^\circ(x) \leqslant C_G(a, x)$ and $\langle a, x \rangle$ is a four-subgroup of S. Thus $K \leqslant H$. □

This shows that 2-generated cores exhibit a kind of 'black-hole' principle, limiting communication between elements of the subgroup $\Gamma_{S,2}(G)$ and its exterior.

LEMMA 3.4. *Let G be a simple K^*-group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G, and let $M = \Gamma_{S,2}(G)$ be the 2-generated core associated with S. If $\mathrm{pr}_2(S) \geqslant 3$ and $M < G$ then $B := M^\circ$ is a maximal proper connected subgroup of G.*

Proof. Let $K < G$ be a connected group containing B. Since $\mathrm{pr}_2(S) \geqslant 3$, Fact 2.12 and Lemma 3.3 yield

$$K \leqslant \langle C_K^\circ(i) : i \in \Omega_1(S^\circ) \rangle \leqslant M. \qquad \square$$

4. *Component analysis*

Our next few lemmas are directed toward the proof that B is solvable. The first of these will allow us to prove that M is strongly embedded when B is non-solvable.

LEMMA 4.1. *Let G be a K-group of finite Morley rank and of odd type with non-solvable connected component, and let i be an involution in G. Then the Sylow$^\circ$ 2-subgroups of $C_G^\circ(i)$ are non-trivial.*

We first recall the following lemma.

FACT 4.2 [**8**, Fact 3.2; **9**, Fact 3.12]. *Let $G = H \rtimes T$ be a group of finite Morley rank with H and T definable. Suppose that T is a solvable π-group of bounded exponent and $Q \lhd H$ is a definable solvable T-invariant $\pi^\perp$-subgroup. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

Proof of Lemma 4.1. We claim that it is enough to prove the statement for $\overline{G} = G/O(G)$. Let $i \in I(G)$, and let us assume that we know the result for $\overline{G}$ and the involution $\bar{i}$ of $\overline{G}$. Let $\overline{S}$ be a non-trivial Sylow$^\circ$ 2-subgroup of $C_{\overline{G}}(\bar{i})$. Since $C_G(i)/C_{O(G)}(i) \cong C_{\overline{G}}(\bar{i})$ by Fact 4.2, there is a non-trivial Sylow$^\circ$ 2-subgroup S of $C_G(i)$, by Fact 2.4. Hence we can assume that $O(G^\circ) = O(G) = 1$.

Let $i \in I(G)$. By Fact 2.11, G° is the central product of finitely many quasi-simple algebraic groups and of a definable connected abelian group $F := F(G)^\circ \lhd G$, say,

$$G^\circ = G_1 * \ldots * G_n * F.$$

Let $L = G_1 * \ldots * G_n$. Since $L \neq 1$ and i normalizes L by Fact 2.9, we can assume that $G = L \rtimes \langle i \rangle$. If i swaps two of the quasi-simple components G_j and G_k, then $\langle ss^i \mid s \in S \rangle$, where S is a Sylow 2-subgroup of G_j, is an infinite 2-subgroup of $C_G(i)$ and the proof is complete. Therefore we may assume that i normalizes each component. This allows us to assume that L is just one component; that is, $G = L \rtimes \langle i \rangle$ and L is quasi-simple algebraic.

By Fact 2.8, we have two cases: i acts on L either as an inner automorphism or as an inner automorphism composed with a graph automorphism, and hence G is algebraic. Since G is of odd type, i is semisimple in G. Therefore $C_G^\circ(i)$ is non-trivial and reductive by [**21**, Theorem 8.1] and hence has an infinite Sylow 2-subgroup. Alternatively, if we scrutinize the table of centralizers of involutive automorphisms of algebraic groups, [**14**, Table 4.3.1] shows that they always have infinite Sylow 2-subgroups. □

The next lemma will be used to contradict strong embedding under the assumption that B is non-solvable.

LEMMA 4.3. *Let G be a K-group of finite Morley rank and of odd type with non-solvable G° and $\mathrm{pr}_2(G) \geqslant 3$. Let S be a Sylow 2-subgroup of G. Then not all the involutions of S° are G-conjugate.*

Notice that the assumption $\mathrm{pr}_2(G) \geqslant 3$ cannot be weakened: if K is an algebraically closed field of characteristic distinct from 2 then the group $G = \mathrm{PSL}_3(K)$ has Prüfer 2-rank 2, and only one conjugacy class of involutions.

