ESSAYS ON CONSISTENCY AND CONVERSE CONSISTENCY IN MATCHING PROBLEMS

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Abstract
This Ph.D. thesis consists of four essays about consistency and converse consistency in matching problems. The first essay is a survey about the use of consistency and converse consistency in the literature for matching problems. The second essay is about characterization of the core by using consistency and converse consistency in two sided one to one matching problems (marriage problems) in general domains which is a joint work with my advisor İpek Özkal-Sanver. In the third essay, I compute maximal conversely consistent subsolution of the Pareto optimal solution for marriage problems. The final essay which is again a joint work with my advisor İpek Özkal-Sanver is about consistency on one-sided one-to-one matching problems, the so called roommate problems.
Essays on Consistency and Converse

Consistency in Matching Problems

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1 Introduction

This Ph.D. thesis consists of four essays about consistency and converse consistency in matching problems.

The first essay is a survey about the use of consistency and converse consistency in the literature. Consistency and converse consistency are two well-known axioms that have recently played fundamental role in axiomatic analysis. Consistency says that if an agreement is made between the both sides of the economy, if some of the agents leave the society with their endowments, payments, partners etc., then the same agreements should be made between the remaining agents. On the other hand, converse consistency says that, the agreement on the payoffs, endowments, partners etc. depends on the agreement for the two agent restricted problems. In this essay, we shortly survey applications of these axioms for many economics problems. We thoroughly present the results for matching problems.

The next three essays which are the main parts of this thesis analyse these axioms for matching problems. Matching processes are usually related to so-called one sided markets and two-sided markets. In one-sided markets coalitions are formed by same type of agents. The most well known example is roommate problem where university rooms are allocated to couples of students, each having preferences over potential mates. In two-sided markets,
coalitions are formed by two types of agents where each type has preferences over the other one. Coalitions can be pairs (one-to-one matching known as marriage markets) or subsets including either one agent of some type and several of the other (many-to-one matching), or several agent of each type (many-to-many matching). In such settings the main addressed question is how to form coalitions in a strategically stable way? In all the matchings models mentioned above stability is a central property. A matching for marriage problems or for roommate problems is stable if each agent in the society is matched with an acceptable agent and no two agent would prefer to matched with each other rather than their current mates. Alvin E. Roth who shared the 2012 Nobel Prize in Economic Sciences with Lloyd S. Shapley said that “Many of the important things that we do in our lives are matching, from getting into a university, from getting married to getting a job.” As Alvin E. Roth emphasized, Matching Theory has many applications for real life situations such as marriage markets, school and university placement, job markets, organ donations.

The second essay is about characterization of the core by using consistency and converse consistency in two sided one to one matching problems, the so called marriage problems in general domains which is a joint work with my advisor İpek Özkal-Sanver. In this essay we characterize the core of marriage problems in general domains. First in a model where agents have
strict preferences over their potential mates, and agents are allowed to stay single, we characterize the core as the unique solution which satisfies individual rationality, Pareto optimality, gender fairness, consistency and converse consistency. Next, relaxing the constraint that agents have strict preferences over their potential mates, we show that there exists no solution satisfying Pareto optimality, anonymity and converse consistency. In this full domain, we characterize the core by individual rationality, weak Pareto optimality, monotonicity, gender fairness, consistency and converse consistency.

In the third essay, I compute maximal conversely consistent subsolution of the Pareto optimal solution for marriage problems. This essay which is very related to the second essay we study marriage problems in a restricted domain where agents have strict preferences over their potential mates, and agents are not allowed to stay single. We consider the most well-known solution concept: the Pareto-optimal solution. The Pareto-optimal solution fails to satisfy converse consistency. In this essay, we compute a maximal conversely consistent subsolution of the Pareto optimal solution. To do this, we introduce the concept of serial men-ordering. Also we show that this result is not valid for general domains where there are unequal number of men and women and being single is allowed.

The final essay which is again a joint work with my advisor İpek Özkal-Sanver is about consistency on one-sided one to one matching problems, the
so called roommate problems. We compute consistent enlargements of the core in roommate problems. By computing it, we evaluate the extent to which the core would have to be expanded in order to be well-defined and consistent. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. We also show that for any fixed order on the set of agents the solution which picks all stable matchings and the serial dictatorship matching with respect to this order is a minimal consistent enlargement of the core. Since for different orders there may be different enlargements, minimal consistent core enlargement is not unique.
2 A Survey on Consistency and Converse Consistency for Matching Problems

2.1 Introduction

The objective of this essay is to present two well-known axioms, consistency and converse consistency that have recently played fundamental role in axiomatic analysis. We briefly survey the applications of these axioms for many economic models. For matching problems, we present applications of these axioms in a more detail way.

Consistency and converse consistency require to be invariant for population changes. Consistency says that if an alternative is acceptable for some problem then it should be an acceptable alternative of its restrictions to all subgroups for the associated reduced problems. On the other hand, converse consistency allows a dual operation, if an alternative is acceptable of its restrictions to all subgroups of cardinality two for the associated reduced problems then this alternative should be acceptable for the whole group. For the definitions of these axioms two notions are crucial; a solution and a reduced problem. A solution picks for each decision problem in some domain one or several of its feasible alternatives. The notion of reduced problem is an important fact, since different formulations of a reduced problem lead to
different definitions of these axioms.

Converse consistency is a kind of two-agent decentralization axiom. Thomson (1996) give an example about the agreement of a proposal in a political convention to emphasize the importance of this axiom. In a political convention, delegates first meet each other in committees of size two and each committee argue on the proposal. If the proposal successfully passes this stage, it is examined by committees of size three. The process is repeated until the proposal is either rejected at some stage by some committee, or finally it is accepted by the whole convention in plenary session. If the decision process satisfies converse consistency, acceptance by all committees of size two will guarantee acceptance at the plenary session. Hence, the formation of the committees of size greater than two will be unnecessary.

First, we survey applications of consistency and converse consistency for many economic problems. The domains in which these axioms are used are classified by Thomson (1996) in four main classes; game theory, public economies and cost allocations, fair allocation, and some other models. The models for game theory include bargaining, games in coalitional form with and without transferable utility and games in strategic form. For public economies and cost allocations there are studies about these axioms for bankruptcy and taxation, quasi linear cost allocation, general cost allocation and pricing. As a third class, these axioms are studies for fair division
in classical private good economies, fair division in economies with single
peaked preferences and fair allocation in economies with indivisible goods.
The last class includes apportionment and matching problems. For each of
these classes, except matching problems without stating formal definitions
of the axioms we briefly present central results of the related literature. For
matching problems in the proceeding sections we analyse these axioms in a
detail way.

