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The Basic Approval Voting Game

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Abstract

We survey results about Approval Voting obtained within the standard framework of game theory. Restricting the set of strategies to undominated and sincere ballots does not help to predict Approval Voting outcomes, which is also the case under strategic equilibrium concepts such as Nash equilibrium and its usual refinements. Strong Nash equilibrium in general does not exist but predicts the election of a Condorcet winner when one exists.

1 Introduction

There is a vast literature which conceives Approval Voting as a mechanism where the approval of voters is a mere strategic action with no intrinsic meaning. As usual, a group of voters who have preferences over a set candidates is considered. Every voter announces the list of candidates which he approves of and the winners are the candidates which receive the highest number of approvals. Assuming that voters take simultaneous and strategic actions, we are confronted to a normal form game whose analysis dates back to Brams and Fishburn (1983). This chapter surveys the main results of this literature.

The problem with this approach is that the main conceptual tool of game theory — Nash equilibrium — is of little help for understanding Approval Voting and most voting rules. By definition, an equilibrium is a vote profile in which no voter can, by changing her vote only, change the outcome of the game in such a way that the new outcome is strictly better for her. In a world where voters are only interested in who wins the election (instrumental and consequentialist voting, opposed to expressive voting), the outcome of the game is just the identity of the elected candidate, or candidates in case of a tie. Then it is almost always the case that no voter can, by changing her vote only, change the outcome of the game. With Approval Voting, as well as with most voting rules, this will happen as soon as one candidate is winning the election with a margin of more than two votes. Therefore, apart cases where several candidates tie or almost tie, almost everything is a Nash equilibrium. In particular, except in some very degenerated cases, any candidate is winning at some Nash equilibrium.

The game-theoretical literature on voting, and in particular on Approval Voting, has therefore focused on the possibility of using more powerful tools than Nash equilibrium in order either to predict the outcome of a voting game or at least to narrow down the set of possible outcomes. To this aim, several routes have been followed.

The first route is to restrict the set of voting strategies that a voter is supposed to possibly use. The natural idea, from the game theoretic perspective, is to suppose that voters do not use dominated strategies. Although this idea reveals very powerful in solving sequential voting games (Farquharson (1969), Moulin (1979, 1983), Banks (1985), Bag et al. (2009)), this is not the case for simultaneous voting games defined by the usual voting rules (Dhillon and Lockwood (2004), Buenrostro and Dhillon (2003), Dellis and Oak (2007)). For Approval Voting, undominated strategies are often called "admissible strategies" and can be characterized: If the voter's preference is strict, she approves her preferred candidate, she does not approve her worse candidate, and no constraint is imposed as to the other, intermediate candidates. (For a precise statement, see Proposition 1). For Approval Voting, another meaningful restriction on the set of strategies is the sincerity requirement, which imposes that when the voter approves a candidate, she also approves all the candidates she strictly to prefers to this one. Brams and Sanver (2006) have described the set of possible outcomes when voters use only undominated ("admissible") and sincere strategies. It turns out that, except in some degenerated situations, all candidates pass this test. (See section 4.1 of this chapter.)

The second route is to come back to a notion of equilibrium and to refine the notion of Nash equilibrium according to the usual concepts of game theory. (See Myerson (1991) or Van Damme (1991) for the general theory and De Sinopoli (2000) for an application to plurality voting.) In comparison with the previous approach, this amounts to give up the idea that the voter's behavior can be restricted *a priori* and to instead consider that each voter is reacting to what she believes are the voting intentions of the other voters. Remark that among the plethora of Nash equilibria of the voting games, most of them are degenerated from the strategic point of view in the sense that no player has any incentive *not* to deviate. In fact, unless she is "pivotal", the voter's choice has indeed no consequence at all on the outcome. This is a clear case for the refinement of equilibrium. One could hope that statements of the kind "A Condorcet loser cannot be elected at equilibrium under Approval Voting" or "Voters vote sincerely at equilibrium under Approval Voting" could be demonstrated when the notion of equilibrium is properly defined. This hope is justified if one allows not only for individual deviations but also for group deviations — hence considers strong equilibrium as the game-theoretic solution concept. (See Proposition 4. about Condorcet-consistency.) But the notion of strong equilibrium has a major drawback as a predictive tool since, in many cases, there is no such equilibrium. On the other hand, for different refinements of Nash equilibrium that yield non-empty predictions in finite normal-form games, De Sinopoli, Dutta and Laslier (2006) have provided counter-examples (reproduced in section 4.2) that kill the hope to make these statements true for any of the classical refinements of Nash equilibrium through concepts such as "perfection", "properness" or "stability".