Proof of Lemma 4.3. Suppose toward a contradiction that the involutions of S° are all G-conjugate. Passing to a quotient, we may suppose that $O(G) = 1$.

By Fact 2.11, G° is a central product of finitely many quasi-simple algebraic groups and of a definable connected abelian group F, say $G^\circ = G_1 * \ldots * G_n * F$. Let $L = G_1 * \ldots * G_n$. Since G_1 has an involution and $L \lhd G$, all the involutions of G are in L.

Case 1: $Z(L)$ has an involution. Then all the involutions of G° are in $Z(L)$. Thus each G_i is a quasi-simple algebraic group with involutions in $Z(G_i)$. From the classification of quasi-simple algebraic groups (for example, [**20**]), it follows that $G_i \simeq \mathrm{SL}_2(K_i)$ for some algebraically closed K_i of characteristic not 2 (see [**14**, Theorem 1.12.5d]). Thus L is a central quotient of $\mathrm{SL}_2(K_1) \times \ldots \times \mathrm{SL}_2(K_n)$. Any non-trivial central quotient of $\mathrm{SL}_2(K_1) \times \ldots \times \mathrm{SL}_2(K_n)$ will introduce new non-central involutions since the involution of $Z(\mathrm{SL}_2(K_i))$ has a non-central square root. Therefore $G^\circ = \mathrm{SL}_2(K_1) \times \ldots \times \mathrm{SL}_2(K_n)$. Since G permutes the components $G_1, \ldots, G_n$ by Fact 2.9, the associated set of involutions $\{i_1, \ldots, i_n\}$, given by $i_j \in I(G_j)$, is G-invariant. Hence i_1 cannot be conjugate to $i_1 i_2$ if $i_1 \neq i_2$. Since $\mathrm{pr}_2(G) \geqslant 3$ and $\mathrm{pr}_2(\mathrm{SL}_2(F_i)) = 1$, there are at least three components, which is a contradiction.

Case 2: $Z(L)$ has no involutions. Passing to a quotient by Fact 2.4, we can assume, without loss of generality, that $Z(L) = 1$ and that each G_i is an algebraic group over an algebraically closed field of characteristic not 2 which is simple as an abstract group. Hence $L = G_1 \times \ldots \times G_n$. Then $S = S_1 \times \ldots \times S_n$ with S_i a Sylow 2-subgroup of G_i. If $n \geqslant 2$ then an involution in S_1 cannot be conjugate to a product of involutions from S_1 and S_2, and so $n = 1$. Thus G acts transitively on the involutions of the simple algebraic group $L = G_1$. Since $\mathrm{pr}_2(L) \geqslant 3$, there are two involutions $t, s \in L$ with $C^\circ(s) \not\cong C^\circ(t)$ by [**14**, Table 4.3.1]. Hence the result follows. □

The following lemmas will be used to show that G is a minimal connected simple group once we have the solvability of $B := M^\circ$. The first is a lifting lemma for 2-generated cores, and the second is a structural result about a group of the form $\mathrm{PSL}_2(K)$.

LEMMA 4.4. *Let G be a group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G. Let $^-$ denote 'image in the quotient $G/O(G)$'. Then*

$$\overline{\Gamma_{S,2}(G)} = \Gamma_{\overline{S},2}(\overline{G}).$$

Proof. For any four-group $A \leqslant S$, the image $\overline{A}$ is still a four-group. Hence the left-hand side is a subgroup of the right-hand side. To prove the reverse inclusion, it is enough to show that, for any four-subgroup E of $\overline{S}$, we have a four-subgroup A of S such that $\overline{A} = E$ and $N_{\overline{G}}(\overline{A}) \leqslant \overline{N_G(A)}$.

Let E be a four-subgroup of $\overline{S}$, and let X be the full preimage of E in G. Since $E \leqslant \overline{S}$, we have $X \leqslant SO(G)$. Let A be a Sylow 2-subgroup of X. By Fact 2.4, $\overline{A} = E$, so $X = AO(G)$ and $A \cong E$. Since $A \leqslant X \leqslant SO(G)$ and S is a Sylow 2-subgroup of $SO(G)$, we may assume that $A \leqslant S$ by conjugating by an element of $O(G)$. Since A is a Sylow 2-subgroup of $AO(G)$, $N_G(AO(G)) \leqslant N_G(A)O(G)$ by Fact 2.2. Therefore, $N_{\overline{G}}(\overline{A}) \leqslant \overline{N_G(A)}$, as desired. □

LEMMA 4.5. *The connected component of a 2-generated core of $\mathrm{PSL}_2(K)$, where K is an algebraically closed field of characteristic distinct from 2, is non-solvable.*

Notice that it follows from Poizat [**17**] that $\mathrm{PSL}_2(K)$ coincides with its 2-generated core, although we do not need the full strength of this result.