In the literature, for each of the classes there are many characterization
results of the well-known solution concepts by using these axioms. In these
results there are also other axioms but the central axioms are consistency and
converse consistency. For bargaining problems by using consistency, Lens-
berg (1988) characterized the Nash solution, Lensberg (1987) characterized
the lexicographic egalitarian solution and the separable additive solution.
For the games in coalitional form with transferable utility whose core is non-
empty there are some characterization results of the core by using different
versions of consistency; Tadenuma (1992) used complement consistency and
Peleg (1986) used max consistency. For this class of problems, by using
complement consistency Moulin (1985) characterized the equal allocation of
non-separable benefits solution. Sobolev (1975) characterized the prenucleous
by using max consistency. There are also some studies by using some versions
of converse consistency in this class of problems. By using max consistency
and max converse consistency, Peleg (1989, 1992) characterized the core for the domain of market games and Peleg (1986) characterized the prekernel. By using another version of consistency, namely self consistency, Hart and Mas-Colell (1989) characterized the Shapley value. For the games in coalitional form without transferable utility, by using complement consistency, Tadenuma (1992) characterized the core whenever it is non-empty. For this class, by using consistency and converse consistency Peleg and Tijs (1996) characterized the Nash equilibrium solution and the coalition proof Nash equilibrium. For bankruptcy and taxation, Young (1987) characterized the continuous parametric solution by using consistency. By a different version of consistency, limited consistency, Chun (1988) characterized the proportional solution. For general cost allocation problems, Moulin and Shenker (1994) characterized the average cost sharing by using consistency and the serial cost sharing by limited consistency. For pricing mechanisms, McLean, Pazgal and Sharkey (2004) characterized the Shapley value by using self consistency. For fair division in economies with single-peaked preferences, Thomson (1994a) characterized the uniform rule by using consistency. For apportionment, Balinski and Young (1982) characterized the divisor solution by consistency.

For matching problems, there are (at least) two approaches dealing with consistency and converse consistency. The first approach is characterization
of well-known solution concepts by these axioms. Since some of the well-known solutions do not satisfy these principals, as a second approach, there are papers about extending or reducing the solutions in such a way that these axioms are satisfied. Alternatively, there are some studies about the identification of conditions under which these solutions satisfy consistency and converse consistency.

This essay proceeds as follows: Section 2.2 presents the basic notations and definitions of the axioms for matching problems. Section 2.3 gives characterization results of the core. Section 2.4 gives the results about the minimal extensions or maximal subsolutions of the solutions. Section 2.5 concludes.

### 2.2 Formal Definitions for Matching Problems

First we define formal definitions of consistency and converse consistency for marriage problems. Let $M$ and $W$ be two disjoint universal sets. Let $M$ be a nonempty and finite subset of $M$. Similarly, let $W$ be a nonempty and finite subset of $W$. A society is a union of some $M \subseteq M$ and some $W \subseteq W$. In the context of marriage, the set $M$ stands for a set of men and the set $W$ for a set of women.

Let $A = M \cup W$ denote the universal set of agents. Let $A = \{M \cup W\}_{M \subseteq M, W \subseteq W}$ be the set of all possible societies. For each society
A = M \cup W \in \mathcal{A} and for each agent \( i \in A \) the set of potential mates of
\( i \), denoted by \( A(i) \), is defined as

\[
A(i) \equiv \begin{cases} 
W & \text{if } i \in M \\
M & \text{if } i \in W.
\end{cases}
\]

Each agent \( i \in A \) has a strict preference relation over \( A(i) \), denoted by
\( P_i \).

Let \( P \) denote the set of all possible preference profiles \( P \equiv (P_i)_{i \in A} \).

A matching is a function \( \mu : A \rightarrow A \) such that for all \( i \in A \), \( \mu(i) \in A(i) \) and for all
\( i \in A \), \( \mu^2(i) = i \). Here, \( \mu(i) \) is the mate of agent \( i \) under
matching \( \mu \). If \( \mu(i) = i \), we say that agent \( i \) is selfmatched or single. Let
\( \mathcal{M}(A) \) denote the set of all matchings for \( A \).

A (matching) problem \( p \) is a pair \( p = (A, P) \), where \( A \) is a society, \( P \)
is the profile of their preferences over potential mates. Let \( \mathcal{P} \) denote the set
of all problems.

Let \( p = (A, P) \in \mathcal{P} \) be an arbitrary problem. A matching \( \mu \in \mathcal{M}(A) \)
is individually rational for \( p \) if for all \( i \in A \), \( \mu(i) \) \( P_i \) \( i \) or \( \mu(i) = i \). Let
\( \mathcal{IR}(p) \) denote the set of all individually rational matchings. A pair of agents
\( (i, j) \) blocks a matching \( \mu \in \mathcal{M}(A) \) if \( j P_i \mu(i) \) and \( i P_j \mu(j) \). A matching
\( \mu \in \mathcal{M}(A) \) is stable for \( p \) if it is individually rational for \( p \) and there is no
pair \( (i, j) \) blocking \( \mu \) at \( p \). Let \( \mathcal{S}(p) \) denote the set of all stable matchings.

Given a problem \( p = (A, P) \in \mathcal{P} \) and two matchings \( \mu, \mu' \in \mathcal{M}(A) \), \( \mu \)
dominates \( \mu' \) if there exists a coalition \( K \subseteq A \) such that for all \( i \in K \),
\(\mu(i) \in K\) and \(\mu(i) \neq \mu'(i)\). A matching \(\mu\) is **undominated** if there exists no matching \(\mu' \in \mathcal{M}(A)\) which dominates \(\mu\). The **core** of \(p\) is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor (1990)). Given a problem \(p = (A, P) \in \mathcal{P}\) and two matchings \(\mu, \mu' \in \mathcal{M}(A)\) with \(\mu' \neq \mu\), \(\mu\) **Pareto dominates** \(\mu'\) if for all \(i \in A\), \(\mu(i) P_i \mu'(i)\) whenever \(\mu(i) \neq \mu'(i)\). A matching \(\mu \in \mathcal{M}(A)\) is **Pareto optimal** for \(p\) if there exists no matching \(\mu' \in \mathcal{M}(A)\) which Pareto dominates \(\mu\). Let \(\mathcal{PO}(p)\) denote the set of all Pareto optimal matchings.

Given a problem \(p = (A, P)\) and a subset of set of agents \(N\), a **reduced problem** of \(p\) with respect to \(N\) is a problem where the preference profile \(P\) is restricted to agents in \(N\). Formally; for all \(p = (A, P) \in \mathcal{P}\) and all \(N \subseteq A\), \(p' = (N, P|_N) \in \mathcal{P}\) is the reduced problem of \(p\) with respect to \(N\). Given a matching \(\mu \in \mathcal{M}(A)\), a **reduced matching** of \(\mu\) with respect to \(N\) is a matching \(\mu|_N : N \cup \mu(N) \rightarrow N \cup \mu(N)\) such that for all \(i \in N\), \(\mu|_N(i) = \mu(i)\).

We define **extension** of a problem \(p' = (M' \cup W', P')\) of \(p = (M \cup W, P) \in \mathcal{P}\) if \(M \subseteq M'\), \(W \subseteq W'\) and \(P'|_{M \cup W} = P\). In particular, if \(M \neq M'\) and \(W = W'\), \(p'\) is \(M\)-extension of \(p\) and if \(M = M'\) and \(W \neq W'\) \(p'\) is \(W\)-extension of \(p\).

A **solution** \(\varphi\) is a correspondence which associates with each \(p\) a non-empty subset \(\varphi(p) \subseteq \mathcal{M}(A)\).
Now, we define axioms on solutions that are used for axiomatic analysis in the literature. The first axiom requires that at each problem the solution recommends individually rational matchings:

**Individual rationality (IR):** For each $p \in \mathcal{P}$, $\varphi(p) \subseteq \mathcal{IR}(p)$.

The second axiom requires that at each problem the solution recommends Pareto optimal matchings:

**Pareto optimality (PO):** For each $p \in \mathcal{P}$, $\varphi(p) \subseteq \mathcal{PO}(p)$.

The third axiom requires that renaming men among men and renaming women among women do not change the result:

**Anonymity (AN):** For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |M'|$ and $|W| = |W'|$, let $\pi : A \to A'$ be a bijection such that $\pi(M) = M'$ and $\pi(W) = W'$. For all $p = (A, P)$, let $P'$ be such that for all $i \in A$ and all $j, k \in A(i)$, $jP'k$ if and only if $\pi(j)P'_{\pi(i)}\pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

The fourth axiom imposes same treatment of men and women, in the sense that renaming men as women and women as men do not change the result.