The third route is to refine the concept of equilibrium following non-standard ideas that would be specific to the voting context. In politics, voting situations often involves large number of players, a fact that raises new difficulties but also new possibilities. This avenue, pioneered by Myerson and Weber (1993) and Myerson (2002) is the object of the survey of Nunez (2010) and is out of the scope of the present chapter.

Section 2 presents the basic notation and concepts. Section 3 deals with undominated and sincere individual strategies. Section 4 deals with the aggregate outcome of the vote. Section 5 concludes.

2 The normal form game

We denote by $\mathbf{I}$ the finite set *voters* (sometime called *individuals* or *players*) and by $\mathbf{X}$ the finite set of *candidates* (sometimes called *alternatives*). We assume $\#\mathbf{I} \geq 2$ and $\#\mathbf{X} \geq 2$. Every voter i has a *preference* over $\mathbf{X}$, expressed by a utility function $u_i : \mathbf{X} \longrightarrow \mathbb{R}$. So given two candidates $x, y \in \mathbf{X}$, voter i finds x at least as good as y iff $u_i(x) \geq u_i(y)$. A candidate x is *high* in u_i iff $u_i(x) \geq u_i(y)$ for all $y \in \mathbf{X}$. We say that x is *low* in u_i iff $u_i(y) \geq u_i(x)$ for all $y \in \mathbf{X}$. We call u_i *null* whenever i is indifferent among all alternatives, i.e., $u_i(x) \geq u_i(y)$ for every $x, y \in \mathbf{X}$. If u_i is null then every candidate is both low and high in u_i. If u_i is not null then the candidates which are high in u_i and those which are low in u_i form disjoint sets.

A *ballot* is any subset of the set of candidates; we denote by $2^{\mathbf{X}}$ the set of ballots. When voter i casts ballot B_i, we say that i *approves* the candidates in B_i. We let $B = (B_i)_{i\in I} \in \left(2^{\mathbf{X}}\right)^{\mathbf{I}}$ stand for a *ballot profile* and write $B = (B_i, B_{-i})$ with $B_{-i} = (B_j)_{j\in\mathbf{I}\setminus\{i\}}$, whenever we wish to highlight the dependency of B with respect to i's ballot. We refer to B_{-i} as *a ballot profile without* i.

Given a ballot profile B, the *score* of candidate x is

$$s(x; B) = \#\{i \in \mathbf{I} : x \in B_i\}$$

and the (non-empty) set of *winning candidates* (under Approval Voting) is

$$W(B) = \{x \in \mathbf{X} : s(x; B) \geq s(y; B) \forall y \in \mathbf{X}\}.$$

Similarly, we write $s(x; B_{-i}) = \#\{j \in \mathbf{I}\setminus\{i\} : x \in B_j\}$.

We suppose that voters vote simultaneously by casting a ballot which is some set of candidates while Approval Voting is used as the outcome function. So we consider a normal form game where the strategy set for any voter i is the set $2^{\mathbf{X}}$ of possible ballots. Hence a ballot profile B is also a strategy profile and the outcome is the set of winning candidates $W(B)$.

As $W(B)$ may contain more than one candidate, our strategic analysis requires the knowledge of voters' preferences over non-empty subsets of $\mathbf{X}$. We assume that ties over outcomes are broken by fair lotteries and that voters evaluate outcomes by expected Von-Neumann Morgenstern utilities. So the utility that voter i attaches to a set S of winning candidates is

$$u_i(S) = \frac{1}{\#S} \sum_{x\in S} u_i(x).$$

Note that we abuse notation and allow u_i to have arguments which are both elements and non-empty subsets of $\mathbf{X}$.