Proof of Lemma 4.5. Let T be the standard maximal torus of $G = \mathrm{PSL}_2(K)$ (that consists of diagonal elements modulo the center of $\mathrm{SL}_2(K)$). Let S be a Sylow 2-subgroup of $G = \mathrm{PSL}_2(K)$ such that $S^\circ \leqslant T$. Then $S = S^\circ \rtimes \langle w \rangle$ for some $w \in I(N_G(T) \setminus T)$. Since w inverts T, it follows that wS° consists entirely of involutions and S is generated by its involutions. Let z be the unique involution of $Z(S) \leqslant S^\circ$. Let $M = \Gamma_{S,2}(G)$. For any involution $t \neq z$ of S, we see that t belongs to the four-subgroup $\langle z, t \rangle$ of S, and so $S \leqslant M$.

Now recall that G is the automorphism group of the projective line $\mathbb{P}^1$ over the field K. Since z and t are involutions and the characteristic is not 2, they have two fixed points each, which we label z_1, z_2 and t_1, t_2, respectively. Since t commutes with z, they stabilize one another's fixed points. Since $z \neq t$, we have $z_1 \neq t_1$ and $z_1 \neq t_2$. Also $z_1^t = z_2$ and $z_2^t = z_1$. Since G acts sharply 3-transitively on $\mathbb{P}^1$, there is an $r \in G$ such that $z_1^r = t_1$, $z_2^r = t_2$, and $t_1^r = z_1$. Since the pointwise stabilizer of t_1 and t_2 is isomorphic to K^*, there is only one involution fixing these two points, and thus $z^r = t$. Since t^r commutes with $z^r = t$, it follows that t stabilizes the fixed point set of t^r. Since t^r fixes $z_1 = t_1^r$, and $z_1^t = z_2$, we find that t^r fixes z_2 too, and thus $t^r = z$. Hence r normalizes $\langle z, t \rangle$ and $r \in M$. Now $t \in S^{\circ r^{-1}}$ since $z \in S^\circ$. Thus $\langle z, t \rangle \leqslant M^\circ$.

Suppose toward a contradiction that M° is solvable. Then $\mathrm{pr}_2(M^\circ) \geqslant 2$ by Fact 2.5, which is a contradiction. □

5. *Proof of the strong embedding theorem*

Let G be a simple K^*-group of finite Morley rank and of odd type with normal 2-rank at least 3 and Prüfer 2-rank at least 2. Let S be a Sylow 2-subgroup of G. Suppose that G has a proper 2-generated core $M = \Gamma_{S,2}(G) < G$. We proceed by first establishing that G is a minimal connected simple group, and then showing that S is connected, which can be used to prove the strong embedding of M.

Let $E \lhd S$ be an elementary abelian 2-subgroup with $m_2(E) \geqslant 3$.

Claim 5.1. *For every $i \in I(S)$, $C_E(i)$ contains a four-group.*

Proof. Since E is normal in S, the involution i induces a linear transformation of the $\mathbb{F}_2$-vector space E. Since $m_2(E) > 2$, the Jordan canonical form of i cannot consist of a single block, and so there are at least two eigenvectors. Since the eigenvalues associated to these eigenvectors must have order 2, the eigenvalues must both be 1, as desired. □

Claim 5.2. *We have $C_G^\circ(i) \leqslant M$ for every $i \in I(M)$.*

Proof. We may assume that $i \in I(S)$ after conjugation. By Claim 5.1, there is a four-group $E_1 \leqslant E$ centralized by i. Thus either E or $\langle E_1, i\rangle$ is an elementary abelian 2-group of rank at least three, which contains i. By Lemma 3.3, we find that $C_G^\circ(i) \leqslant M$. □

Claim 5.3. *We have $C_G(i) \leqslant M$ for any $i \in I(M)$ for which $C_M^\circ(i)$ has an infinite Sylow 2-subgroup.*

