CHARACTERIZATION OF MAJORITY RULES

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Characterization of Majority Rules
(Çoğunluk Kurallarının Karakterize Edilmesi)

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Toplam Sayfa Sayısı: 18

Anahtar Kelimeler (Türkçe)	Anahtar Kelimeler (İngilizce)
1) Göreceli Çoğunluk	1) Relative Majority
2) Mutlak Çoğunluk	2) Absolute Majority
3) M_k Çoğunluk	3) M_k Majority
4) Mutlak q-Çoğunluk	4) Absolute q-Majority
5) Sosyal Seçme	5) Social Choice

Abstract

This paper examines different characterizations of the Relative Majority Rule, Absolute Majority Rule, and variations of those two, chronologically. Basically, Relative Majority Rule, as the name suggests, concerns with the relative number of supporters of the different alternatives. Whereas Absolute Majority Rule requires an alternative to be chosen more than half of the supporters to win. Relative Majority may cause an alternative with very poor support to win (i.e. one alternative has only one supporter and the other alternative has no supporters with 1000 voters will end up with the decision of this single voter at a two-alternative world.); on the other hand, it may for most of the time be difficult to have one of the alternatives to win via Absolute Majority Rule. Therefore, need for some moderate rules arises. M_k Majority helps us here as the number of supporters of the winner needs to exceed number of supporters of the other alternative by k voters. We may also need a more strict rule (If an alternative will win then for some cases, we may require it to have much more wider support). Then it is better to use Absolute q-Majority Rule, where we can chose q among the numbers, which is more than the number of half of the society. In the first section, necessary definitions and axioms are illustrated, and in the next section, Majority Rule characterizations and related theorems are given with their detailed proofs.

(Özet

Bu çalışma, Göreceli Çoğunluk Kuralı, Mutlak Çoğunluk Kuralı ve bunların farklı varyasyonlarını kronolojik olarak incelemektedir. Temel olarak Göreceli Çoğunluk Kuralı, adından da anlışlacağı gibi, farklı alternatiflerin destekçilerinin göreceli sayıları ile belirlenir. Diğer taraftan Mutlak Çoğunluk Kuralı, bir alternatifin seçilebilmesi için oyların mutlak çoğunluğunu almasını gerektirir. Göreceli Çoğunluk Kuralı, çok az desteğe sahip olan alternatifin kazanmasına neden olabilirken (2 alternatifli bir dünyada, 1000 kişilik bir grupta bir kişinin desteklediği alternatif, diğerinin hiç destekçisi olmaması durumunda oylamayı kazanacaktır.) Mutlak Çoğunluk Kuralı kullanılarak herhangi bir alternatifin oylamayı kazanması çoğu zaman zordur. Bu sebeplerden daha güçlü ve daha belirleyici orta seviyedeki seçim kurallarına ihtiyaç doğmuştur. Bu noktada bir alternatifin kazanması için diğerinden k sayıda fazla destekçi gerektiren M_k Çoğunluk Kuralı devreye girer. Bazı durumlarda çok daha katı bir kurala gereksinim de duyabiliriz. Bu durumlarda, bir alternatifin kazanması için toplumun yarı nüfusundan büyük sayılar arasından seçilen q sayısından fazla detekçiye gereksinim duyan Mutlak q-Çoğunluk Kuralı'nı kullanabiliriz. İlk bolümde gerekli tanım ve aksiyomlar, ikinci bölümde ise karakterizasyon ve ilgli diğer teoremler ayrıntılı ispatlarıyla birlikte sunulmuştur.)

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Introduction

Majority Rule is a very important Social Choice Rule as it adresses the preferences of each individual to aggregate decision of the whole society with the sence of Majority. May (1952) characterized majority rule via anonimity, neurality and positive responsiveness. After May's work Aşan-Sanver (2002), Woeginger (2003), Miroiu (2004), Woeginger (2005), Llamazares (2006), Aşan-Sanver (2006), Sanver (2006) defined characterizations for various types of majority rules. Characterizations made later than May's characterization, which was critisized for using the too much strong condition positively respensiveness, drop positively responsiveness and also anonimity in some cases and use instead conditions like pareto optimality and weakly path independence (Aşan-Sanver), reducibility to subsocieties (Woeginger), additive possitive responsiveness and subset decomposibility (Miroiu), cancellativeness (Llamazares), maskin monotonicity (Aşan-Sanver).

This paper collects major characterizations of different types of majority rules at a two alternatives world, where all the individuals have complete and transitive preferences over these two alternatives: x and y.

Basic Notations

Let $N = \{j_1, ...j_n\}$be a society and assume that each individual $j \in N$ has complete and transitive preference relations on a set of alternatives $A = \{x, y\}$.

We say $R_i = 1$ when the individual i prefers the alternative x to y; $R_i = -1$ when the individual i prefers the alternative y to x; and $R_i = 0$ when the individual i is indifferent between the alternatives x and y.

Let H be a subsociety of N such that $H = \{k_1, ...k_h\}$.

SWF (Social Welfare Function): $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$.

$$E^j = e_i^j = \left\{ \begin{array}{ll} 1, & if \quad i \leq j, \\ 0, & otherwise \end{array} \right\}$$

$$U^+ = \{R \in \{-1, 0, 1\}^n : n_+(R) = n - 1, \quad n_-(R) = 0\}$$

$$U^- = \{R \in \{-1, 0, 1\}^n : n_+(R) = 0, \quad n_-(R) = n - 1\}$$

$U = U^+ \cup U^-$ here $\cup$ does not mean the union in the common sense, but it means simply "or".

The operation: $R \oplus R' = (R_1, ...R_n, R_{n+1}, ...R_{n+n'}) \in \{-1, 0, 1\}^{n+n'}$, for given any $R \in \{-1, 0, 1\}^n$ and $R' \in \{-1, 0, 1\}^{n'}$.

$$p = n_+(R) := \#\{i \in N \mid R_i = 1\},$$

$$m = n_-(R) := \#\{i \in N \mid R_i = -1\}$$

$$z := n_0(R) := \#\{i \in N \mid R_i = 0\}$$

(Note that p+m+z=n)

Axioms and Independence

PR (Positively Responsiveness): For any profiles $R, R' \in \{-1,0,1\}^n$ with $R'_i \geq R_i$ (resp. $\leq$) $\forall i \in N$ and $R'_j > R_j$ (resp. $<$) for some $j \in S$, we have that $F(R) \geq 0$ (resp. $\leq$) $\Longrightarrow F(R') = 1$ (resp. -1).

APR (Additive Positive Responsiveness): For any $R \in \{-1,0,1\}^n$ with $F(R) \geq 0$ and $R_i = 1$, $i \notin N$(resp. $F(R) \leq 0$ and $R_i = -1$) we have $F(R \oplus R_i) = 1$ (resp. $F(R \oplus R_i) = -1$).

M (Maskin Monotonicity): For any profiles $R, R' \in \{-1,0,1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$ ($\leq$ resp.) $\forall i \in N$, we have $F(R) \geq 0 \Longrightarrow F(R') \geq 0$.($\leq$ resp.).

NE (Neutrality): For any $R \in \{-1,0,1\}^n$, we've $F(-R) = -F(R)$

A (Anonimity): For any $R \in \{-1,0,1\}^n$, and any permutation function $\pi : N \longrightarrow N$, we've $F(R_1, ...R_n) = F(R_{\pi(1)}, ...R_{\pi(n)})$

PO (Pareto Optimal): For any $R \in \{-1,0,1\}^n$, with $R_i \geq 0$ (resp. $R_i \leq 0$), $\forall i \in N$ and $R_j = 1$ (resp. $R_j = -1$) for some $j \in N$, we've $F(R) = 1$ (resp. $F(R) = -1$)

WPO (Weakly Pareto Optimal): If $R_i = 1$ holds for all $i \in N$, then $F(R) = 1$.

PI (Path Independence): Let $N = \{1, ...n\}$, $N' = \{n+1, ...n+n'\}$. A SWF F is said to be PI iff for any $R \in \{-1,0,1\}^n$ and $R' \in \{-1,0,1\}^{n'}$, we've $F(R \oplus R') = F(F(R) \oplus F(R'))$.

WPI (Weak Path Independence): F is said to be WPI iff for any $R \in \{-1,0,1\}^n$ and any $R' \in \{-1,0,1\}^{n'}$ with $|F(R) - F(R')| \neq 2$, we've $F(R \oplus R') = F(F(R) \oplus F(R'))$

RS (Reducibility to Subsocieties): For any profile $R \in \{-1,0,1\}^n$ with $n \geq 2$, we've $F(R) = F(F(R^{-1}), ...F(R^{-n}))$. Where $R^{-i} \in \{-1,0,1\}^{n-1}$ denote the profile that results from removing the ith voter from profile R.

SD (Subset Decomposability): A SWF F for a society $H = \{k_1, ...k_h\}$ $(h \geq 1)$ is SD iff $F(H) = F(F(H_1), ...F(H_m))$ where H_j $(1 \leq j \leq 2^h - 1)$ is a proper subset of H.