3 Admissibility and sincerity

3.1 Admissible strategies

Following the game-theoretical vocabulary, for any voter i with preference u_i, we say that the ballot B_i (*weakly*) *dominates* the ballot B_i' if and only if $u_i(W(B_i, B_{-i})) \geq u_i\ (W(B_i', B_{-i}))$ for all B_{-i} and $u_i(W(B_i, B_{-i})) > u_i$

$(W(B_i', B_{-i}))$ for some B_{-i}. A ballot is *undominated* if and only if it is dominated by no ballot. Following Brams and Fishburn (1983), we qualify undominated ballots as *admissible* and use either words. The following proposition characterizes admissible ballots.

Proposition 1 *(i) If u_i is null then all ballots are admissible for voter i.*

(ii) Let the number of voters be at least three. If u_i is not null then the ballot B_i is admissible for voter i if and only if B_i contains every candidate who is high in u_i and no candidate who is low in u_i.

Proof. (i) directly follows from the definitions. To show the "only if" part of (ii), consider a ballot B_i which fails to contain a candidate y who is high in u_i. Let $B_i' = B_i \cup \{y\}$. We will prove that B_i' dominates B_i.

Given any B_{-i}, all candidates except y have the same score at $(B_i; B_{-i})$ and $(B_i'; B_{-i})$ while the score of y is raised by one unit at the latter ballot profile. Therefore, regarding the sets of winning candidates $Y = W(B_i; B_{-i})$ and $Y' = W(B_i'; B_{-i})$, the following three cases are exhaustive:

1. $y \notin Y$ and $Y' = Y$;
2. $y \notin Y$ and $Y' = Y \cup \{y\}$;
3. $y \in Y$ and $Y' = \{y\}$.

In all three cases, $u_i(Y') \geq u_i(Y)$. Now fix some $k \in \mathbf{I} \backslash \{i\}$ and consider B_{-i} where $B_j = \emptyset$ for all $j \in \mathbf{I} \backslash \{i, k\}$ and $B_k = \{z\}$ for some candidate z who is not high in u_i. If $z \notin B_i$ then $W(B_i; B_{-i}) = B_i \cup \{z\}$ and $W(B_i'; B_{-i}) = B_i' \cup \{z\} = B_i \cup \{y, z\}$, hence $u_i(W(B_i'; B_{-i})) > u_i(W(B_i; B_{-i}))$. If $z \in B_i$, then $W(B_i; B_{-i}) = \{z\}$, $W(B_i'; B_{-i}) = \{y, z\}$ and we have $u_i(W(B_i'; B_{-i})) > u_i(W(B_i; B_{-i}))$. This proves that B_i' dominates B_i, and we conclude that an undominated ballot must contain all candidates who are high in u_i. Similar arguments show that an undominated ballot cannot contain a candidate who is low in u_i.

We now show the "if" part of (ii). Consider a ballot B_i that contains every candidate high in u_i and no candidate low in u_i. In order to show that B_i is undominated, we consider any distinct ballot B_i' and establish the existence of some B_{-i} where $u_i(W(B_i; B_{-i})) > u_i(W(B_i'; B_{-i}))$.

First let B_i' contain a candidate y who is low in u_i. Let B_{-i} be such that $B_j = \{y\}$ for some voter $j \in \mathbf{I} \backslash \{i\}$ and $B_k = \emptyset$ for every voter $k \in \mathbf{I} \backslash \{i, j\}$. So $W(B_i; B_{-i}) = B_i \cup \{y\}$, $W(B_i'; B_{-i}) = \{y\}$ and $u_i(W(B_i; B_{-i})) > u_i(W(B_i'; B_{-i}))$.

Now let B_i' fail to contain all candidates high in u_i. So the set Y of candidates in $B_i \backslash B_i'$ who are high in u_i is non-empty. Let L be the set of candidates who are low in u_i. Let B_{-i} be such that $B_j = Y \cup L$ for some voter $j \in \mathbf{I} \backslash \{i\}$ and $B_k = \emptyset$ for every voter $k \in \mathbf{I} \backslash \{i, j\}$. So $W(B_i; B_{-i}) = Y$ and at $(B_i'; B_{-i})$, the score of every candidate who is high in u_i is at most one and the score of some candidates who are low in u_i is one. Thus, $W(B_i'; B_{-i})$ contains candidates who are not high in u_i. Hence $u_i(W(B_i; B_{-i})) > u_i(W(B_i'; B_{-i}))$.