**Gender fairness (GF):** For all $A = M \cup W$ and all $A' = M' \cup W'$ with $|M| = |M'|$ and $|W| = |M'|$, let $\pi : A \to A'$ be a bijection such that $\pi(M) = W'$ and $\pi(W') = M'$. For all $p = (A, P)$, let $P'$ be such that for all
\[ i \in A \text{ and all } j, k \in A(i), j P_i k \text{ if and only if } \pi(j) \overset{P_{\pi(i)}}{\succ} \pi(k). \] And also for all \[ \mu \in \mathcal{M}(A) \text{ define } \pi_\mu \in \mathcal{M}(A') \text{ by setting for all } i \in A, \pi_\mu(i) = \pi(\mu(\pi^{-1}(i))). \] If \( \mu \in \varphi(A, P), \) then \( \pi_\mu \in \varphi(A', P'). \)

The fifth axiom says that if there exists a matching which is preferred by every agent it should be the unique recommendation.

**Weak unanimity (WU):** For each \( p = (M \cup W, P) \in \mathcal{P}, \) if there exists a matching \( \mu \in \mathcal{M}(M \cup W) \) such that \( \mu(a) \) is most preferred by each agent \( a \in M \cup W, \) then \( \varphi(p) = \{\mu\}. \)

The sixth axiom which is stronger than weak unanimity requires that a pair of mutually best agents to be matched at every solution outcome.

**Mutually best (MB):** For each \( p = (M \cup W, P) \in \mathcal{P}, \) for any \( m \in M \) and \( w \in W \) if \( m \) and \( w \) most prefer each other then \( \mu(m) = w \) for each \( \mu \in \varphi(p). \)

The next axiom requires that if the number of agents in one side of the market increases while the opposite sides remains fixed no incumbent on the same side as the entrants is strictly better off.

**Population Monotonicity (PMON):** For each \( p = (M \cup W, P) \in \mathcal{P} \) and each \( M \)-extension \( p' \) of \( p \) if \( \mu \in \varphi(p), \) then there exists \( \mu' \in \varphi(p') \) such that \( \mu(m)R_m\mu'(m) \) for each \( m \in M. \) If \( p' \) is a \( W \)-extension of \( p, \) there exists \( \mu' \) satisfying a symmetric requirement for each \( w \in W. \)

The next axiom says that if the preference profile changes in such a way
that ranking does not decrease in each agents preferences then the outcome is still recommended.

Let \( p = (M \cup W, P) \in \mathbf{P} \) be a problem and \( \mu \in \mathcal{M}(M \cup W) \). For each \( m \in M \), define \( L(\mu, R_m) = \{a \in W \mid \mu(m)R_m a\} \).

Similarly, for each \( w \in W \), \( L(\mu, R_w) \) is defined. It is said that \( p' = (M \cup W, P') \) is obtained from \( p \) by a monotonic transformation at \( \mu \) if \( L(\mu, R_a) \subseteq L(\mu, R'_a) \) for each \( a \in M \cup W \).

**Maskin Monotonicity (MMON.):** For each \( p = (M \cup W, P) \in \mathbf{P} \), and \( \mu \in \mathcal{M}(M \cup W) \) if \( p' = (M \cup W, P') \) is obtained from \( p \) by a monotonic transformation at \( \mu \) then \( \mu \in \varphi(p') \).

Now we define our main axioms; consistency and converse consistency.

Consider some problem \( p \) and some solution \( \varphi \). Take any matching \( \mu \) recommended by \( \varphi \) at \( p \). If the reduced matching of \( \mu \) with respect to each subgroup of matched pairs is among the recommendations made by the solution \( \varphi \) for the reduced problem of \( p \) with respect to this subgroup of matched pairs, then we say that the solution \( \varphi \) is consistent. More formally:

**Consistency (CON):** For each \( p = (A, P) \in \mathbf{P} \), each \( \mu \in \varphi(p) \) we have \( \mu |_N \in \varphi(N, P|_N) \) for any society \( N \subseteq A \) such that \( \mu(N) = N \).

We define converse consistency only imposing the requirement on the subgroups formed exactly by two matched pairs: Consider some problem \( p \) and some solution \( \varphi \). Take any matching \( \mu \). The requirement is that if the
reduced matching of $\mu$ with respect to each subgroup of two matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to the subgroup of these two matched pairs, then $\mu$ must be recommended by $\varphi$ at the original problem $p$.

**Converse Consistency (CCON):** For each $p = (A, P) \in P$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu|_N \in \varphi(N \cup \mu(N), P|_{N\cup\mu(N)})$, then $\mu \in \varphi(p)$.

In the following chapter, we consider three versions of converse consistency.\(^1\) In a restricted domain where the number of men and women coincide and agents are not allowed being single, all these three versions are equivalent.

We also analyse the roommate problems. Now, we give formal definitions of the axioms that are used for roommate problems.

To define the following axioms, let $p = (A, P)$ be a roommate problem and let $p' = (A', P')$ be an extension of $p$ where $A' = A \cup \tilde{A}$.

The next axiom, competition sensitivity which is first defined by Klaus Thomson (2008) requires that if two incumbents are newly matched after a set of newcomers arrived, then one of them suffers.

**Competition Sensitivity (CS):** Each $\mu \in \varphi(p)$ there is $\mu' \in \varphi(p')$ such

\(^1\)Thomson (2004, 2009) introduced and studied the converse consistency axiom for various economic models.
that for any $i, j \in A$ with $\mu(i) \neq j$ and $\mu'(i) = j$ we have either $\mu(i)P_i\mu'(i)$ or $\mu(i)P'_i\mu'(i)$.

A solution is **weakly competition sensitive** if competition sensitivity is satisfied when we only add one newcomer at a time.

Resource sensitivity requires that if two incumbents are unmatched after a set of newcomers arrived, then one of them benefits. Formally,

**Resource Sensitivity (RS):** For each $\mu' \in \varphi(p')$ there is $\mu \in \varphi(p)$ such that for any $i, j \in A$ with $\mu(i) = j$ and $\mu'(i) \neq j$ we have either $\mu'(i)P_i\mu(i)$ or $\mu'(i)P'_i\mu(i)$.

A solution is **weakly resource sensitive** if resource sensitivity is satisfied when we only add only one newcomer at a time.

### 2.3 Characterization Results

In the literature, there are many studies about the characterization of the core for matching problems by using consistency and converse consistency. For the sake of completeness we state these results as theorems:

Consistency and converse Consistency are introduced to the matching theory literature by Sasaki and Toda (1992) who characterize the core of marriage (two sided, one-to-one matching) problems as the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse
consistency.

**Theorem 2.1** (Sasaki and Toda, 1992) The core is the unique solution satisfying Pareto optimality, anonymity, consistency and converse consistency.

For marriage problems, Toda (2006) give another two characterization of the core by using weak unanimity, population monotonicity, Maskin monotonicity and consistency.

**Theorem 2.2** (Toda, 2006) The core is the unique solution satisfying weak unanimity, population monotonicity and Maskin monotonicity.

**Theorem 2.3** (Toda, 2006) The core is the unique solution satisfying weak unanimity, population monotonicity and consistency.

For marriage problems when agents have weak preferences over their potential mates Toda (2006) give a characterizaiton of the core without using converse consistency.