C (Cancelative): A SWF F is is cancelative if for all pair of profiles $R, R' \in \{-1,0,1\}^n$ such that $R_i = 1,\ R_j = -1$ and $R'_i = R'_j = 0$ for some $i, j \in N$, and $R'_l = R_l\ \forall l \in N \backslash \{i,\ j\}$, we have $F(R') = F(R)$.

p-Pareto: Given $p \in \{0, 1, ...n-1\}$, F is p-Pareto if:

a) For any $R \in \{-1,0,1\}^n$,

i) If $n_+(R) > p$ and $n_-(R) = 0$, then $F(R) = 1$.

ii) If $n_-(R) > p$ and $n_+(R) = 0$, then $F(R) = -1$.

b) If $p \in \{1, ...n-1\}$, $\exists\ R \in \{-1,0,1\}^n$ such that it satisifes one of the following conditions:

i) $n_+(R) = p$, $n_-(R) = 0$ and $F(R) < 1$.

ii) $n_-(R) = p$, $n_+(R) = 0$ and $F(R) > -1$.

q-Stable: Given $q \in \{0, 1, ...n-1\}$, a SWF F is q-Stable if it satisfies the following conditions:

i) $\forall R, R' \in \{-1, 0, 1\}^n$ such that $\# \{i \in N \mid R_i \neq R'_i\} \leq q$,

$F(R) = 1 \Longrightarrow F(R') \geq 0; \qquad F(R) = -1 \Longrightarrow F(R') \leq 0$

ii) $\exists R, R' \in \{-1, 0, 1\}^n$ such that $\# \{i \in N \mid R_i \neq R'_i\} = q + 1$ satisfying $F(R) = 1$ and $F(R') = -1$.

NSA (Null Society Assumption): If $H = \varnothing$, then $F(H) = 0$.

M_k Majority: Given $k \in \{0, 1, ...n-1\}$, the M_k Majority is the SWF defined by

$$M_k(R) = \left\{ \begin{array}{cc} 1, & if \sum_{i=1}^{n} R_i > k, \\ -1, & if \sum_{i=1}^{n} R_i < -k, \\ 0, & otherwise. \end{array} \right\}$$

or by N $M_k(R) = 1 \Longleftrightarrow n_+(R) > n_-(R) + k$.

MR (Relative Majority Rule): The MR is defined at each $R \in \{-1, 0, 1\}^n$ as ($MR(R) = 1 \Longleftrightarrow n_+(R) > n_+(R)$) and ($MR(R) = -1 \Longleftrightarrow n_-(R) > n_+(R)$).

UNA (Unanimous Majority Rule): A SWF that assigns $UNA(R) = 1$ (-1 resp.) only if all voters $i \in N$ have $R_i = 1$ (-1 resp.) and $UNA(R) = 0$ in all other cases.

Absolute q-majority: Let n^* be the lowest integer exceeding $n/2$. Picking some $q \in \{n^*, ..., n+1\}$, absolute q-majority rule is defined as an aggregation rule F such that $\forall R \in \{-1, 0, 1\}^n$ we have $F(R) = 1 \Longleftrightarrow n_+(R) \geq q$ and $F(R) = -1 \Longleftrightarrow n_-(R) \geq q$.

AMR (Absolute Majority Rule): Let n^* be the lowest integer exceeding $n/2$. The AMR is defined at each $R \in \{-1, 0, 1\}^n$ as ($AMR(R) = 1 \iff n_+(R) \geq n^*$) and ($AMR(R) = -1 \iff n_-(R) \geq n^*$).

Below theorem shows that PR and APR are not the same and neither implies the other.

Theorem 1 (Woeginger, 2005) *There exists a SWF F_1 that satisfies A and PR, but not APR. There exists a SWF F_2 that satisfies A and APR, but not PR.*

Proof

(A and PR, but not APR) F_1 is defined as $F_1(1) = F_1(0) = 1$ and $F_1(-1) = 0$ for $n = 1$, and $F_1 = 1$ for $n \geq 2$ voters. F_1 is A as for $n \geq 2$, we always have $F_1 = 1$, no matter what permutation we take. F_1 is also PR as $F_1(R) \leq 0$ is only the case when $F_1(-1) = 0$ where the voter is being the most negative. As there is no other profile satisfying $F_1(R) \leq 0$, we cannot establish any further negative profile, either. On the other hand F_1 is not APR since $F_1(-1) = 0$ and if we add voter -1 to the society $F_1(-1, -1)$ should be equal to -1 whereas here we have $F_1(-1, -1) = 1$.

(A and APR, but not PR) F_2 is defined as $F_2(1) = F_2(-1) = 1$ and $F_2(0) = 0$ for $n = 1$ and for $n \geq 2$, F_2 is defined as:

$F_2(R) = 0$, if $n_+(R) = n_-(R) = 0$

$F_2(R) = 1$, if $n_-(R) = 0$ and $n_+(R) > 0$

$F_2(R) = -1$, if $n_+(R) = 0$ and $n_-(R) > 0$

$F_2(R) = 1$, if $n_+(R) > 0$ and $n_-(R) > 0$

F_2 does not satisfy PR since $F_2(0) = 0$ implies $F_2(-1) = -1$ by PR, but here we have $F_2(-1) = 1$. F_2 is clearly A. To see that F_2 satisfies APR:

First let $F_2(R) \leq 0$ then either:

$n_+(R) = n_-(R) = 0$, and if we add -1 to the society, we have $n_+(R) = 0$ and $n_-(R) > 0$, which gives us $F_2(R) = -1$. Or

$n_+(R) = 0$ and $n_-(R) > 0$, and if we add -1 to the society, we again have $n_+(R) = 0$ and $n_-(R) > 0$, which gives us $F_2(R) = -1$.

Second let $F_2(R) \geq 0$ then either:

$n_+(R) = n_-(R) = 0$, and if we add 1 to the society, we have $n_-(R) = 0$ and $n_+(R) > 0$, which gives us $F_2(R) = 1$. Or

$n_-(R) = 0$ and $n_+(R) > 0$, and if we add 1 to the society, we again have $n_-(R) = 0$ and $n_+(R) > 0$, which gives us $F_2(R) = 1$. Or

$n_+(R) > 0$ and $n_-(R) > 0$, and if we add 1 to the society, we have $n_-(R) > 0$ and $n_+(R) > 0$, which gives us $F_2(R) = 1$. ■

Theorem 2 (Woeginger, 2003) *There exists a SWF F:*$\{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ *that is not the majority rule and that satisfies the axioms:*

a) N, A, RS, but not PO: $F(R) \equiv 0$ is N, A, RS, but not PO. Since $F(1, ..., 1) \neq 1$.

b) N, A, PO, but not RS: $F(R) = \left\{ \begin{array}{c} 1, \text{ if } R_j = 1 \text{ for some } j \in N \text{ and } R_i \geq 0, \ \forall i \in N, \\ -1, \text{ if } R_j = 1 \text{ for some } j \in N \text{ and } R_i \leq 0, \ \forall i \in N, \\ 0, \text{ otherwise.} \end{array} \right\}$

Since $F(1,1,-1) = 0 \neq F(0,0,1) = F(F(1,-1), F(1,-1), F(1,1))$ F is not RS.

c) A, PO, RS, but not N: $F(R) = \left\{ \begin{array}{c} 0, \text{ if } R_i = 0, \ \forall i \in N, \\ 1, \text{ if } R_j = 1 \text{ for some } j \in N \text{ and } R_i \geq 0 \, \forall i \in N, \\ -1, \text{ otherwise.} \end{array} \right\}.$

Since $-F(1,-1) = -(-1) = 1 \neq -1 = F(-1,1)$, it is not N. ■

Results

1) Impossibilities

Theorem 3 (Aşan-Sanver, 2006) *There exists no SWF, which satisfies M and PO.*

Proof.

Suppose for a contradiction, F is an aggregation rule which satisfies M and PO. Take some $R \in \{-1,0,1\}^n$ and WLOG consider the case $F(R) \in \{0,1\}$. Define another profile $R' \in \{-1,0,1\}^n$ be such that $R'_i = 0 \iff R_i \in \{0,1\}$ and $R'_i = -1 \iff R_i = -1, \forall i \in N$. $F(R') = -1$ by PO and $F(R') \in \{0,1\}$ by M gives us a contradiction. ■

2) Characterizations

First, see below some MR implications:

M1) A SWF $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ is MR $\Longrightarrow$ It is NE: If $F(R) = 0$ then $n_+(R) = n_-(R)$ and $F(-R) = 0$ as well. If $F(R) = 1$ then $n_+(R) > n_-(R)$ meaning when we multiply R by -1, we get $n_-(R) > n_+(R) \Longrightarrow F(-R) = -1 = -F(R)$. The case of $F(R) = -1$ is symmetrical to $F(R) = -1$. Hence F is NE.