Finally let B_i' contain every candidate who is high in u_i and no candidate which is low in u_i. First consider the case where there exists a candidate y in B_i not in B_i'. Let B_{-i} be such that for two (distinct) voters $j, k \in \mathbf{I} \backslash \{i\}$ we have $B_j = B_k = \{y, z\}$ where z is low in u_i and $B_l = \emptyset$ for every voter $l \in \mathbf{I} \backslash \{i, j, k\}$. So $W(B_i; B_{-i}) = \{y\}$, $W(B_i'; B_{-i}) = \{y, z\}$ and $u_i(W(B_i; B_{-i})) > u_i(W(B_i'; B_{-i}))$. Now consider the case where B_i is a proper subset of B_i'. Take some $y \in B_i' \setminus B_i$. Note that y is not high in u_i. Take some candidate z which is high in u_i and let B_{-i} be such that for two (distinct) voters $j, k \in \mathbf{I} \backslash \{i\}$ we have $B_j = B_k = \{y, z\}$ and $B_l = \emptyset$ for every voter $l \in \mathbf{I} \backslash \{i, j, k\}$. So $W(B_i; B_{-i}) = \{y, z\}$, $W(B_i'; B_{-i}) = \{y\}$ and $u_i(W(B_i; B_{-i})) > u_i(W(B_i'; B_{-i}))$. ■

3.2 Sincerity

Following Brams and Fishburn (1983), a strategy (or ballot) B_i of voter i with preference P_i is said to be *sincere* iff for all candidates x, $y \in \mathbf{X}$,

$$y \in B_i \text{ and } u_i(x) > u_i(y) \Rightarrow x \in B_i.$$

So under a sincere strategy B_i, if i approves of a candidate y then she also approves of any candidate x which she strictly prefers to y. With K candidates, if voter i is never indifferent between two distinct candidates, she has at her disposal $K + 1$ sincere strategies, including the *full ballot* $B_i = \mathbf{X}$ which consists of approving of all candidates, and the *empty ballot* $B_i = \emptyset$ which consists of approving of none.

Proposition 1 does not make any statement about candidates which are neither high nor low. In fact, for a voter i with preference u_i, every non-sincere ballot which contains every candidate which is high in u_i and no candidate which is low in u_i. is an undominated strategy for i. So admissible ballots need not be sincere, nor sincere ballots have to be admissible.[1] On

[1] Nevertheless, if there are precisely three candidates, then every admissible ballot is sincere.

the other hand, sincere and non-sincere ballots can be discriminated through the fact that every ballot profile B_{-i} without i admits at least one sincere ballot B_i as a best-response of i. In other words, the set of best responses of i to B_{-i} cannot consist of insincere ballots only.

Proposition 2 *Given any voter i with preference u_i and any ballot profile B_{-i} without i, there exists a sincere ballot $B_i \in 2^{\mathbf{X}}$ such that $u_i(W(B_i; B_{-i})) \geq u_i(W(B'_i; B_{-i}))$ for every ballot $B'_i \in 2^{\mathbf{X}}$.*

Proof. Take any voter i with preference u_i and any ballot profile B_{-i} without i. Let Y be the (non-empty) set of candidates which receive the highest number of approvals at B_{-i}. Let Z be the (possibly empty) set of candidates who receive at B_{-i} precisely one approval less than the highest number of approvals. The outcome set $W(B_i, B_{-i})$ when B_i vary can take two forms: if $B_i \cap Y \neq \emptyset$, then $W(B_i, B_{-i}) = B_i \cap Y$, and if $B_i \cap Y = \emptyset$, then $W(B_i, B_{-i}) = Y \cup Z'$, for $Z' = B_i \cap Z$. Denote by u_i^* the maximum utility obtained by i. Then $u_i^* \geq \max_{y \in Y} u_i(y)$, and $u_i^* \geq \max_{Z' \subseteq Z} u_i(Y \cup Z')$, with one of these two inequalities being an equality. Let $y^* \in Y$ be such that $u_i(y^*) = \max_{y \in Y} u_i(y)$. Let $B_i^1 = \{x \in X : u_i(x) \geq u_i(y^*)\}$. This is a sincere ballot, so if $B_i^1$is a best reponse, we are done.