Proof. Let R be a Sylow° 2-subgroup of $C_G^\circ(i)$. We may assume that $\langle R, i\rangle \leqslant S$ after conjugation.We claim that $N_{C_G(i)}(R) \leqslant M$. If $i \notin S^\circ$ then $m_2(\langle \Omega_1(R), i\rangle) \geqslant 2$, and so

$$N_{C_G(i)}(R) \leqslant N_G(\langle \Omega_1(R), i\rangle) \leqslant M.$$

If $i \in S^\circ$ then $m_2(\Omega_1(R)) \geqslant 2$ since $\mathrm{pr}_2(G) \geqslant 2$, and so

$$N_{C_G(i)}(R) \leqslant N_G(\Omega_1(R)) \leqslant M.$$

Now Fact 2.2 and Claim 5.2 yield

$$C_G(i) = C_G^\circ(i) N_{C_G(i)}(R) \leqslant M.$$

□

Claim 5.4. *The group $B := M^\circ$ is solvable.*

Proof. Suppose toward a contradiction that B is non-solvable. Then, by Lemma 4.1, for every involution $i \in M$, the Sylow 2-subgroups of $C_M(i)$ are infinite. By Claim 5.3, we have $C_G(i) \leqslant M$. Therefore, So M is strongly embedded, by Fact 3.1, and any two elements of $E^\#$ are G-conjugate, by Fact 3.2.

We observe that $N_G(S^\circ) \leqslant M$ since $\mathrm{pr}_2(S) \geqslant 2$. By Fact 2.3, $N_M(S^\circ)$ controls M-fusion in $C_S(S^\circ)$, and so all involutions in E are conjugate in $N_M(S^\circ)$. Hence we have $E \leqslant \Omega_1(S^\circ)$ and $\mathrm{pr}_2(S) \geqslant 3$, in contradiction with Lemma 4.3. □

Claim 5.5. *The group G is a minimal connected simple group.*

Proof. Suppose toward a contradiction that G has a proper definable non-solvable connected subgroup. Let K be a minimal proper definable non-solvable connected subgroup of G, and let $\overline{K} = K/O(K)$. By Fact 2.11, $\overline{K}$ is a central product of quasi-simple algebraic groups over algebraically closed fields of characteristic not 2 and of one definable connected abelian group. By minimality of K, it holds that $\overline{K}$ must actually be one quasi-simple algebraic group. Now $\overline{K}$ must be isomorphic to either $\mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ for some algebraically closed field F of characteristic not 2, and also by minimality of K (Fact 2.7).

A 2-generated core of K is a subgroup of a 2-generated core of G. Therefore the connected component of a 2-generated core of K is also solvable. By Lemma 4.4, the connected component of a 2-generated core of $\overline{K}$ is also solvable. By Lemma 4.5, we have $\overline{K} \not\simeq \mathrm{PSL}_2(F)$, and so $\overline{K} \simeq \mathrm{SL}_2(F)$.

Now $\overline{K}$ has a central involution $\bar{z}$ in the connected component of a Sylow 2-subgroup. By Fact 4.2, we see that $C_K(z)O(K)/O(K) = C_{K/O(K)}(\bar{z})$ for some involution $z \in S^\circ$. By Claim 5.2 applied to E, we have $C_G^\circ(z) \leqslant B$, in contradiction with Claim 5.4. □

Now suppose for the moment that S is connected. Then S is abelian and $C_G(i) \leqslant M$ for every $i \in M$, by Claim 5.3. Hence M is strongly embedded, by Fact 3.1. Since $\mathrm{pr}_2(S) = n_2(S) \geqslant 3$ too, B is a Borel subgroup, by Lemma 3.4. This means that we can dedicate the remainder of the argument to showing that S is connected.

Claim 5.6. *We have $N_G(B) = M$.*

Proof. We observe that S° is now a Sylow 2-subgroup of B, by Fact 2.5. Since $\mathrm{pr}_2(S) \geqslant 2$, it follows that $N_G(S^\circ) \leqslant N_G(\Omega_1(S^\circ)) \leqslant M$. By Fact 2.2, we have

$$N_G(B) = BN_{N_G(B)}(S^\circ) \leqslant M. \qquad \square$$

Claim 5.7. *For any $I(B \cap B^g) = \emptyset$, we have $g \notin N_G(B)$.*

Proof. Suppose towards a contradiction that there is an $i \in I(B \cap B^g)$. We may assume that $i \in I(S)$ after conjugation. Since B is solvable, by Claim 5.4, the Sylow 2-subgroups of B are connected, by Fact 2.5. As $i \in B$ and B is of odd type, $S^\circ \leqslant C_G^\circ(i)$. Since $i \in M^g$, it follows that $C_G^\circ(i) \leqslant M^g$, by Claim 5.2. Since $\mathrm{pr}_2(S) \geqslant 2$, we find that $S \leqslant N_G(\Omega_1(S^\circ)) \leqslant M^g$. Since S is now a Sylow 2-subgroup of M^g, we have $M^g = \Gamma_{S,2}(G) = M$. □

Claim 5.8. *The union $\cup B^G$ is generic in G.*

For this, we employ the following fact from [**11**] and a general lemma.