M2) A SWF $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ is MR $\Longrightarrow$ It is PO: Take any $R \in \{-1,0,1\}^n$ such that $R_i \geq 0$ (respectively $R_i \leq 0$) $\forall i \in N$ and $R_j = 1$ (respectively $R_j = -1$) for some $j \in N$ then $n_+(R) > 0$ and $n_-(R) = 0 \Longrightarrow n_+(R) > n_-(R) \Longrightarrow F(R) = 1$ (respectively $n_+(R) = 0$ and $n_-(R) > 0 \Longrightarrow n_-(R) > n_+(R) \Longrightarrow F(R) = -1$).Hence F is PO.

M3) A SWF $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ is MR $\Longrightarrow$ It is WPI: Take any $k, k' > 0$ and consider two disjoint societies $K = \{1, ..., k\}$ and $K' = \{k+1, ..., n\}$ (where $k + k' = n$) and take any $R \in \{-1,0,1\}^k$, and any $R' \in \{-1,0,1\}^{k'}$ with $|F(R) - F(R')| \neq 2$ then

a) If $n_+(R) = n_-(R)$ (i.e. $F(R) = 0$) and $n_+(R') = n_-(R')$ (i.e. $F(R') = 0$) $\Longrightarrow n_+(R \oplus R') = n_-(R \oplus R') \Longrightarrow F(R' \oplus R'') = 0$.

If $n_+(R \oplus R') = n_-(R \oplus R')$ and as $|F(R) - F(R')| \neq 2$ we must have $n_+(R) = n_-(R)$ (i.e. $F(R) = 0$) and $n_+(R') = n_-(R')$ (i.e. $F(R') = 0$)$\Longrightarrow F(F(R) \oplus F(R')) = 0$.

b) If $n_+(R) > n_-(R)$ (i.e. $F(R) = 1$) then either we have $n_+(R') > n_-(R')$ or $n_+(R') = n_-(R')$ (i.e. $F(R') \geq 0$) $\Longrightarrow n_+(R \oplus R') > n_-(R \oplus R') \Longrightarrow F(R \oplus R') = 1$.

If $n_+(R \oplus R') > n_-(R \oplus R')$ and as $|F(R) - F(R')| \neq 2$ we must have both $n_+(R) \geq n_-(R)$ and $n_+(R') \geq n_-(R')$; furthermore, one of the inequalities should be strict, say WLOG $n_+(R') > n_-(R') \Longrightarrow F(F(R) \oplus F(R')) = 1$.The reverse case can similarly be shown.

Hence F is WPI.

M4) A SWF $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is MR $\Longrightarrow$ It is A: F is A as the only concern of MR is the relative number of supporters of the two alternatives, it does not matter who supports whom.

M5) A SWF $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is MR $\Longrightarrow$ It is PR: Take any profiles $R, R' \in \{-1, 0, 1\}^n$ with $R'_i \geq R_i \; \forall i \in N$ and $R'_j > R_j$ for some $j \in N$, and let $F(R) \geq 0$ then $n_+(R) \geq n_-(R)$ but as $R'_j > R_j$ for some $j \in N$ we have $n_+(R') > n_-(R')$ which consequently implies $F(R') = 1$. Hence F is PR.

M6) A SWF $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is MR $\Longrightarrow$ It is RS: Take any $R \in \{-1, 0, 1\}^n$:

$F(R) = 0 \Longleftrightarrow n_+(R) = n_-(R)$. As F is MR, and so A, let $R_i = 1$, $R_{i+1} = -1 \; \forall i \leq k$ for some odd interger k. (Note that $k < n$) Then we have $F(F(R^{-1}), ...F(R^{-n})) = F(-1, 1, -1, 1, ... -1, 1, 0, ..., 0) = 0 = F(R)$.

$F(R) = 1$ such that $n_+(R) = n_-(R) + 1$. Then $F(R^{-i}) \geq 0 \forall i \in N$ and $F(R^{-j}) = 1$ for some $j \in N$. By A we can set the order such that $F(1, ..., 1, 0, ..., 0) = 1 = F(R)$.

$F(R) = 1$ such that $n_+(R) > n_-(R) + 1$.Then $F(R^{-i}) = 1 \forall i \in N \Longrightarrow F(1, ..., 1) = 1 = F(R)$.

The case $F(R) = -1$ can similarly be shown as $F(R) = 1$.

Hence F is RS.

M7) A SWF $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is MR $\Longrightarrow$ It is WPO: Take any $R \in \{-1, 0, 1\}^n$ such that $R_i = 1 \; \forall i \in N$ then $n_+(R) = n$ meaning definitely $n_+(R) > n_-(R) \Longrightarrow F(R) = 1$. Hence F is WPO.

M8) A SWF $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ is MR $\Longrightarrow$ It is C: MR is obviously C as we are interested in $n_+(R)$ and $n_-(R)$ only relatively; therefore, cancelling out one x supporter and one y supporter does not change the result.

Proposition 4 (May, 1952) *A social welfare function (SWF)* $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ *satisfies A, NE, PR* $\Longleftrightarrow$ *It is the MR.*

Proof.

($\Longleftarrow$): Take any SWF F which is MR, then it is A by M4, NE by M1 and PR by M5.

($\Longrightarrow$): Take a SWF F which is A, NE and PR. Consider:

$p = m$: Then $F(R) = 0$. For a contradiction assume that $F(R) = 1$ then by NE $F(-R) = -1$ but $F(-R) = F(R)$ by A.

$p = m + 1$: Then $F(R) = 1$ by PR and above case. We can iterate this case further and by the same reasoning we will get $F(R) = 1$ for $p = m + k$ where $0 < k \leq n - m$.

Hence $p > m$ implies $F(R) = 1$.

$m = p + 1$: Then $F(R) = -1$ by PR and above case. We can iterate this case further and by the same reasoning we will get $F(R) = -1$ for $m = p + k$ where $0 < k \leq n - p$.

Hence $m > p$ implies $F(R) = -1$. Which completes the proof that F is the MR.

■

After May was criticised by using a too strong condition, PR, in his characterization some other characterizations including more than two alternatives were established also by May. However, we will next look at the below theorem formed by Aşan and Sanver, which replaces PR with PO and WPI, still concerning two alternatives.

Theorem 5 (Aşan-Sanver, 2002) *A SWF* $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ *satisfies A, NE, PO and WPI* $\Longleftrightarrow$ F *is the MR.*

Proof.

($\Longleftarrow$): Take a SWF F which is MR. Then F is A by M4, NE by M1, PO by M2 and WPI by M3.

($\Longrightarrow$): Take any SWF F which is A, N, PO and WPI. Take any $n > 0$ for $N = \{1, ...n\}$.

To show F is MR, we need to show $\forall R \in \{-1,0,1\}^n$, we have:

1) $n_+(R) = n_-(R) \Longrightarrow F(R) = 0$

For a contradiction, assume WLOG $F(R) = 1$ but $F(-R) = 1$ by A, and $F(-R) = -1$ by N, which gives a contradiction.

2) $n_+(R) > n_-(R) \Longrightarrow F(R) = 1$

Take any $R \in \{-1,0,1\}^n$ with $n_+(R) > n_-(R)$ and let $k := n_+(R) - n_-(R)$. Now take some $K \subset \{i \in N : R_i = 1\}$ such that $\mid K \mid = k$. Consider K and $N - K$ with $R' \in \{-1,0,1\}^k$ and $R'' \in \{-1,0,1\}^{n-k}$ defined as $Ri' = Ri \, \forall i \in K$, $Ri'' = Ri$ $\forall i \in N - K$

Hence $n_+(R') = k \Longrightarrow F(R') = 1$ by PO, and $n_+(R'') = n_-(R'') \Longrightarrow F(R'') = 0$ by NE and A of F. Thus $|F(R') - F(R'')| \neq 2$, and $F(R' \oplus R'') = F(F(R') \oplus F(R''))$ by WPI. As we constructed R, R' so as $R' \oplus R'' = R \Longrightarrow F(R) = F(1,0)$ which is equal to 1 as F is PO.

3) $n_+(R) < n_-(R) \Longrightarrow F(R) = -1$

This part of proof is similar to (2) ■

Theorem 6 (Woeginger, 2003) *Let $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ be a SWF that satisfies NE, RS, and the anchor condition $F(0,1) = 1$. Then F satisfies WPO $\Longleftrightarrow$ F satisfies PO.*

Proof.

($\Longrightarrow$): Let F be N, RS, $F(0,1) = 1$ and WPO, we need to show F is PO as well. For proof by induction, first let $n = 1$ then $F(1) = 1$ and $F(-1) = -1$ by WPO. For $n = 2$, $F(1,1) = 1$ and $F(-1,-1) = -1$ by again WPO. We assumed at the begining, $F(0,1) = 1$ by N $F(0,-1) = -1$ and by RS $F(0,-1) = F(-1,0) = -1$ and $F(0,1) = F(1,0) = 1$. Then F is PO for $n \leq 2$.