Notice that B_i^1 brings at least the level of utility $u_i(y^*)$; so if $B_i^1$is a not best reponse, it must be the case that $u_i(y^*) < u_i^*$ and that $u_i^* = \max_{Z' \subseteq Z} u_i(Y \cup Z')$. In that case, let $Z^* = \{z \in Z : u_i(z) \geq u_i(Y)\}$. Recall that the utility for a subset is the average utility of its elements; as one can easily check, it follows that $u_i(Y \cup Z^*) = \max_{Z' \subseteq Z} u_i(Y \cup Z')$. Let $B_i^2 = \{x \in X : u_i(x) \geq u_i(Y \cup Z^*)\}$ This is again a sincere ballot. Moreover, in that case, $B_i^2 \cap Y = \emptyset$ so that the ballot B_i^2 brings the utility $u_i(Y \cup (B_i^2 \cap Z))$. Here, $B_i^2 \cap Z = \{z \in Z : u_i(z) \geq u_i(Y \cup Z^*)\}$ and $u_i(z) \geq u_i(Y \cup Z^*)$ if and only if $u_i(z) \geq u_i(Y)$, so that $B_i^2 \cap Z = Z^*$, and B_i^2 brings the maximal utility u_i^*. We again found a sincere best response. ■

Proposition 1 slighty differs from the existing results of the literature regarding the way preferences over sets.are handled. In fact, it makes the same statement as Corollary 2.1 in Brams and Fishburn (2007) which is shown under more general assumptions for extending preferences over sets. On the other hand, the result announced by Proposition 2 has no analogous in Brams and Fishburn (1983, 2007)), as it fails to hold under these more general assumptions.[2].

[2] To see this, let voter i have the preference $u_i(x_1) > u_i(x_2) > u_i(x_3) > u_i(x_4) > u_i(x)$

Proposition 2 has no analogous for insincere ballots. In other words, the best response of i to B_{-i} can consist of sincere ballots only.[3] As a result, one may be tempted to assume — as we do in Section 4.1 — that voters restrict their strategies to those which are admissible and insincere. On the other hand, in Section 4.2, we see that such an assumption is not totally innocuous.

4 Approval Voting outcomes

4.1 Admissible and sincere outcomes

Brams and Sanver (2006) study the set of candidates which are chosen under Approval Voting at a given preference profile, assuming that voters use admissible and sincere strategies. For a formal expression of their findings, let $u = (u_i)_{i \in \mathbf{I}}$ be a preference profile. Write

$$\alpha(u) = \left\{ B \in \left(2^{\mathbf{X}}\right)^{\mathbf{I}} : \forall\, i \in \mathbf{I},\, B_i \text{ is admissible and sincere with respect to } u_i \right\}.$$

We define

$$AV(u) = \{x \in X : x \in W(B) \text{ for some } B \in \alpha(u)\}$$

as the set of (admissible and sincere) Approval Voting outcomes at u. So candidate x is an Approval Voting outcome at u if and only if there exists a profile of sincere and admissible strategies B where x is a (possibly tied) winning candidate under Approval Voting.

Note that a voter who strictly ranks K candidates has exactly $K-1$ admissible and sincere strategies which consist of approving her first $k \in \{1, ..., K-1\}$ best candidates. This is a drastic reduction of a voter's strategy space which originally contained 2^K strategies. Nevertheless, this does not restrict much the size of $AV(u)$ which Brams and Sanver (2006) characterize, assuming that voters are never indifferent between any two candidates, i.e., $u_i(x) \neq u_i(y)\ \forall i \in \mathbf{I}, \forall x, y \in \mathbf{X}$.