**Theorem 2.4** (Toda, 2006) If agents have weak preferences then the core is the unique solution satisfying weak unanimity, population monotonicity, Maskin monotonicity and consistency.

In a similar way, Toda (2006) uses consistency in characterizing the core of two sided many-to-one matching problems (College admissions problems).
Theorem 2.5 (Toda, 2006) The core of college admissions problems is the unique solution satisfying weak unanimity, population monotonicity and consistency.

For matching problems with money which is introduced first by Shapley and Shubik (1972), Sasaki (1995) characterized the core for these problems by using a weaker version of consistency, namely separation independence.

Theorem 2.6 (Sasaki, 1995) On the domain of matching problems with money the core is the unique solution satisfying Pareto optimality, continuity, individual rationality, couple rationality, separation independence and worth monotonicity.

On the domain of one sided one-to-one matching problems (roommate problems) Özkal-Sanver (2010) showed that no solution satisfies Pareto optimality, anonymity and converse consistency. Klaus (2013) characterizes the core by using consistency and converse consistency on the domain of no odd rings roommate problems. She also proves that extending her result to the domain of solvable roommate problems is not possible. Can and Klaus (2012) give two characterizations of the core of roommate problems by using consistency for no odd rings domains and for solvable domains.

Theorem 2.7 (Can and Klaus, 2012)
a) On the domain of no odd rings roommate problems the core is the unique solution satisfying weak unanimity, weak competition sensitivity and consistency.

b) On the domain of solvable roommate problems the core is the unique solution satisfying weak unanimity, competition sensitivity and consistency.

**Theorem 2.8** *(Can and Klaus 2012)*

a) On the domain of no odd rings roommate problems the core is the unique solution satisfying weak unanimity, weak resource sensitivity and consistency.

b) On the domain of solvable roommate problems the core is the unique solution satisfying weak unanimity, resource sensitivity and consistency.

By using these axioms, Can and Klaus (2012) get an impossibility result for all roommate problems. They show that, on the domain of all roommate problems there exists no solution satisfying weak unanimity, competition sensitivity and consistency.

### 2.4 Minimal Extensions and Maximal Subsolutions

If a solution does not satisfy consistency or converse consistency, we would like to know how serious violations of these axioms. There are two procedures which are developed by Thomson (1994b, 1996) to recover these properties.
The first attempt is minimally enlarge the solution so that consistency or converse consistency are satisfied. Since consistency and converse consistency are preserved under intersections Minimal consistent extension of a solution (or minimal conversely consistent extension) is defined by intersection of consistent solutions which include that solution. More formally, given a solution $\varphi$ the minimal consistent extension of $\varphi$, denoted by $MCE_\varphi$, is defined by Thomson (1996) as follows:

$$MCE_\varphi = \bigcap_{\psi \in \Psi} \psi$$

where $\Psi = \{\psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is consistent}\}$

Similarly, the minimal conversely consistent extension of a solution $\varphi$ can be defined as follows:

$$MCCE_\varphi = \bigcap_{\psi \in \Psi} \psi$$

where $\Psi = \{\psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is conversely consistent}\}$

As a second attempt, Thomson (1996) introduced the concept of maximal consistent subsolution. By computing maximal consistent subsolution we reduce the solution maximally to get consistency. Formally, given a solution $\varphi$ that contains at least one consistent subsolution, the maximal consistent subsolution of $\varphi$, denoted by $MXCS_\varphi$, is defined as follows:

$$MXCS_\varphi = \bigcup_{\psi \in \Psi} \psi$$

where $\Psi = \{\psi \in \Phi : \psi \subseteq \varphi, \psi \text{ is consistent}\}$

As Thomson (1996) noticed, union of two conversely consistent solutions may not be conversely consistent. So, we can not carry the concept of maxi-
mal consistent subsolution directly to the conversely consistent subsolution. Therefore, for a given solution $\varphi$ we define maximal conversely consistent subsolution $MXCCS_{\varphi}$ as a maximal conversely consistent solution that includes $\varphi$.

In the literature, there are few papers computing the minimal consistent extension of solution, as well as the minimal conversely consistent extension of solutions for different economic problems. For instance, for the domain of fair allocation problems Bevia (1996), Kolm (1973), for bargaining problems Thomson (1995) and Korthues (2000). For matching problems, Özkal-Sanver (2012) compute the minimal conversely consistent extension of the men-optimal solution. For this aim, she introduced the concept of men-barterproofness. Men barter-proof solution $MB$ is defined in the following way: A matching is men-barterproof whenever there is no such a pair of men who benefit from switching their mates among themselves. More formally, a matching $\mu \in \mathcal{M}(A)$ is **men-barterproof** for $p = (A, P) \in \mathcal{P}$, if there exists no pair of men $\{m, m'\} \subset M$ bartering at $\mu$. Let $MB(p)$ denote the set of the men-barterproof matchings for $p$. The **men-barterproof solution** is the correspondence $MB$ that associates with each problem $p$ the set of men-barterproof matchings $MB(p)$.

**Theorem 2.9** (Özkal-Sanver, 2013) The minimal conversely consistent
extension of the men-optimal solution is \( MCCE_{MO}(p) = MO(p) \cup [S(p) \cap MB(p)] \) for all \( p \in P \).

Also, there are some studies for roommate problems about consistency. As Gale and Shapley (1962) showed, the core may be empty for roommate problems. Özkal-Sanver (2010) define the concept of core extension as a solution which picks the core whenever it is non-empty. Formally, a core extension is a solution \( \varphi \) such that for any \( p \in P \) with \( S(p) \neq \emptyset \), \( \varphi(p) = S(p) \). She showed that no core extension is consistent.

**Theorem 2.10** (Özkal-Sanver, 2010) No core extension is consistent.

If a solution does not satisfy consistency and converse consistency, as an alternative approach there are some studies which provide necessary and sufficient conditions for the solution to satisfy these axioms. For instance, for College Admissions problems, Klaus and Klijn (2011) established necessary and sufficient conditions for student optimal solution to be conversely consistent.

### 2.5 Concluding Remarks

In this essay, we survey applications of consistency and converse consistency in the literature. First, we state both informal and formal definitions of these
axioms. Then we present results of related literature. There are several related open questions to be investigated further: In the proceedings chapters we deal with some of them. First, we characterize the core of marriage problems in general domains. Then, we compute a maximal conversely consistent subsolution of the Pareto optimal solution in marriage problems. Finally, we study on consistent enlargements of the core in roommate problems. For many to one matching problems (college admission problems) characterization of the core by using converse consistency is still open questions worth to be investigated.
3 Characterization of the Core in Full Domain Marriage Problems

Duygu Nizamoğulları and İpek Özkal-Sanver

3.1 Introduction

Aim of this essay is to characterize the core of two-sided one-to-one matching problems in general domains. Sasaki and Toda (1992)’s characterization result is valid for a domain where agents have strict preferences over their potential mates and agents are not allowed to be single.\(^2\)

First we consider a model where agents are allowed to stay single, but still agents have strict preferences over their potential mates. We show that the core is the unique solution satisfying individual rationality, Pareto optimality, gender fairness, consistency and converse consistency.