($\Longleftarrow$): Now take any $R \in \{-1,0,1\}^n$ with $n \geq 3$ such that $R_i \geq 0$, $\forall i \in N$ and $R_j = 1$, for some $j \in N$. Then $m = 0$, $p \geq 1$ and $z = n - p$.

First consider $p = 1$ then $z = n-1 \geq 2$. $n-1$ of the subsocieties have one voter prefering 1 and other voters prefering 0. And in the remaining one society everybody votes for 0. Then by induction and RS, we get $F(R) = F(R')$ (i.e. every subsociety provides PO), where R' has $(p', z', m') = (n-1, 1, 0)$. then $F(R') = 1$.

Second, consider $p \geq 2$. Then every subsociety has at least one voter prefering 1.whereas the remaining voters either prefer 1 or 0. As we assumed for the induction hypothesis, every subsociety has 1 as aggregate decision. As F is RS, $F(R) = F(R'')$ for R'' such that $(p'', z'', m'') = (n, 0, 0)$ and $F(R'') = 1$ by WPO.

Hence $F(R) = 1$. The reverse case is symmetrical. ■

Theorem 7 (Woeginger, 2003) *A SWF $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ satisfies NE, PO and RS $\Longleftrightarrow$ it is the MR.*

Proof.

($\Longleftarrow$): Take a SWF F, which is MR. NE, PO and RS were proved at M1, M2 and M6.

($\Longrightarrow$): By induction:

$n = 1 \Longrightarrow F(1) = 1 = MR(1)$ and $F(-1) = -1 = MR(-1)$ by PO and $F(0) = 0 = MR(0)$ by N.

Hence for $R \in \{-1, 0, 1\}^1$, we have $F(R) = MR(R)$

$n \geq 2 \Longrightarrow$ Assume $\forall R' \in \{-1, 0, 1\}^{n-1}$, $F(R') = MR(R')$. Consider $R \in \{-1, 0, 1\}^n$ and let $p := n_+(R)$, $m := n_-(R)$, $z := n_0(R)$ (Note that p+m+z=n). There are three cases to look at:

If $p = m \Longrightarrow F(R^{-n}) = MR(R^{-n}) = -R_n$ (As we exclude the nth person.) $\forall n \in S$ by induction hypothesis. $F(R) = F(F(R^{-1}), ..., F(R^{-n})) = F(-R)$ by RS and $F(-R) = -F(R)$ by N. Hence $F(R) = -F(R)$, meaning $F(R) = 0 = MR(R)$.

If $p \geq m+1$ holds, then eliminating any single individual will lead $p \geq m$ for all individuals, and $p > m$ for some individuals.By induction, $F(R') = MR(R') \forall R' \in \{-1, 0, 1\}^{n-1}$. (i.e. it may look like $F(1, ..., 1, 0, ..., 0)$). And by PO $F(R) = 1 = MR(R)$.

If $p \leq m - 1$ holds then $F(R) = MR(R)$ and the proof is symmetric to the previous one. ■

Theorem 8 (Woeginger, 2003-4) *Let* $F : \{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ *be a SWF that satisfies NE, WPO, and RS:*

a) If $F(0, 1) = 1$, then F is the MR.

b) If $F(0, 1) = 0$, then F is the UNA.

Proof.

a) If $F(0, 1) = 1$, then $WPO \equiv PO$ Hence, the previous two theorems together shows the result.

b) Then $F(0, -1) = 0$ and $F(0, 0) = F(-1, 1) = 0$ by NE, and $F(1, 1) = 1$ and $F(-1, -1) = -1$ by WPO. Therefore, $F(R) = UNA(R)$ for $n \leq 2$.

To prove by induction, assume we have $F(R) = UNA(R)$ for $n - 1$ voters and $n \geq 3$. Consider $R \in \{-1, 0, 1\}^n$:

If $p = n$, then WPO yields $F(R) = UNA(R) = 1$.

If $p = n - 1$, then also by RS $F(R) = F(R')$ where R' has $(p', z', m') = (1, n-1, 0)$. applying RS and induction again one more time, we have $F(R') = F(R'')$ where R'' has $(p'', z'', m'') = (0, n, 0)$. As F is NE, $F(R'') = 0$, hence $F(R) = UNA(R) = 0$.

If $2 \leq p \leq n - 2$, then RS and induction together imply $F(R) = F(R')$ where R' has $(p', z', m') = (0, n, 0)$. Hence $F(R) = UNA(R) = 0$.

The cases where $m \geq 2$ can be proved symmetrically.

Finally, if $p \leq 1$ and $m \leq 1$, then $z \geq n - 2 \geq 1$. Then the function F gives 0 for all subsocieties of R. Then $F(R) = UNA(R) = 0$ by RS and N. ■

Lemma 9 (Miroiu, 2004) *If a SWF F satisfies NSA, AR, and NE, then $F(R_j) = R_j$.*

Proof.

$F(\varnothing) = 0$ by NSA

Now let $R_j = 1$ (resp. $R_j = -1$) then $F(\varnothing \cup R_j) = F(R_j) = 1$ by AR.

For $R_j = 0$, $F(R_j) = 0$ by NE. (ie. Assume for a contradiction WLOG $F(R_j) = 1 \implies 1 = F(R_j) = F(-R_j) = -F(R_j) = -1\#$) ■

Lemma 10 (Miroiu, 2004) *If a SWF F satisfies NSA, AR, and NE, then it satisfies PO.*

Proof.

For subsociety H let $R_i \geq 0 \forall i$ and $R_i = 1$ for some$i \in H$. By induction,

$\underline{n = 1} \implies F(H) = 1$ by above Lemma.

$\underline{n \geq 2} \implies$ Assume F is PO for $n-1$ voters and $|H| = n-1$ and $j \notin H$, where $R_j \geq 0$. Then $|H \cup \{j\}| = n$. (Note that if $R_i = 0, \forall i$ then $F(R) = 0$ by N.)

If $R_j = 1$ then since $F(H) \geq 0$, we get $F(H \cup \{j\}) = 1$ by AR.

If $R_j = 0$ then as we supposed in the definiton, there was some $j' \in H$ such that $R_{j'} = 1$. Now let $H' = (H \cup \{j\}) - \{j'\} \implies F(H') \geq 0$ as now we have $n-1$ people again.

$F(H \cup \{j\}) = F(((H \cup \{j\}) - \{j'\}) \cup \{j'\}) = 1$, by AR. ■

Theorem 11 (Miroiu, 2004) *If NSA holds, then a SWF F satisfies AR, NE, and SD $\Longleftrightarrow$ it is the MR.*

Proof.

($\Longleftarrow$): NE was proved at M1. To show F is AR is easy. Let H be such that $F(H) \geq 0$, and let $j \notin H$ and $R_j = 1$. Then as $F(H) \geq 0$ $n_+(H) \geq n_-(H)$, if we add the person j with $R_j = 1$ we get $n_+(H) > n_-(H)$. Then $F(H \cup \{j\}) = 1$ by MR.(The reverse case is similar to show.) Hence F is AR.

$(MR \Longrightarrow SD)$: Suppose that F is MR and $|H| = k$ and $k = p + z + m$. We need to show $F(H) = F(F(H_1), ...F(H_r))$ with H_j $(1 \leq j \leq r = 2^k - 1)$ the subsets of H. Define $P_s(H)$ be the set of all sets H^* such that $|H^*| = s$. Obviously, is $s = 0$, then $F(H^*) = 0$. Consider the cases:

$p > m$: Then for each s $\exists$ no majority of sets H^* such that $|H^*| = s$ and $F(H^*) = -1$. (If $\exists$ no such H^* at all, then $F(H^*) \geq 0$ for all H^*, and since $p > 0$ $\exists$ at least one H^* such that $F(H^*) = 1$, then as NSA, AR, and NE, implies PO by above lemma, we will get $F(H) = 1$.) Even is there exists some H^* such that $F(H^*) = -1$ they are not majority of the whole subsets; therefore, MR will give the result $F(H) = 1$. Hence $F(H) = F(F(H_1), ...F(H_r))$.

The other two cases where $p = m$ and $p < m$ are similar to show.

($\Longrightarrow$): To prove the reverse implication, we will again use induction:

$n = 1$: by Lemma 9, $F(R_j) = R_j = MR(R_j)$,

$n \geq 2$: Supppose F is MR for all $n' < n$. Consider the cases:

$p > m$: Then there exists at least one individual j such that $R_j = 1$. For the society $H - \{j\}$, we have $p \geq m$ and by the induction hypothesis $F(H - \{j\}) \geq 0$. By AR, $F((H - \{j\}) \cup \{j\}) = F(H) = 1 = MR(H)$.

$p < m$: is similar to the above case.