$\forall x \in \mathbf{X} \backslash \{x_1, x_2, x_3, x_4\}$ and let B_{-i} be such that $s(x_2; B_{-i}) = s(x_4; B_{-i}) > s(x_1; B_{-i}) = s(x_3; B_{-i}) > s(x; B_{-i}) \forall x \in \mathbf{X} \backslash \{x_1, x_2, x_3, x_4\}$ while $s(x_2; B_{-i}) - s(x_1; B_{-i}) = 1$. The ballot $B_i = \{x_1, x_3\}$ which yields $\{x_1, x_2, x_3, x_4\}$ can be a best-response under the Brams and Fishburn (1983, 2007) assumptions while there is no sincere ballot for voter i which yields the same outcome. Endriss (2009) identifies the assumptions on preferences over sets which rule out incentives to vote insincerely.

[3] Consider four voters and four candidates where each of voters 2, 3 and 4 approve of precisely one candidate; say x, y and z respectively. Let the fourth candidate w be ranked last in the preference of voter 1 whose unique admissible best response is to approve of the candidate he prefers the most.

Proposition 3 *Given a preference profile u with no indifferences, a candidate x is not in $AV(u)$ if and only if there exists a candidate $y \in \mathbf{X} \backslash \{x\}$ such that according to u, the number of voters who rank y as the best and x as the worst candidate exceeds the number of voters who prefer x to y.*

Based on Proposition 3, $AV(u)$ may contain Pareto dominated alternatives[4] as well as Condorcet losers. Moreover, at every preference profile u, a Condorcet winner (whenever it exists); all scoring rule outcomes; the Majoritarian Compromise winner; the Single Transferable Vote winner are always in $AV(u)$. We refer the reader to Brams and Sanver (2006) for a more detailed and formal expression of these results. Nevertheless, we can right away conclude that, in our game theoretic framework, assuming that voters restrict their strategies to those which are admissible and sincere does not suffice to have a fine prediction of the election result under Approval Voting.

4.2 Equilibrium outcomes

The model can be more predictive, when admissible and sincere strategy profiles are required to pass certain stability tests. A profile of sincere and admissible strategies B is *strongly stable* at preference profile u iff given any other profile of admissible and sincere strategies B', there exists a voter i with $B_i \neq B'_i$ while $u_i(W(B)) \geq u_i(W(B'))$. So B is strongly stable at u iff there exists no coalition of voters whose members can all be better-off by switching their strategies to another admissible and sincere one (which may differ among the members of the coalition). Let $AV^*(u) = \{x \in X : x \in W(B)$ for some $B \in \alpha(u)$ which is strongly stable$\}$ be the set of strongly stable AV outcomes at u. Clearly, $AV^*(u)$ refines $AV(u)$ and the reduction is indeed dramatic:

Proposition 4 *Given a preference profile u, a candidate x is strongly stable at u if and only if x is a weak Condorcet winner at u.*

Note that the definition of a Condorcet winner is a weak one: x is a weak Condorcet winner at u iff given any other candidate y, the number of voters who prefer x to y is at least as much as the number of voters who prefer y to x. So, in some cases, u may admit more than one weak Condorcet winner. Of course, u may admit no weak Condorcet winner,

[4] In the environment we consider, if a Pareto dominates b and $b \in AV(u)$, then $a \in AV(u)$ as well.

hence no strongly stable profile of admissible and sincere strategies. This last observation is not surprising, as strong stability — which corresponds to strong Nash equilibrium — is a rather demanding condition. The interested reader can see Sertel and Sanver (2004) for a more general treatment of strong equilibrium outcomes of voting games.

The complete proof of Proposition 4 can be found in Brams and Sanver (2006). However, we wish to give a simple and instructive description of the proof. If an outcome x is not a weak Condorcet winner, it means that there exists another outcome y which is prefered to x by some majoritarian coalition of voters which can block any strategy profile which yields x as the Approval Voting outcome. If x is a weak Condorcet winner, then no coalition can block the strategy profile where voters for whom x is not low approve x but do not approve anything below x and voters for whom x is low approve only their high candidate.[5]

We now present two results from de Sinopoli et al. (2006) which advise caution in interpreting Proposition 4 and Proposition 2.