Consistency and converse consistency are two fundamental properties of solutions for allocation problems, in variable population models. The consistency axiom is some kind of independence of irrelevant alternatives axiom.\(^2\)

Consistency of a solution imposes that if some matched pairs are leaving

\(^2\)Sasaki and Toda (1992) characterize the core by Pareto optimality, anonymity, consistency and converse consistency. Toda (1993) showed that there exists some other solution than the core satisfying individual rationality, Pareto optimality, anonymity, consistency, and a weaker version of converse consistency; when we allow that agents stay single.
the society; the reduced matching is among the matchings recommended by
the solution for the reduced problem. Consistency is trivially adapted to
the model where agents may stay single. However; there are several possible
ways to adapt the converse consistency to this model. Converse consistency
is a kind of decentralization axiom. Thomson (2004) illustrates the notion
of converse consistency by a jigsaw puzzle as “correct positioning of pieces
two-by-two guarantees correct positioning altogether.” This axiom allows us
to deduce from the simpler calculations on all of the meaningfully smallest
subproblems whether an alternative should be chosen for the original big
problem. We discuss alternative definitions of converse consistency in this
more general framework. Gender fairness is a stronger version of anonymity.3
Roughly speaking, if a solution is gender fair, by renaming men as women,
and women as men, and applying the solution, we end up with the outcome
permuted accordingly.

Next we consider what would happen if we further relax the condition
that agents have strict preferences over their potential mates. If we allow in-
differences on preferences, the core fails to satisfy Pareto optimality. We show
that there exists no solution satisfying Pareto optimality, anonymity and con-
verse consistency. We characterize the core as the unique solution satisfying
individual rationality, weak Pareto optimality, gender fairness, consistency,

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3See Özkal-Sanver (2004).
converse consistency and monotonicity. Another characterization of the core can be found in Toda (2006) where the core of many-to-one matching problems is characterized by weak unanimity, consistency, Maskin monotonicity and population monotonicity.\footnote{The characterization result of Toda (2006) can be carried to one-to-one matching problems.}

This essay proceeds as follows: Section 3.2 presents the basic notations and definitions. Section 3.3 discusses converse consistency of a solution. Section 3.4 gives the results of the first model where agents are allowed to stay single, and agents have strict preferences over their potential mates. Section 3.5 gives the results of the second model where agents may have weak preferences. Section 3.6 concludes.

### 3.2 Notations and Definitions

Let $M$ and $W$ be two disjoint universal sets. Let $M$ be a nonempty and finite subset of $M$. Similarly, let $W$ be a nonempty and finite subset of $W$. A society is a union of some $M \subset M$ and some $W \subset W$. In the context of marriage, the set $M$ stands for a set of men and the set $W$ for a set of women.

Let $A = M \cup W$ denote the universal set of agents. Let $\mathcal{A} = \{M \cup W\}^{M \subset M, W \subset W}$ be the set of all possible societies. For each society $A = M \cup W \in \mathcal{A}$ and for each agent $i \in A$ the set of potential mates of
i, denoted by \( A(i) \), is defined as
\[
A(i) \equiv \{i\} \cup \begin{cases} 
W & \text{if } i \in M \\
M & \text{if } i \in W.
\end{cases}
\]

Each agent \( i \in A \) has a strict preference relation over \( A(i) \), denoted by \( P_i \). Let \( P \) denote the set of all possible preference profiles \( P \equiv (P_i)_{i \in A} \).

A matching is a function \( \mu : A \rightarrow A \) such that for all \( i \in A \), \( \mu(i) \in A(i) \) and for all \( i \in A \), \( \mu^2(i) = i \). Here, \( \mu(i) \) is the mate of agent \( i \) under matching \( \mu \). If \( \mu(i) = i \), we say that agent \( i \) is selfmatched or single. Let \( \mathcal{M}(A) \) denote the set of all matchings for \( A \).

A (matching) problem \( p \) is a pair \( p = (A, P) \), where \( A \) is a society, \( P \) is the profile of their preferences over potential mates. Let \( \mathcal{P} \) denote the set of all problems.

Let \( p = (A, P) \in \mathcal{P} \) be an arbitrary problem. A matching \( \mu \in \mathcal{M}(A) \) is individually rational for \( p \) if for all \( i \in A \), \( \mu(i) \) or \( \mu(i) = i \). Let \( \mathcal{IR}(p) \) denote the set of all individually rational matchings. A pair of agents \( (i, j) \) blocks a matching \( \mu \in \mathcal{M}(A) \) if \( j P_i \mu(i) \) and \( i P_j \mu(j) \). A matching \( \mu \in \mathcal{M}(A) \) is stable for \( p \) if it is individually rational for \( p \) and there is no pair \( (i, j) \) blocking \( \mu \) at \( p \). Let \( \mathcal{S}(p) \) denote the set of all stable matchings.

Given a problem \( p = (A, P) \in \mathcal{P} \) and two matchings \( \mu, \mu' \in \mathcal{M}(A) \), \( \mu \) dominates \( \mu' \) if there exists a coalition \( K \subseteq A \) such that for all \( i \in K \), \( \mu(i) \in K \) and \( \mu(i) \) or \( \mu(i) \) for all \( \mu(i) \). A matching \( \mu \) is undominated if there exists
no matching $\mu' \in \mathcal{M}(A)$ which dominates $\mu$. The core of $p$ is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor (1990)). Given a problem $p = (A, P) \in \mathcal{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, $\mu$ Pareto dominates $\mu'$ if for all $i \in A$, $\mu(i) \succ P_i \mu'(i)$ whenever $\mu(i) \neq \mu'(i)$. A matching $\mu \in \mathcal{M}(A)$ is Pareto optimal for $p$ if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates $\mu$. Let $\mathcal{PO}(p)$ denote the set of all Pareto optimal matchings.

Given a problem $p = (A, P)$ and a subset of set of agents $N$, a reduced problem of $p$ with respect to $N$ is a problem where the preference profile $P$ is restricted to agents in $N$. Formally; for all $p = (A, P) \in \mathcal{P}$ and all $N \subseteq A$, $p' = (N, P|_N) \in \mathcal{P}$ is the reduced problem of $p$ with respect to $N$.

Given a matching $\mu \in \mathcal{M}(A)$, a reduced matching of $\mu$ with respect to $N$ is a matching $\mu|_N : N \cup \mu(N) \rightarrow N \cup \mu(N)$ such that for all $i \in N$, $\mu|_N(i) = \mu(i)$.

A solution $\varphi$ is a correspondence which associates with each $p$ a non-empty subset $\varphi(p) \subseteq \mathcal{M}(A)$. The core solution is the correspondence $\mathcal{S}$ which associates with each $p$ its set of stable matchings $\mathcal{S}(p)$.

Now, we are ready to define our axioms on solutions:

The first axiom requires that at each problem the solution recommends individually rational matchings:

**Individual rationality (IR):** For each $p \in \mathcal{P}$, $\varphi(p) \subseteq \mathcal{IR}(p)$. 

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The second axiom requires that at each problem the solution recommends Pareto optimal matchings:

**Pareto optimality (PO):** For each \( p \in P \), \( \varphi(p) \subseteq PO(p) \).

The third axiom requires that renaming men among men and renaming women among women do not change the result:

**Anonymity (AN):** For all \( A = M \cup W \) and all \( A' = M' \cup W' \) with \( |M| = |M'| \) and \( |W| = |W'| \), let \( \pi : A \to A' \) be a bijection such that \( \pi(M) = M' \) and \( \pi(W) = W' \). For all \( p = (A, P) \), let \( P' \) be such that for all \( i \in A \) and all \( j, k \in A(i) \), \( j P' k \) if and only if \( \pi(j)P'_{\pi(i)}\pi(k) \). And also for all \( \mu \in \mathcal{M}(A) \) define \( \pi_{\mu} \in \mathcal{M}(A') \) by setting for all \( i \in A \), \( \pi_{\mu}(i) = \pi(\mu(\pi^{-1}(i))) \). If \( \mu \in \varphi(A, P) \), then \( \pi_{\mu} \in \varphi(A', P') \).