$p = m$: If $p = m = 0$, then $F(H) = F(0, ..., 0) = 0 = MR(H)$. Now let $p = m > 0$, we have two subcases to consider:

$z = 0$: Take any subsociety H_1 ($|H_1| = k_1$) such that p_1 of the voters have $R_j = 1$ and m_1 of the voters have $R_j = -1$ and $k_1 = p_1 + m_1$, and there is one and only one subsociety H_2 ($|H_2| = k_1$) such that m_1 of the voters have $R_j = 1$ and p_1 of the voters have $R_j = -1$ and $k_1 = p_1 + m_1$. By N, $F(H_1) = -F(H_2)$. But as H_1 was arbitrary, this is true for all subsocieties, they by N $F(F(H_1), ...F(H_r)) = 0$. Therefore, $F(H) = 0 = MR(H)$ by SD.

$z \geq 1$: Then there is at least one j such that $R_j = 0$ and $|H - \{j\}| = n - 1$ meaning by induction hypothesis $F(H - \{j\}) = 0 \Longrightarrow F((H - \{j\}) \cup \{j\}) = 0 = F(H) = MR(H)$. ■

Theorem 12 (Woeginger, 2005) *A SWF F satisfies NE, A, and APR $\Longleftrightarrow$ it is the MR.*

Proof.

($\Longleftarrow$): We already proved that if F is MR then it also satsfies NE, A at M1 and M4. Proof of APR is similar to AR.

($\Longrightarrow$):By induction, Take any SWF F satisfiying NE, A, and APR:

$\underline{n=1}\Longrightarrow F(0)=0=MR(0)$ by NE. Suppose for a contradiction$F(1)=\alpha\leq 0$, then by NE $F(-1)=-\alpha\geq 0$. The society $\{-1,1\}$ on the one hand by adding the new voter -1 to the society $\{1\}$ which brings the result $F(1,-1)=-1$ by APR. On the other hand, if we add voter 1 to the society $\{-1\}$, we get $F(1,-1)=1$ by again APR, which leads to a contradiction. Hence $F(1)=1=MR(1)$ and furthermore, $F(-1)=-1=MR(-1)$ by N.

$\underline{n\geq 2}\Longrightarrow$ Assume $F(R')=MR(R')$ for all R' with $n-1$ voters. Now consider a profile R with n voters:

$p=m$: Then $F(R)=0=MR(R)$ by A and NE.

$p\ \geq\ m+1$: Then by the induction hypothesis we have $p'\ \geq\ m'$ and $F(R')\geq 0$. Hence, $F(R)=1=MR(R)$ by APR.

$p\leq m-1$: This case is similar to prove the second case. Here we have $F(R)=-1=MR(R)$.

Hence F is MR. ■

The above theorem is very similar to May's majority characterization since Woeginger only replaces PR by APR, but he later shows that PR and APR are independent of each other, which was shown in the "Independence" Part.

Remark1 (Woeginger, 2005) As $SD\Longrightarrow A$ Woeginger replaced SD with A in Miroiu's characterization.

Theorem 13 (Woeginger, 2005) *There exists a SWF F_3 that is not MR and that satisfies NE, A, PO, and SD.*

Proof.

F_3 is defined as:

$F_3(R) = 0$, if $n_+(R) = n_-(R) = 0$ (I)

$F_3(R) = 1$, if $n_-(R) = 0$ and $n_+(R) > 0$ (II)

$F_3(R) = -1$, if $n_+(R) = 0$ and $n_-(R) > 0$ (lII)

$F_3(R) = 0$, if $n_+(R) > 0$ and $n_-(R) > 0$ (IV)

F_3 is not MR as $F_3(1, 1, -1) = 0 \neq 1 = MR(1, 1, -1)$. F_3 clearly satisfies A, NE, and PO. To show that F_3 satisfies SD, consider $n \geq 2$ and let $R_1,$ ....R_k with $k = 2^n - 1$ be an enumeration of all proper multi-subsets of R. Consider the cases:

If $n_+(R) = n_-(R) = 0$, then all multi-subsets R_i of R consists of indifferent voters only; therefore, the result will be indifference as well. Hence, $F_3(F_3(R_1),$ ....., $F_3(R_k)) = 0 = F_3(R)$.

If $n_-(R) = 0$ and $n_+(R) > 0$, then all multi-subsets consists of indifferent or positive voters only, meaning $F_3(R_i) \geq 0 \; \forall i \in k$. And $F_3(R_j) = 1$ for some multi-subsets R_j. Therefore, $F_3(F_3(R_1),$ ....., $F_3(R_k)) = 1 = F_3(R)$.

If $n_+(R) = 0$ and $n_-(R) > 0$, then the proof is similar to the previous case.

If $n_-(R) > 0$ and $n_+(R) > 0$, then there exist one negative say "i" and one positive say "j" voters, satisfying $F_3(R_i) = -1$ and $F_3(R_j) = 1$. Therefore,

$F_3(F_3(R_1), ..., F_3(R_k)) = 0 = F_3(R)$. Therefore F is SD, and the proof is completed. ■

Remark2 (Llamazares, 2006) If F is NE, then:

(1) $F(E^0) = 0$ since $F(E^0) = F(-E^0) = -F(E^0) = 0$ by NE.

(2) F is characterized by the set $F^{-1}(\{1\})$, since

$$F^{-1}(\{-1\}) = \{R \in \{-1, 0, 1\}^n : -R \in F^{-1}(\{1\})\},$$

$$F^{-1}(\{0\}) = \{-1, 0, 1\}^n \diagdown (F^{-1}(\{1\}) \cup F^{-1}(\{-1\})).$$

Proposition 14 (Llamazares, 2006) *If F is a C SWF, then $F(R_\sigma) = F(R)$ $\forall R \in \{-1, 0, 1\}^n \backslash U$ and all permutation σ on N.*

Proof.

For any $R \in \{R \in \{-1, 0, 1\}^n \backslash U : n_+(R) = n_-(R) + m\}$, with $m \geq 0, m \neq n-1$ (since if we had the equality, our collection of profiles would be U^+), it is enough to prove that $F(R) = F(E^m)$ since R amd m are arbitrary. (the reverse case, $n_+(R) < n_-(R)$ is similar to prove.) Since $n_+(R_\sigma) = n_+(R)$ and $n_-(R_\sigma) = n_-(R)$ for all permutation σ on N, we will have $F(R_\sigma) = F(E^m) = F(R)$.

Let's pick another profile $R^* \in \{-1, 0, 1\}^n \backslash U$ satisfying $n_+(R^*) = m$ and $n_-(R^*) = 0$ such that $F(R) = F(R^*)$ as F is C. Therefore, WLOG it is possible to assume $n_+(R^*) = m$ and $n_-(R^*) = 0$:

(1) If $m = n$ or $m = 0$, then $R = E^m$ the result is either $F((1, ..., 1))$ or $F((0, ..., 0))$.

(2) If $1 \leq m \leq n-2$ and $R \neq E^m$, then $\exists j \in \{1, ..., m\}$ and $l \in \{m+1, ..., n\}$ such that $R_j = 0$ and $R_l = 1$.As $n_+(R) \leq n-2, \exists r \in N, r \neq j$ such that $R_r = 0$ Now we consider the profiles R' and R'' such that:

$$R'_i = \left\{ \begin{array}{c} 1, \text{ if } i = j, \\ -1, \text{ if } i = r, \\ R_i, \text{ otherwise.} \end{array} \right\}, \qquad R''_i = \left\{ \begin{array}{c} 0, \text{ if } i = r, l, \\ R'_i, \text{ otherwise.} \end{array} \right\}$$

Since F is C we have $F(R) = F(R') = F(R'')$. If $R'' \neq E^m$, then we can repeat the previous process until we find such a profile. Hence $F(R) = F(E^m)$ even if $R \neq E^m$. ■

Corollary 15 (Llamazares, 2006) *If F is a C SWF, then F is completely determined by its values in the set $U \cup \{E^j : j \in N \cup \{0\}\} \cup \{-E^j : j \in N\}$.*

Corollary 16 (Llamazares, 2006) *Let F be a C SWF:*

(1) If F is A, then F is completely determined by its values in the set $\{E^j : j \in N \cup \{0\}\} \cup \{-E^j : j \in N\}$ (By A, U^+ is included in the first set here, and U^- is included in the second set.)

(2) If F is NE, then F is completely determined by its values in the set $U^+ \cup \{E^j : j \in N \cup \{0\}\}$ (We can reach the result just by taking the minus of these sets.)

(3) If F is A and NE, then F is completely determined by its values in the set $\{E^j : j \in N \cup \{0\}\}$ (Reasoning is simple when we combine (1) and (2))

(4) If F is p-Pareto, then F is completely determined by its values in the set

(a) $\{E^0\}$, if $p = 0$ ($\{(0, ..., 0)\}$).

(b) $\{E^j : j \in \{0, 1, ..., p\}\} \cup \{-E^j : j \in N \cup \{1, ...p\}\}$ if $p \in \{1, ..., n-2\}$ (i.e. $\{E^0, E^1, ..., E^p, \}$ for $p \in \{1, ..., n-2\}$).