1. There may exist non-trivial equilibria where a Condorcet winner obtains no vote;

2. There may exist non-trivial equilibria with some voters voting non-sincerely.

Example 5 (Condorcet in-consistency) *There are four candidates:* $\mathbf{X} = \{a, b, c, d\}$ *and three voters* $\{1, 2, 3\}$ *with utility:*

$$u_1(a) = 10,\ u_1(b) = 0,\ u_1(c) = 1,\ u_1(d) = 3,$$
$$u_2(a) = 0,\ u_2(b) = 10,\ u_2(c) = 1,\ u_2(d) = 3,$$
$$u_3(a) = 1,\ u_3(b) = 0,\ u_3(c) = 10,\ u_3(d) = 3.$$

Candidate d is the Condorcet winner of this utility profile. Consider the following strategy profile:

- *voter* 1 *votes* $\{a\}$*;*
- *voter* 2 *votes* $\{b\}$*;*
- *voter* 3 *votes* $\{c\}$*.*

[5] We take the occasion to claim that Proposition 4 remains valid when strong stability is further strengthened so as to allow non-admissible and non-sincere strategies.

In such a situation there is a tie among the candidates a, b, and c, so that the payoff to each player is 11/3*. Starting from this situation each player is playing a unique best response: any other choice would lead to a strictly lower payoff. In this strict equilibrium, the Condorcet winner receives no vote.*

The question of sincerity is raised by considering the possibility that players use mixed strategies. A mixed strategy is a probability distribution over the set of pure strategies. Here the set of mixed strategies is thus the simplex $\Delta(2^X)$ with 2^K vertices, that is an affine space of dimension $2^K - 1$. We denote by

$$\sigma_i \in \Delta(2^X)$$

a mixed strategy of voter i and by σ_{-i} a profile of mixed strategies for the other voters. Payoffs are defined in the usual ways as expected values. For a mixed strategy profile σ, $\sigma(B)$ is the probability of the pure-strategy profile B under σ. Players are supposed to randomize independently the ones from the others so that;

$$\sigma(B) = \prod_{i \in \mathbf{I}} \sigma_i(B_i)$$

and

$$u_i(\sigma) = \sum_{B \in (2^X)^{\mathbf{I}}} u_i(B)\sigma(B) = \sum_{B \in (2^X)^{\mathbf{I}}} \frac{1}{\#W(B)} \sum_{x \in W(B)} u_i(x)\sigma(B).$$

Example 6 (A non-sincere equilibrium) *There are four candidates:* $\mathbf{X} = \{a, b, c, d\}$ *and three voters* $\{1, 2, 3\}$ *with utility:*

$$u_1(a) = 1000,\ u_1(b) = 867,\ u_1(c) = 866,\ u_1(d) = 0,$$
$$u_2(a) = 115,\ u_2(b) = 1000,\ u_2(c) = 0,\ u_2(d) = 35,$$
$$u_3(a) = 0,\ u_3(b) = 35,\ u_3(c) = 115,\ u_3(d) = 1000.$$

Candidate d is the Condorcet winner of this utility profile. Consider the following strategy profile:

- *voter* 1 *votes* $\{a, c\}$*;*
- *voter* 2 *votes* $\{b\}$ *with probability 1/4 and* $\{a, b\}$ *with probability 3/4.*
- *voter* 3 *votes* $\{d\}$ *with probability 1/4 and* $\{c, d\}$ *with probability 3/4.*

Note that voter 1 *is not voting sincerely. Nevertheless, this strategy profile is an equilibrium and De Sinopoli et al (2006) show that it forms a singleton-stable set, an important refinement of Nash equilibrium.*

5 Conclusion

The analysis above raises three issues:

- An a priori restriction of voters' strategies based on a reasonable intuition such as undominated and sincere voting is not sufficient to restrict the set of possible outcomes of an Approval Voting election.
- Many refinements of Nash equilibrium, when applied to Approval Voting games, ensure the existence of equilibrium but the outcome of these equilibria do not seem to behave particularly well with respect to social choice requirements.
- Strong Nash equilibrium predicts Condorcet winners as the only Approval Voting outcomes but equilibrium fails to exist when there is no Condorcet winner.

These essentially negative theoretical results call for developing a finer understanding of how a voter chooses a ballot under Approval Voting. This analysis could rely on some general, game-theoretic principles such as the ones just described, but should probably also embody some elements specific to real voting situations such as the large size of the electorate, the specific structures of Approval Voting strategies, or the specificities of political information.

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