The fourth axiom imposes same treatment of men and women, in the sense that renaming men as women and women as men do not change the result:

**Gender Fairness (GF):** For all \( A = M \cup W \) and all \( A' = M' \cup W' \) with \( |M| = |M'| \) and \( |W| = |M'| \), let \( \pi : A \to A' \) be a bijection such that \( \pi(M) = W' \) and \( \pi(W) = M' \). For all \( p = (A, P) \), let \( P' \) be such that for all \( i \in A \) and all \( j, k \in A(i) \), \( j P' k \) if and only if \( \pi(j)P'_{\pi(i)}\pi(k) \). And also for all \( \mu \in \mathcal{M}(A) \) define \( \pi_{\mu} \in \mathcal{M}(A') \) by setting for all \( i \in A \), \( \pi_{\mu}(i) = \pi(\mu(\pi^{-1}(i))) \). If \( \mu \in \varphi(A, P) \), then \( \pi_{\mu} \in \varphi(A', P') \).

\textsuperscript{5}Note that gender fairness is stronger than anonymity.
The next axiom is central to our analysis. Consider some problem \( p \) and some solution \( \varphi \). Take any matching \( \mu \) recommended by \( \varphi \) at \( p \). If the reduced matching of \( \mu \) with respect to each subgroup of matched pairs is among the recommendations made by the solution \( \varphi \) for the reduced problem of \( p \) with respect to this subgroup of matched pairs, then we say that the solution \( \varphi \) is consistent.

**Consistency (CON):** For each \( p = (A,P) \in \mathcal{P} \), each \( \mu \in \varphi(p) \) we have \( \mu_{|N} \in \varphi(N,P_{|N}) \) for any society \( N \subseteq A \) such that \( \mu(N) = N \).

The next section is devoted to converse consistency axiom.

### 3.3 Converse Consistency

In this section we consider three versions of “converse consistency”\(^6\). In a restricted domain where the number of men and women coincide and agents are not allowed being single, all these three versions are equivalent to its definition used in Sasaki and Toda (1992).

First we define "weakest converse consistency". Consider some problem \( p \) and some solution \( \varphi \). Take any matching \( \mu \). The requirement is that if the reduced matching of \( \mu \) with respect to each subgroup consisting of at most two men and two women matched at \( \mu \) is among the recommendations made

\(^6\)Thomson (2004, 2009) introduced and studied the converse consistency axiom for various economic models.
by the solution $\varphi$ for the reduced problem of $p$ with respect to this subgroup, then $\mu$ must be recommended by $\varphi$ at the original problem $p$. More formally,

**Weakest Converse Consistency (WWCON):** For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $M' \cup W' \subseteq A$ with $|M'| \leq 2$ and $|W'| \leq 2$, $\mu_{|M'\cup W'} \in \varphi(M' \cup W', P_{M'\cup W'})$, then $\mu \in \varphi(p)$.$^7$

WWCON will consider a subgroup of agents consisting of two self-matched women and two selfmatched men; as well as a subgroup of agents consisting of a pair matched to each other, a selfmatched woman and a selfmatched man. From our point of view, the former subgroup consists of four matched pairs, and the latter subgroup consists of three matched pairs.

Next we define "weak converse consistency". It is stronger than WWCON, in the sense that the subgroups consisting of three and four matched pairs are not considered here, but only the subgroups consisting of one and two matched pairs are taken into account: Consider some problem $p$ and some solution $\varphi$. Take any matching $\mu$. The requirement is that if the reduced matching of $\mu$ with respect to each subgroup consisting of at most two matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to this subgroup, then $\mu$ must be recommended by $\varphi$ at the original problem $p$.

**Weak Converse Consistency (WCCON):** For each $p = (A, P) \in \mathbf{P}$

$^7$Toda (1993) uses this version of converse consistency.
and each $\mu \in \mathcal{M}(A)$, if for each subset $N \subseteq A$ with $|N| = 2$, $\mu_{|N} \in \varphi(N \cup 
abla(N), P_{|N\cup\mu(N)})$, then $\mu \in \varphi(p)$.

Finally, we define converse consistency, only imposing the requirement on the subgroups formed exactly by two matched pairs: Consider some problem $p$ and some solution $\varphi$. Take any matching $\mu$. The requirement is that if the reduced matching of $\mu$ with respect to each subgroup of two matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to the subgroup of these two matched pairs, then $\mu$ must be recommended by $\varphi$ at the original problem $p$.

**Converse Consistency (CCON):** For each $p = (A, P) \in P$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu_{|N} \in \varphi(N \cup \mu(N), P_{|N\cup\mu(N)})$, then $\mu \in \varphi(p)$.

Throughout the essay, we will use the strongest version of converse consistency, which we call converse consistency. Indeed, since we deal with individually rational and Pareto optimal solutions, weak converse consistency and converse consistency axioms are equivalent.

**Proposition 3.1** If a solution satisfies IR and PO, then CCON is equivalent to WCCON.

**Proof.** Clearly, converse consistency implies weak converse consistency. For the other direction, let $\varphi$ be a solution which satisfies individual ratio-
nality, Pareto optimality and weak converse consistency but fails to satisfy converse consistency. Then there is a problem $p = (A, P)$ and a matching $\mu \in \mathcal{M}(A)$ such that $\mu \notin \varphi(p)$ and $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for any subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$. Since $\varphi$ satisfies weak converse consistency we have $\mu|_N \notin \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for some $\{i, j\} = N \subseteq A$ with $\mu(i) = j$. Say $i = m$ and $j = w$. If $w P_m m$ and $m P_w w$, then by Pareto optimality of $\varphi$, $\{\mu|_N\} = \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$. Hence, there are three possibilities, we have either (i) $m P_m w$ and $m P_w w$ or (ii) $w P_m m$ and $w P_w m$ or (iii) $m P_m w$ and $w P_w m$. For all these three cases, take any agent $\hat{m} \in M \setminus \{m\}$ and let $N' = \{m, \hat{m}\}$. By individual rationality of $\varphi$, we have $\mu|_N \notin \varphi(N' \cup \mu(N'), P|_{N' \cup \mu(N')})$, a contradiction. 

**Remark 3.1** There exists some IR, PO and WWCCON solution, which fails to satisfy WCCON. To see that take the solution $\bar{\varphi}$ defined for each $p = (A, P)$ as $\bar{\varphi}(p) = \{\mu \in IR(p) \cap PO(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m \text{ and } \mu(w) \neq w\}$ and consider the example below:

**Example 3.1** Let $\bar{\varphi}$ be defined for each $p = (A, P)$ as $\bar{\varphi}(p) = \{\mu \in IR(p) \cap PO(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m \text{ and } \mu(w) \neq w\}$. By its definition, $\bar{\varphi}$ is an individually rational and Pareto optimal solution. Toda (2003) shows that $\bar{\varphi}$ is weakest conversely consistent. Since for each $p \in P$, $\bar{\varphi}(p) \subseteq IR(p) \cap PO(p)$, by Proposition 3.1, it is suffi-
cient to show that $\tilde{\varphi}$ fails to satisfy converse consistency. Let $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$. Let $P$ be defined as follows:

<table>
<thead>
<tr>
<th>$P_{m_1}$</th>
<th>$P_{m_2}$</th>
<th>$P_{w_1}$</th>
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<tr>
<td>$w_2$</td>
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<tr>
<td>$m_1$</td>
<td>$w_1$</td>
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</tr>
</tbody>
</table>

Let $p = (M \cup W, P)$. Consider the matching $\mu \in \mathcal{M}(M \cup W)$ where $\mu(m_1) = w_1, \mu(m_2) = m_2$ and $\mu(w_2) = w_2$. Since $\mu$ is Pareto dominated by $\mu^*$ where $\mu^*(m_1) = w_2$ and $\mu^*(m_2) = w_1, \mu \notin \tilde{\varphi}(p)$. One can easily check that for any proper subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$.