(c) $U \cup \{E^j : j \in \{0, 1, ..., n-2\}\} \cup \{-E^j : j \in \{1, ..., n-2\}\}$ if $p = n-1$.

Theorem 17 (Llamazares, 2006) *A SWF F is a M_k majority if and only if it is A, NE, M, weak Pareto and C.*

Proof.

($\Longrightarrow$) : Let's first check that any M_k majority satisfies the properties. It satisfies A, as social consequence of M_k majority only depends on the numbers $n_+(R)$ and $n_-(R)$ but not on which profile belond to which individual. If we multiply each profile by minus one then we get the opposite of the winner when it was 1 or -1, and 0 when it was already 0, hence M_k majority is N. Take any $R, R' \in \{-1, 0, 1\}^n$ such that $R' \geq R \Longrightarrow n_+(R') \geq n_+(R) \Longrightarrow F(R') \geq F(R)$ proving M_k majority is M. M_k majority is weak Pareto as the collective profile E^n has $n_+(E^n) = n > k$ since $k \in \{0, 1, ..., n-1\}$. M_k majority is obviously C as we are interested in $n_+(R)$ and $n_-(R)$ only relatively.

($\Longleftarrow$) : Reciprocally, suppose that F is anonymous, neutral, monotonic, weak Pareto and cancellative. By Corollary 16(3) F is determined by the set $\{E^j : j \in N \cup \{0\}\}$.

Since F is weak Pareto, $F(E^n) = 1$, and by Remark 2(1) we also have $F(E^0) = 0$. Moreover, $F(E^i) \geq F(E^j)\ \forall i > j$ because F is monotonic. Therefore, $\exists\ k \in \{1, ..., n-1\}$ such that $F(E^{k+1}) = 1$ and $F(E^k) = 0$; meaning, F is the M_k majority. ■

Remark3 (Llamazares, 2006)

(1) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1, \text{ if } R = E^n, \\ -1, \text{ if } R = -E^n, \\ R_1, \text{ if } R \in U, \\ 0, \text{ otherwise.} \end{array} \right\}$ is NE, M, weak P, and C, but not A. (Not A, as the first person is the dictator when $R \in U$.)

(2) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1, \text{ if } R \in U^+ \cup \{E^n\}, \\ -1, \text{ if } R = -E^n, \\ 0, \text{ otherwise.} \end{array} \right\}$ is A, M, weak P, and C, but not NE. (Not N, as $F(-R) \neq -F(R)$)

(3) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1, \text{ if } R \in U^- \cup \{E^n\}, \\ -1, \text{ if } R \in U^+ \cup -\{E^n\}, \\ 0, \text{ otherwise.} \end{array} \right\}$ is NE, A, weak P, and C, but not M. (Take $R' \in U^+$ and $R \in U^-$ then $R' \geq R$, but $F(R') \ngeq F(R)$ hence F is not M.)

(4) The null SWF, ie.i $F(R) = 0\ \forall R \in \{-1, 0, 1\}^n$, is A, NE, M and C, but not weak P (as $F(E^n) \neq 1$ and $F(-E^n) \neq -1$). By above theorem, the null SWF is the only one that satisfies these conditions.

(5) The AMR, $AMR(R) = \left\{ \begin{array}{c} 1, \text{ if } n^+(R) > \frac{n}{2}, \\ -1, \text{ if } n^-(R) > \frac{n}{2}, \\ 0, \text{ otherwise.} \end{array} \right\}$ is NE, M, weak P, and A but not C. (let for the profile R, $n^+(R) = n^*$ where $R_k = 1$ and $R_l = -1$ then $AMR(R) = 1$, furthermore let another profile R'such that $R'_k = 0$ and $R'_l = 0$ and

$R_i = R'_i \; \forall i \in N - \{k, l\}$ then $AMR(R) = 0$, which according to C should have been equal to 1 again. Hence AMR is not C.)

Theorem 18 (Llamazares, 2006) *A SWF F is the MR if and only if it is C, strong Pareto and $F(E^0) = 0$.*

Proof.

$(\Longrightarrow)$: MR implies C by M8. F is strong Pareto and $F(E^0) = 0$ is obvious.

$(\Longleftarrow)$: Now take any F such that it is cancellative, strong Pareto and $F(E^0) = 0$. By Corollary 16(4a) any C and Strong Pareto SWF is determined by its value in the profile E^0. Hence, F is MR if $F(E^0) = 0$. ■

Remark4 (Llamazares, 2006)

(1) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1, \text{ if } n^+(R) > 0 \text{ and } n^-(R) = 0, \\ -1, \text{ if } n^-(R) > 0 \text{ and } n^+(R) = 0, \\ R_1, \text{ otherwise.} \end{array} \right\}$ is PO and $F(E^0) = 0$, but not C.

(2) The SWF F defined by $F(R) = -M_0(R) \; \forall R \in \{-1, 0, 1\}^n$ is C and $F(E^0) = 0$, but not PO.

(3) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1, \text{ if } n^+(R) \geq n^-(R), \\ -1, \text{ if } n^-(R) > n^+(R), \end{array} \right\}$ is PO and C, but $F(E^0) \neq 0$.

Corollary 19 (Llamazares, 2006) *A SWF F is MR if and only if it is C, strong Pareto and NE. (i.e.$F(E^0) = 0$ is replaced by NE of previous theorem.)*

Proof.

As we have NE implies $F(E^0) = 0$, it is enough to consider the proof of the previous theorem. ■

Proposition 20 (Llamazares, 2006) *Given $k \in \{0, 1, ..., n-1\}$, the M_k majority is k-stable.*

Proof.

We are going to prove that the M_k majority satisfies the conditions of being q-Stable. Let $R, R' \in \{-1, 0, 1\}^n$ such that $\#\{i \in N : R_i \neq R'_i\} \leq k$. Since $n^+(R') \geq n^+(R) - k$ and $n^-(R') \leq n^-(R) + k$, we have:

$n^-(R') \leq n^-(R) + k < n^+(R) \leq n^+(R') + k$, i.e. $M_k(R') \geq 0$.

The case $M_k(R) = -1$ comes from the NE of M_k.

On the other hand, in order to prove the second condition of Definition of q-Stable F it is sufficient to consider E^{k+1} and -E^{k+1}. ■

Theorem 21 (Llamazares, 2006) *Given $k \in \{1, ..., n-2\}$, a SWF F is the M_k majority if and only if it is NE, M, C and k-Pareto.*

Proof.

$(\Longrightarrow)$: M_k majority satisfies NE and C as MR satisfies all by M1 and M8. For any profiles $R, R' \in \{-1,0,1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$ ($\leq$ resp.) $\forall i \in N$, assume we have $F(R) \geq 0$ ($\leq$ resp.).then as F is M_k majority $n^-(R) \ngtr n^+(R)+k$ so as $n^-(R') \ngtr n^+(R')+k$ we have $F(R') \geq 0$. Hence F is M. M_k majority is k-Pareto by above proposition.

$(\Longleftarrow)$: Reciprocally, suppose that F is NE, M, C and k-Pareto. By Proposition 14, $F(R_\sigma) = F(R)\ \forall\ R \in \{-1,0,1\}^n \setminus U$ and all permutation σ on N. Moreover, $F(R) = 1\ \forall\ R \in U$ because F is k-Pareto. Therefore, F is A and by Theorem 17 we have that F is a M_j majority for some $j \in \{1,...,n-2\}$. Finally, F is the M_k majority because it is k-Pareto. ■

Remark5 (Llamazares, 2006) Let $k \in \{1,...,n-2\}$:

(1) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1,\ \text{if}\, n^+(R) > n^-(R)+k, \\ -1,\ \text{if}\, n^-(R) > n^+(R), \\ 0,\ \text{otherwise.} \end{array} \right\}$ is M, C and k-Pareto, but not NE.

(2) The SWF F defined by

$$F(R) = \left\{ \begin{array}{c} 1,\ \text{if}\, n^+(R) > n^-(R)+k \text{ or } n^+(R)+k\ \geq n^-(R) > n^+(R), \\ -1,\ \text{if}\, n^-(R) > n^+(R)+k \text{ or } n^-(R)+k\ \geq n^+(R) > n^-(R), \\ 0,\ \text{otherwise.} \end{array} \right\}$$

is NE, C and k-Pareto, but not M.

(3) The SWF F defined by $F(R) = \left\{ \begin{array}{c} 1,\ \text{if}\, n^+(R) > n^-(R)+k, \\ -1,\ \text{if}\, n^-(R) > n^+(R)+k, \\ R_i,\ \text{otherwise.} \end{array} \right\}$ is M, NE and k-Pareto, but not C.

(4) The SWF given in Remark 3(1) is NE, M and C, but not k-Pareto.

Theorem 22 (Llamazares, 2006) *A SWF F is the unanimous majority if and only if it is A, NE, M and (n-1)-Pareto.*

Proof.