Fact 5.9 [**11**, Lemma 3.3]. *Let G be a connected group of finite Morley rank. Let B be a definable subgroup of finite index in its normalizer. Suppose that there is a definable subset Q of B, not generic in B, such that $B \cap B^g \subseteq Q$ whenever $g \notin N_G(B)$. Then $\cup B^G$ is generic in G.*

Lemma 5.10. *Let H be a connected solvable group of finite Morley rank and of odd type. Let $\mathcal{F}$ be a uniformly definable family of $2^\perp$-subgroups of H. Then there is a definably characteristic definable $2^\perp$-subgroup Q of H containing $\cup\mathcal{F}$.*

Here, 'definably characteristic' means 'invariant under definable automorphisms'.

Proof. Lemma 3.2 of [**11**] says that the quotient $\bar{H} := H/O(H)$ is divisible abelian, since H is connected solvable of odd type. Therefore $\bar{F} \lhd \bar{H}$ for any $F \in \mathcal{F}$. By Fact 2.1, $\bar{F}$ is a $2^\perp$-subgroup of $\bar{H}$ for any $F \in \mathcal{F}$. Since $\mathrm{O}(\bar{H}) = 1$ and $\bar{H}$ is abelian, $\bar{F}$ is finite for any $F \in \mathcal{F}$. Since the family $\{\bar{F} : F \in \mathcal{F}\}$ is uniformly definable, there is a bound on $|\bar{F}|$ by [**6**, Axiom D, p. 57]. Therefore $m = \mathrm{lcm}\{|\bar{F}| : F \in \mathcal{F}\} < \infty$ is odd. Since $\bar{H}$ is abelian, $\bar{Q} := \{h \in \bar{H} : h^m = 1\}$ is a characteristic $2^\perp$-subgroup of $\bar{H}$ containing $\bar{F}$ for all $F \in \mathcal{F}$. Therefore the pullback Q of $\bar{Q}$ is a suitable definably characteristic $2^\perp$-subgroup of H. □

Proof of Claim 5.8. By Claim 5.7 and Lemma 5.10, there is a definably characteristic definable $2^\perp$-subgroup Q of B which contains $B \cap B^g$ for any $g \notin N_G(B)$. Since B has non-trivial Sylow 2-subgroup, we have $Q < B$. The subgroup B has finite index in its normalizer by Claim 5.6. Now $\cup B^G$ is generic in G by Fact 5.9. □

We observe that conclusions (5), (6), and (7) of the strong embedding theorem follow from the previous three claims.

Consider a pair B_1, B_2 of definable subgroups of G. We say that a definable subgroup H of G is (B_1, B_2)-*bi-invariant* if H is (A_1, A_2)-invariant for some four-groups $A_1 \leqslant B_1$ and $A_2 \leqslant B_2$. We may simply say that H is *bi-invariant* when the choice of B_1 and B_2 is clear from the context. Similarly, we say that a collection of definable subgroups $\mathcal{H}$ is *simultaneously bi-invariant* if all $H \in \mathcal{H}$ are (A_1, A_2)-invariant for the same choice of A_1 and A_2.

Claim 5.11. *Any connected definable (B, B^g)-bi-invariant subgroup K of G is contained in $B \cap B^g$; and hence is a $2^\perp$-group when $g \notin N_G(B)$, by Claim 5.7.*

Proof. By Fact 2.12 and Claim 5.2, we have

$$K = \langle C_K^\circ(a) : a \in A_1 \rangle \leqslant M,$$

and similarly $K \leqslant M^g$. □

We claim that S is connected. Suppose toward a contradiction that S is disconnected. We fix an $i \in S - S^\circ$ with $i^2 \in S^\circ$. We also define

$$X := \{x \in iB : x \in (\langle i \rangle B)^g \text{ for some } g \notin N_G(B)\}.$$

Claim 5.12. *There is a $j \in X$ with $j^2 = 1$.*

For this, we employ the following fact from [**11**].