To see that, consider the following three subproblems: First, letting $N = \{m_1, m_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$. Indeed, $(m_2, w_1)$ forms a blocking pair, but $\mu(m_2) = m_2$. Next, letting $N = \{m_1, w_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$. Indeed, $(m_1, w_2)$ forms a blocking pair, but $\mu(w_2) = w_2$. Finally, letting $N = \{m_2, w_2\}$, note that $\mu|_N \in \tilde{\varphi}(N \cup \mu(N), P|_{N \cup \mu(N)})$, by individual rationality of $\tilde{\varphi}$. Hence, $\tilde{\varphi}$ is not conversely consistent. By Proposition 3.1, $\tilde{\varphi}$ is not weak conversely consistent.
3.4 Results

In this section, we consider a model where agents have strict preferences and agents may stay single. In this more general model, the characterization result of Sasaki and Toda (1992) is not valid. The core is no more characterized by IR, PO, AN, CON and CCON. To see that take the solution $\varphi^*$ defined for each $p = (A, P)$ as $\varphi^*(p) = \{\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p) \mid \text{there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m\}$ and consider the example below.

Example 3.2 Let $M = \{m_1, m_2\}$ and $W = \{w_1\}$. Let $w_1 P_{m_1} m_1, w_1 P_{m_2} m_2$ and $m_1 P_{w_1} m_2 P_{w_1} w_1$. Let $p = (M \cup W, P)$. Consider the matchings $\mu_1 \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$ where $\mu_1(m_1) = w_1, \mu_1(m_2) = m_2$ and $\mu_2 \in \mathcal{IR}(p) \cap \mathcal{PO}(p)$ where $\mu_2(m_2) = w_1$ and $\mu_2(m_1) = m_1$. Note that $S(p) = \{\mu_1\}$ and $\varphi^*(p) = \{\mu_1, \mu_2\}$.8

For our characterization result, we use a strong anonymity condition which is called gender fairness. This axiom does not allow that a matching rule favors a gender at the expense of the other. It imposes same treatment of men and women, in the sense that renaming men as women and women as men does not change the result. Since gender fairness implies anonymity we also utilize from anonymity for the proposition below.

8Refer to Lemma 7.1 in the Appendix I to see that $\varphi^*$ satisfies CCON.
Proposition 3.2 Let \( \varphi \) be a solution satisfying IR, PO, GF and CCON. If \( \mu_{|N} \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)}) \) for any \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \), then for any \( \mu \in \varphi(A, P) \) we have \( \mu_{|N} \in \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)}) \).

**Proof.** Let \( \varphi \) be a solution satisfying IR, PO, GF and CCON. Let \( \mu = (A, P) \) be a problem. Take \( \mu \in \varphi(p) \) such that \( \mu_{|N} \in \varphi(N \cup \mu(N), P_{|N \cup \mu(N)}) \) for all \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \), but \( \mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)}) \) for some \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \).

Let \( N = \{i, j\} \). First we assume that \( \mu(i) = i \) and \( \mu(j) = j \). Wlog, say \( i = m_1 \) and \( j = w_1 \). Then \( \mu(m_1) = m_1 \) and \( \mu(w_1) = w_1 \). Since \( \mu_{|N} \in \mathcal{TR}(N \cup \mu(N), P_{|N \cup \mu(N)}) \) and \( \mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)}) \), \( (m_1, w_1) \) forms a blocking pair. We have \( w_1 \, P_{m_1} \, m_1 \) and \( m_1 \, P_{w_1} \, w_1 \). Then by Pareto optimality of \( \varphi \), we have \( \mu_{|N} \notin \varphi(N \cup \mu(N), P_{|N \cup \mu(N)}) \), a contradiction.

Next, let \( \mu(i) = i \) and \( \mu(j) \neq j \). Again, say \( i = m_1 \) and \( j = w_1 \). Then \( \mu(m_1) = m_1 \) and \( \mu(w_1) = m_2 \). Since \( \mu_{|N} \in \mathcal{TR}(N \cup \mu(N), P_{|N \cup \mu(N)}) \) and \( \mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)}) \), \( (m_1, w_1) \) forms a blocking pair. Hence, there is only one possible preference profile:

\[
\begin{array}{ccc}
P_{m_1} & P_{m_2} & P_{w_1} \\
| & | & |
\hline
w_1 & w_1 & m_1 \\
| & | & |
\hline
m_1 & m_2 & m_2 \\
| & | & |
\hline
w_1 & & \\
\end{array}
\]
Introduce an agent $w_2$ and extend the preferences of the agents of $N$ to the larger set $N \cup \{w_2\}$ of agents in the following way:

$$
\begin{array}{c|c|c|c|c}
\hline
P^*_m & P^*_w & P^*_m & P^*_w \\
\hline
w_1 & w_2 & m_1 & m_2 \\
\hline
w_2 & w_1 & m_2 & w_2 \\
\hline
m_1 & m_2 & w_1 & m_1 \\
\hline
\end{array}
$$

Consider the matching $\mu^*$ where $\mu^*(m_2) = w_1, \mu^*(m_1) = m_1$ and $\mu^*(w_2) = w_2$. Since $\mu^*$ is Pareto dominated by the matching $\mu'$ where $\mu'(m_1) = w_1$ and $\mu'(m_2) = w_2$ we have $\mu^* \not\in \varphi(M^*, W^*, P^*)$. However, $\mu^*_{|N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N'\cup\mu^*(N')})$ for each $N' = \{i, j\} \subset A$ with $\mu^*(i) \neq j$.

To see that, consider the following three subproblems: Letting $N' = \{m_1, m_2\}$, we have $P^*_{|N'\cup\mu^*(N')} = P^*_{|N'\cup\mu(N)}$ and $\mu^*_{|N'} = \mu_{|N} \in \varphi(N' \cup \mu^*(N'), P^*_{|N'\cup\mu^*(N')})$. (Indeed, $N \cup \mu(N) = N' \cup \mu^*(N')$). Next, letting $N' = \{m_1, w_2\}$, by individual rationality of $\varphi$, we have $\mu^*_{|N'} = \varphi(N' \cup \mu^*(N'), P^*_{|N'\cup\mu^*(N')})$. Finally, let $N' = \{m_2, w_2\}$. Let $\pi(m_1) = w_2, \pi(m_2) = w_1$ and $\pi(w_1) = m_2$. Let $P'$ be the preference profile in the permuted problem.

Note that $P' = P_{|N'\cup\mu(N)}$, $\pi_{\mu}(m_2) = w_1$ and $\pi_{\mu}(w_2) = w_2$. Since $\pi_{\mu} = \mu^*_{|N'}$, then by gender fairness we have $\mu^*_{|S} \in \varphi(N' \cup \mu^*(N'), P^*_{|N'\cup\mu^*(N')})$. Finally, a similar argument works for the symmetric cases where $\mu(i) \neq i$ and $\mu(j) = j$. 