($\Longrightarrow$) : Unanimous majority satisfies NE and C as MR satisfies all by M1 and M8. For any profiles $R, R' \in \{-1, 0, 1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$ ($\leq$ resp.) $\forall i \in N$, assume we have $F(R) \geq 0$ ($\leq$ resp.).then as F is unanimous majority $n^-(R) \ngtr n^+(R) + (n-1)$ so as $n^-(R') \ngtr n^+(R') + (n-1)$ we have $F(R') \geq 0$. Hence F is M. Unanimous majority is obviously (n-1)-Pareto, i.e. replace k by (n-1).

($\Longleftarrow$) : Reciprocally, suppose that F is A, NE, M and (n-1)-Pareto. By the last property, $F(E^n) = 1$, $F(-E^n) = -1$ and $\exists R^* \in U^+$ such that $F(R^*) < 1$ or $R^* \in U^-$ with $F(R^*) > -1$. Assume that $R^* \in U^+$ (the case $R^* \in U^-$ can symmetrically be shown). Since F is N, by of Remark 2(1), $F(E^0) = 0$, and by M of F, $F(R^*) = 0$. Because F is A, $F(R) = 0 \; \forall R^* \in U^+$,and by the N, $F(R)$ is also zero $\forall R \in U^-$. Now, given $R \in \{-1, 0, 1\}^n \setminus \{E^n, -E^n\}$, $\exists R' \in U^-$ and $R'' \in U^+$ such that $R' \leq R \leq R''$. Consequently, by M of F, $F(R) = 0$. ■

Proposition 23 (Llamazares, 2006) *Let F be a no constant SWF such that $F^{-1}(\{1\})$ be not empty. Then there exist $R, R' \in \{-1, 0, 1\}^n$ such that $\#\{i \in N : R_i \neq R'_i\} = 1$, $F(R) = 1$ and $F(R') < 1$.*

Proof.

Since $F^{-1}(\{1\}) \neq 0$, there exists $R^n \in \{-1,0,1\}^n$ such that $F(R^n) = 1$. Moreover, since F is not constant, there exists $R^0 \in \{-1,0,1\}^n$ such that $F(R^0) < 1$. For $j \in \{1,...,n-1\}$ we consider the profile R^j defined by

$$R_i^j = \left\{ \begin{array}{l} R_i^n, \text{ if } i \leq j, \\ R_i^0, \text{ if } i \leq j. \end{array} \right\}$$

Since $F(R^0) < 1$ and $F(R^n) = 1$, there exists $j \in \{1,...,n-1\}$ such that $F(R^j) < 1$ and $F(R^{j+1}) = 1$. Furthermore, $\#\{i \in N : R_i^j \neq R_i^{j+1}\} = 1$. ■

Proposition 24 (Llamazares, 2006) *Let $q \in \{0,1,...,n-1\}$, F be a NE, M and q-stable SWF and $R \in \{-1,0,1\}^n$. Then:*

(1) If $F(R) = 1$, then $n^+(R) > q$.

(2) If $F(R) = -1$, then $n^-(R) > q$.

Proof.

(1) Suppose for a contradiction $n^+(R) \leq q$ and consider another profile $R' \in \{-1,0,1\}^n$ defined by

$$R_i' = \left\{ \begin{array}{l} -1, \text{ if } R_i = 1, \\ R_i, \text{ otherwise.} \end{array} \right\}$$

Since $F(R) = 1$, $-R \geq R', \#\{i \in N : R_i \neq R_i'\} \leq q$ and F is NE, M and q-Stable we have:

$-1 = F(-R) \geq F(R') \geq 0$, gives us a contradiction.

(2) Since F is NE, then $F(-R) = 1$, and by the previous case, $n^-(R) = n^+(-R) > q$. ■

Remark6 (Llamazares, 2006)

(1) The SWF F defined by

$$F(R) = \left\{ \begin{array}{c} 1, \text{ if } n^+(R) > n^-(R) + (n-2), \\ -1, \text{ if } R = -E^n, \\ 0, \text{ otherwise.} \end{array} \right\}$$ is M and (n-1)-Stable, but not NE.

(2) The SWF F defined by $F(R) = -M_{n-1}(R)$ for all $R \in \{-1, 0, 1\}^n$ is NE and (n -1)-Stable, but not M.

(3) M_0 is NE and M, but not (n-1)-Stable.

Theorem 25 (Llamazares, 2006) *A SWF F is the unanimous majority iff it is NE, M and (n-1)-Stable.*

Proof.

$(\Longrightarrow)$: Unanimous majority satisfies NE and C as MR satisfies all by M1 and M8. For any profiles $R, R' \in \{-1, 0, 1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$ ($\leq$ resp.) $\forall i \in N$, assume we have $F(R) \geq 0$ ($\leq$ resp.).then as F is unanimous majority $n^-(R) \ngtr n^+(R) + (n-1)$ so as $n^-(R') \ngtr n^+(R') + (n-1)$ we have $F(R') \geq 0$. Hence F is M. Unanimous majority is obviously (n-1)-Stable.

$(\Longleftarrow)$:Reciprocally, suppose that F is NE, M and (n-1)-stable. By the last property, there exist $R, R' \in \{-1, 0, 1\}^n$ such that $F(R) = 1$ and $F(R') = -1$. By Proposition 23, $R = E^n, R' = -E^n$ and $F(R) = 0$ for all $R \in \{-1, 0, 1\}^n \setminus \{E^n, -E^n\}$, i.e., F is the unanimous majority. ■

If F is an A and M SWF, then for all pair of profiles $R, R' \in \{-1, 0, 1\}^n$ with the same number of non-indifferent voters, i.e.,

$n^+(R) + n^-(R) = n^+(R') + n^-(R')$, we have: $n^+(R) \leq n^+(R') \Longrightarrow F(R) \leq F(R')$.

Remark7 (Llamazares, 2006)

Let $k \in \{0, 1, ..., n-2\}$

(1) The SWF F where there exists an oligarchy constituted by the first $k+1$ individuals, i.e.,
$F(R) = \left\{ \begin{array}{c} 1, \text{ if } R_i = 1 \;\; \forall i \leq k+1, \\ -1, \text{ if } R_i = -1 \;\; \forall i \leq k+1, \\ 0, \text{ otherwise.} \end{array} \right\}$ is NE, M and k-Stable, but not A.
If we consider $R \in \{-1, 0, 1\}^n$ defined by $R_i = \left\{ \begin{array}{c} 1, \text{if } i \leq k+1, \\ -1, \text{ otherwise.} \end{array} \right\}$ then $F(R) = 1$ and $M_k(R) < 1$.

(2) The SWF F defined by
$F(R) = \left\{ \begin{array}{c} 1, \text{ if } n^+(R) > n^-(R) + (k-1), \\ -1, \text{ if } n^-(R) > n^+(R) + k, \\ 0, \text{ otherwise.} \end{array} \right\}$ is A, M and k-Stable, but not NE. In this case we have $F(E^k) = 1$ and $M_k(E^k) = 0$.

(3) The SWF defined by $F(R) = -M_k(R) \; \forall R \in \{-1, 0, 1\}^n$ is A, NE and k-Stable, but not M. Here we have $F(-E^n) = 1$ and $M_k(-E^n) = -1$.

(4) If $k \in \{0, 1, ..., n-2\}$, M_0 is A, NE and M, but not k-Stable. In this case, $M_0(E^1) = 1$ and $M_k(E^1) = 0 \; \forall k \in \{0, 1, ..., n-2\}$.

Theorem 26 (Aşan-Sanver, 2006) *A SWF F satisfies A, NE, and M $\Longleftrightarrow$ F is a q-majority rule for some $q \in \{n^*, ..., n+1\}$.*

Proof.

($\Longleftarrow$): Take any F,which is q-majority rule, we will show F satisfies A, NE, and M.

When our rule is q-majority, for a candidate to be the winner the number of supporters has to exceed the number q, so it has nothing to do who supports and who the candidate is; q-majority rule is easily A and NE. It is left to show F is M. Take any $R, R' \in \{-1, 0, 1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$, $i \in N$ and assume $F(R) \geq 0$, then we need to show that $F(R') \geq 0$. For a contradiction assume that $F(R') = -1$. Then $F(R') = -1$ means $n_-(R') \geq q$, which contradicts $n_+(R') \geq n_+(R) \geq q$, the reverse case is similar.

($\Longrightarrow$): Take any F satisfying A, NE, and M, to show F is a q-majority rule:

(1) $\forall R \in \{-1, 0, 1\}^n$. If $F(R) = 1$ then $n_+(R) \geq q$ for some $q \in \{n^*, ..., n+1\}$. Similarly, if $F(R) = -1$ then $n_-(R) \geq q$ for some $q \in \{n^*, ..., n+1\}$.