Fact 5.13 [**11**, Lemma 3.5]. *Let G be a connected group of finite Morley rank. Let B be a proper definable subgroup of finite index in its normalizer such that $\cup B^G$ is generic in G. Suppose that $z \in N_G(B) - B$ has order $n > 1$ modulo B, and let $\langle z \rangle B$ be the union $B \cup zB \cup z^2B \cup \ldots \cup z^{n-1}B$. Then the following subset X of zB is generic in zB:*

$$X := \{x \in zB : x \in (\langle z \rangle B)^g \text{ for some } g \notin N_G(B)\}.$$

Proof of Claim 5.12. The subset X is generic in iB, by Fact 5.8 and Claim 5.8. Therefore there is some $x \in X$. Then $x \in iB \cap (\langle i \rangle B)^g$ for some $g \notin N_G(B)$ and $x^2 \in B \cap B^g$. Thus $K := \{1, x\}(B \cap B^g)$ is a definable group, and $B \cap B^g \lhd K$. By Fact 2.1, there is a non-trivial 2-element $j \in x(B \cap B^g) \leqslant X$. Now $j^2 = 1$ since $j^2 \in B \cap B^g$ and $I(B \cap B^g) = \emptyset$ by Claim 5.7. □

For the next portion of our argument, we fix an involution $j \in X$ and some $g \notin N_G(B)$ with $j \in (\langle i \rangle B)^g$.

Claim 5.14. *We know that $C_G^\circ(j)$ is non-trivial and is (B, B^g)-bi-invariant.*

Proof. Since G is non-abelian, $C_G^\circ(j)$ is non-trivial [**6**, Example 13, p. 79] Since $j \in iB \leqslant M$, it follows that $j \in S^b$ for some $b \in M$ by conjugacy. By Claim 5.1, there is a four-group $A_1 \leqslant E^b$ centralizing j. The existence of a suitable $A_2 \leqslant B^g$ follows similarly. □

Now consider a maximal proper definable connected (B, B^g)-bi-invariant subgroup H of G. The subgroup H is non-trivial, by Claim 5.14. Let C be a Carter subgroup of H (which exists by Fact 2.6(1)).

From this point on, we will have no more need of the assumption that S is disconnected, or the involution j. Instead, we proceed by general arguments involving the groups B, B^g, H, and C.

Claim 5.15. *The subgroups C and H are simultaneously (B, B^g)-bi-invariant.*

Proof. Let A denote one of the two groups with respect to which H is bi-invariant, that is, either A_1 or A_2. By Fact 2.6-3, $HA = HN_{HA}(C)$. Since $I(H) = \emptyset$ by Claim 5.11, A is a Sylow 2-subgroup of HA and $N_{HA}(C)/C \cong HA/H$ is a four-group. By Fact 2.4, $N_{HA}(C)$ contains a Sylow 2-subgroup A_0 of HA. Thus H and C are clearly A_0-invariant and A_0 lives in B or B^g, as appropriate. □

Claim 5.16. *We have $N_G^\circ(C) = C$, and hence C is a Carter subgroup of any Borel subgroup containing it.*

Proof. The two groups H and $N_G^\circ(C)$ are simultaneously bi-invariant, by Claim 5.15, and so $\langle H, N_G^\circ(C) \rangle \leqslant B \cap B^g$ is proper and bi-invariant. Hence $N_G^\circ(C) \leqslant H$ by maximality. Thus $N_G^\circ(C) = C$; and C is a Carter subgroup of any Borel subgroup which contains it, by Fact 2.6(2). □

The following general lemma now shows that C contains a conjugate of S°.

Lemma 5.17. *Let B be a connected solvable group of finite Morley rank and of odd type, and let C be a Carter subgroup of B. Then C contains a conjugate of the Sylow 2-subgroup of B.*

Proof. Let S be a Sylow 2-subgroup of B. By Fact 2.6(4), $N_B(S)$ contains a Carter subgroup C_1 of B. Since C_1 is connected by Fact 2.6(2), $C_1 \leqslant N_B^\circ(S) \leqslant C_B(S)$ by [**6**, Lemma 6.16]. Therefore $S \leqslant N_B(C_1) = C_1$. By Fact 2.6(3), C_1 is conjugate to C. □

By Lemma 5.17 above, C contains a conjugate of S°; in contradiction to Claim 5.11. Thus S is connected and all of our claims follow.

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