To show (1), take any $R \in \{-1, 0, 1\}^n$ such that $F(R) = 1$ and suppose for a contradiction $n_+(R) < n^*$. Now set a preference profile $R' \in \{-1, 0, 1\}^n$ such that $R'_i = 1 \Longleftrightarrow R_i = 1$ and $R'_i = 0 \Longleftrightarrow R_i \in \{-1, 0\}$. By M $F(R') = 1$

as well. Now take a set of voters K with cardinality $n_+(R)$ such that $i \in K \implies R_i \in \{-1, 0\}$, which is possible as we assumed $n_+(R) < n^*$. Consider another profile $R'' \in \{-1, 0, 1\}^n$ such that $R''_i = 1 \iff R_i = 1$ and $R''_i = -1 \iff i \in K, \forall i \in N$. By A and N $F(R'') = 0$ (ie. $n_+(R'') = n_-(R'')$), which combined with $F(R) = 1$, contradicts M, as when $R'' \leq 0 \implies R \leq 0$ we should have $F(R'') \leq 0 \implies F(R) \leq 0$ (but not $F(R) = 1$ as here is the case). The case of $F(R) = -1$ is similar.

(2) $\forall R, R' \in \{-1, 0, 1\}^n$. If $F(R) = 1$ and $n_+(R') \geq n_+(R)$ then $F(R') = 1$. If $F(R) = -1$ and $n_-(R') = n_-(R)$ then $F(R') = -1$.

To show (2), take any $R, R' \in \{-1, 0, 1\}^n$ such that $F(R) = 1$ and $n_+(R') \geq n_+(R)$. Suppose for a contradiction, $F(R') \in \{-1, 0\}$. Now pick another profile $Q \in \{-1, 0, 1\}^n$ such that $Q_i = 1 \iff R_i = 1$ and $Q_i = 0 \iff R_i \in \{-1, 0\}$, $\forall i \in N$. By M $F(Q) = 1$ as when $R \geq 0 \implies Q \geq 0$ we have $F(R) \geq 0 \implies F(Q) \geq 0$. Similarly, pick $Q' \in \{-1, 0, 1\}^n$ such that $Q'_i = 1 \iff R'_i = 1$ and $Q'_i = 0 \iff R'_i \in \{-1, 0\}$, $\forall i \in N$. Again by M $F(Q') \in \{-1, 0\}$. Next pick $Q'' \in \{-1, 0, 1\}^n$ such that $n_+(Q'') = n_+(Q')$ thus by A $F(Q'') \in \{-1, 0\}$, which together with $F(Q) = 1$ contradicts with M, as when $Q'' \leq 0 \implies Q \leq 0$ we should have $F(Q'') \leq 0 \implies F(Q) \leq 0$ (but not $F(Q) = 1$ as here is the case).

(1) and (2) together shows that $\exists q_1, q_2 \in \{n^*, \ldots.., n+1\}$ such that $\forall R \in \{-1, 0, 1\}^n$ we have $(F(R) = 1 \iff n_+(R) \geq q_1)(F(R) = -1 \iff n_-(R) \geq q_2)$. By A and N of F q_1 and q_2 must be equal and therefore, F is an absolute q-majority rule. ■

Side Results

Proposition 27 (Sanver, 2006) *The MR is not Maskin Monotonic.*

Proof.

Take some $R \in \{-1,0,1\}^n$ such that $R_1 = 1, R_2 = -1$ and $R_i = 0$, $\forall i \in N\backslash\{1,2\}$. Then since $p = m$, we have $MR(R) = 0$. Now take another profile $R' \in \{-1,0,1\}^n$ such that $R'_1 = 0$, and $R'_i = R_i$, $\forall i \in N\backslash\{1\}$. Here since $n_-(R) > n_+(R)$, we have $MR(R') = -1$. While we have $R_i \geq 0 \Longrightarrow R'_i \geq 0\ \forall i \in N$, we do not have $MR(R) \geq 0 \Longrightarrow MR(R') \geq 0$ (but we have $MR(R) \geq 0$ and $MR(R') < 0$ instead), which proves that MR is not M. ■

Theorem 28 (Sanver, 2006) *The AMR (absolute majority rule) is the minimal monotonic extension of the MR.*

Proof.

For this it is useful to interpret the outcomes $1,-1,0$ as $\{a\},\{b\}\,\{a,b\}$ respectively.

First, to show that AMR is M, take any $R, R' \in \{-1,0,1\}^n$ such that $R_i \geq 0 \Longrightarrow R'_i \geq 0$, $i \in N$ and assume $AMR(R) \geq 0$, then we need to show that $AMR(R') \geq 0$. For a contradiction assume that $AMR(R') = -1$ and let n^* be the lowest integer exceeding $n/2$. Then $AMR(R') = -1$ means $n_-(R') \geq n^*$, which contradicts $n_+(R') \geq n_+(R) \geq n^*$, the reverse case is similar.

Second, to show $MR(R) \subseteq AMR(R)$, cosider cases:

If $MR(R) = \{a\}$ then $n_+(R) > n_-(R)$ leading either $AMR(R) = \{a\}$ or $AMR(R) = \{a,b\}$. Hence $MR(R) \subseteq AMR(R)$.

If $MR(R) = \{b\}$ then $n_-(R) > n_+(R)$ leading either $AMR(R) = \{b\}$ or $AMR(R) = \{a,b\}$. Hence $MR(R) \subseteq AMR(R)$.

If $MR(R) = \{a,b\}$ then $n_+(R) = n_-(R)$ leading $AMR(R) = \{a,b\}$ as none of $n_+(R)$ or $n_-(R)$.can exceed n^*. Hence $MR(R) \subseteq AMR(R)$.

Last, it is left to show its minimality. Suppose for a contradiction $\exists$ a M SWF $F : \{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ such that $MR(R) \subseteq F(R) \subseteq AMR(R), \forall R \in \{-1,0,1\}^n$ and $F(R^\sim) \neq AMR(R^\sim)$ for some $R^\sim \in \{-1,0,1\}^n$. Therefore for $AMR(R^\sim)$ to have a proper subset, it should be equal to$\{a,b\}$, otherwise it could have no proper subset and WLOG assume $F(R^\sim) = \{a\}$, so we set it as a proper subset of $AMR(R^\sim)$. By definition, of $\{a,b\}$, $n_+(R^\sim) < n^*$ and $n_-(R^\sim) < n^*$. Also as $MR(R) \subseteq F(R)\ \forall R \in \{-1,0,1\}^n$, we also have $MR(R^\sim) = \{a\}$, which means $n_+(R^\sim) > n_-(R^\sim)$. Now let another preference profile $R^{\sim\prime} \in \{-1,0,1\}^n$ be such that $\forall i \in N$ if $R^\sim_i = 0$, then $R^{\sim\prime}_i = -1$; $\forall j \in N$ if $R^\sim_j \in \{-1,1\}$, then $R^{\sim\prime}_j = R^\sim_j$. For this new profile, as all 0's became -1, we now have either $n_+(R^{\sim\prime}) = n_-(R^{\sim\prime})$, or $n_-(R^{\sim\prime}) > n_+(R^{\sim\prime})$; therefore, definitely $b \in MR(R^{\sim\prime})$, and so $b \in F(R^{\sim\prime})$ as well. However, $b \notin F(R^\sim)$, which contradicts with F being M as when $R^{\sim\prime} \leq 0 \Longrightarrow R^\sim \leq 0$, we should have $F(R^{\sim\prime}) \leq 0$ (which is the case as either we have $F(R^{\sim\prime}) = \{a,b\}$ or $F(R^{\sim\prime}) = \{b\}) \Longrightarrow F(R^\sim) \leq 0$ (which is not the case as $F(R^\sim) = \{a\}$). ■

Concluding Remarks

Within the thesis, we have characterized all from simple majority rule to unanimous majority rule, which are the two extreme examples of M_k majority rules, and further more we have characterized absolute majority rule, which requires lowest number of supporters, for an alternative to win , among absolute q-majority rules. As the next step, it could be interesting to consider more than two alternatives, for which Yi (2005) has done certain work.

References

Aşan, G., Sanver, M.R., 2002. Another characterization of the majority rule. Economics Letters 75, 409– 413.

Aşan, G., Sanver, M.R., 2006. Maskin monotonic aggregation rules. Economics Letters 91, 179-183.

Llamazares, B., 2006. The forgotten decision rules: Majority rules based on difference of votes. Mathematical Social Sciences 51, 311–326.

May, O.K., 1952. A set of Independent Necessary and Sufficient Conditions for Simple Majority Decision. Econometrica 20-4, 680-684.

Miroiu, A., 2004. Characterizing majority rule: from profiles to societies. Economics Letters 85, 359– 363.

Sanver, M.R., 2006. Nash implemantation of the majority rule. Economics Letters 91, 369– 372.

Woeginger, G.J., 2003. A new characterization of the majority rule. Economics Letters 81, 89–94.

Woeginger, G.J., 2005. More on the majority rule: Profiles, societies, and responsiveness. Economics Letters 88, 7–11.