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Now, assume that $\mu(i) \neq i$ and $\mu(j) \neq j$. Say $i = m_1$ and $j = m_2$. Then $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Since $\mu_{|N} \in I \mathcal{R}(N \cup \mu(N), P_{|N \cup \mu(N)})$ and $\mu_{|N} \notin \mathcal{S}(N \cup \mu(N), P_{|N \cup \mu(N)})$, then there is a blocking pair, wlog assume $(m_1, w_2)$ forms a blocking pair. Then we have nine possible preferences profiles, four of them are the same as Sasaki and Toda (1992) where each agent puts being single at the bottom. The other five possible preference profiles are:

\begin{table}
\begin{center}
\begin{tabular}{|l|l|}
\hline
$P_{m_1}$ & $P_{m_2}$ \\
\hline
$w_2$ & $w_2$ \\
\hline
$w_1$ & $m_2$ \\
\hline
$m_1$ & $w_1$ \\
\hline
\end{tabular}
\begin{tabular}{|l|l|}
\hline
$P_{w_1}$ & $P_{w_2}$ \\
\hline
$m_1$ & $m_1$ \\
\hline
$m_2$ & $m_2$ \\
\hline
$w_1$ & $w_2$ \\
\hline
\end{tabular}
\end{center}
\end{table}

\begin{table}
\begin{center}
\begin{tabular}{|l|l|}
\hline
$P_{m_1}$ & $P_{m_2}$ \\
\hline
$w_2$ & $w_2$ \\
\hline
$w_1$ & $m_2$ \\
\hline
$m_1$ & $w_1$ \\
\hline
\end{tabular}
\begin{tabular}{|l|l|}
\hline
$P_{w_1}$ & $P_{w_2}$ \\
\hline
$m_2$ & $m_1$ \\
\hline
$m_1$ & $m_2$ \\
\hline
$w_1$ & $w_2$ \\
\hline
\end{tabular}
\end{center}
\end{table}

\begin{table}
\begin{center}
\begin{tabular}{|l|l|}
\hline
$P_{m_1}$ & $P_{m_2}$ \\
\hline
$w_2$ & $w_1$ \\
\hline
$w_1$ & $w_2$ \\
\hline
$m_1$ & $m_2$ \\
\hline
\end{tabular}
\begin{tabular}{|l|l|}
\hline
$P_{w_1}$ & $P_{w_2}$ \\
\hline
$m_1$ & $m_1$ \\
\hline
$w_1$ & $m_2$ \\
\hline
$m_2$ & $w_2$ \\
\hline
\end{tabular}
\end{center}
\end{table}

43
For the case (i), two new agents $m_3$ and $w_3$ enter the society. We extend the preferences of the agents of $N$ to the larger set $N \cup \{m_3, w_3\}$ of agents in the following way:

Consider the matching $\mu^*$ where $\mu^*(m_1) = w_1, \mu^*(m_2) = w_2$ and $\mu^*(m_3) = w_3$. Since $\mu^*$ is Pareto dominated by the matching $\mu'$ where $\mu'(m_1) = w_2, \mu'(m_2) = w_3$ and $\mu'(m_3) = w_1$, we have $\mu^* \notin \varphi(M^*, W^*, P^*)$. 

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However, \( \mu^*_{|N'} \in \varphi(N' \cup \mu^*(N')), P^*_{|N' \cup \mu^*(N')} \) for each \( N' = \{i, j\} \subset A \) with \( \mu^*(i) \neq j \).

To see that, consider the following three subproblems: Letting \( N' = \{m_1, m_2\} \), we have \( P^*_{|N' \cup \mu^*(N')} = P_{|N' \cup \mu(N)} \) and \( \mu^*_{|N'} = \mu_{|N} \in \varphi(N' \cup \mu(N')), P_{|N' \cup \mu^*(N')} \).

Next, letting \( N' = \{m_1, m_3\} \), note that \( P^*_{|N' \cup \mu^*(N')} \) is

<table>
<thead>
<tr>
<th>( P^*_{m_1} )</th>
<th>( P^*_{m_3} )</th>
<th>( P^*_{w_1} )</th>
<th>( P^*_{w_3} )</th>
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<tbody>
<tr>
<td>( w_1 )</td>
<td>( w_1 )</td>
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<td>( w_3 )</td>
<td>( m_3 )</td>
<td>( w_1 )</td>
<td>( w_3 )</td>
</tr>
</tbody>
</table>

Let \( \pi(m_1) = m_3, \pi(m_2) = m_1, \pi(w_1) = w_3 \) and \( \pi(w_2) = w_1 \). Let \( P' \) be the preference profile in the permuted problem. Note that \( P' = P_{|N' \cup \mu(N)} \). Since \( \pi_{|\mu} = \mu^*_{|N'} \), by anonymity of \( \varphi \), we have \( \mu^*_{|N'} \in \varphi(N' \cup \mu^*(N')), P^*_{|N' \cup \mu^*(N')} \).

Finally, letting \( N' = \{m_2, m_3\} \), note that \( P^*_{|N' \cup \mu^*(N')} \) is

<table>
<thead>
<tr>
<th>( P^*_{m_2} )</th>
<th>( P^*_{m_3} )</th>
<th>( P^*_{w_2} )</th>
<th>( P^*_{w_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_3 )</td>
<td>( w_3 )</td>
<td>( m_2 )</td>
<td>( m_2 )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( m_3 )</td>
<td>( m_3 )</td>
<td>( m_3 )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( w_2 )</td>
<td>( w_2 )</td>
<td>( w_3 )</td>
</tr>
</tbody>
</table>

Let \( \pi(m_1) = m_2, \pi(m_2) = m_3, \pi(w_1) = w_2 \) and \( \pi(w_2) = w_3 \). Let \( P' \) be the preference profile in the permuted problem. Note that \( P' = P_{|N' \cup \mu(N)} \). Since
\[ \pi|_{\mu} = \mu^*|_{N}, \text{ by anonymity of } \varphi, \text{ we have } \mu^*|_{N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}). \]

Similar arguments work for the case (ii), (iii) and (iv).

For the case (v), two new agents \(m_3\) and \(w_3\) enter the society. We extend the preferences of the agents of \(N\) to the larger set \(N \cup \{m_3, w_3\}\) of agents in the following way.

\[
\begin{array}{ccc|ccc}
 P^*_m & P^*_m & P^*_m & P^*_w & P^*_w & P^*_w \\
 w_2 & w_3 & w_1 & m_3 & m_1 & m_2 \\
 w_1 & w_2 & w_3 & m_1 & m_2 & m_3 \\
 m_1 & m_2 & m_3 & w_1 & w_2 & w_3 \\
 w_3 & w_1 & w_2 & m_2 & m_3 & m_1 \\
\end{array}
\]

Consider the matching \(\mu^*\) where \(\mu^*(m_1) = w_1, \mu^*(m_2) = w_2\) and \(\mu^*(m_3) = w_3\). Since \(\mu^*\) is Pareto dominated by the matching \(\mu'\) where \(\mu'(m_1) = w_2, \mu'(m_2) = w_3\) and \(\mu'(m_3) = w_1\), we have \(\mu^* \not\in \varphi(M^*, W^*, P^*)\).

However, \(\mu^*|_{N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}\) for each \(N' = \{i, j\} \subset A\) with \(\mu^*(i) \neq j\).

To see that, consider the following three subproblems: Letting \(N' = \{m_1, m_2\}\), we have \(P^*_{|N' \cup \mu^*(N')} = P_{|N' \cup \mu(N)}\) and \(\mu^*|_{N'} = \mu|_{N} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}\).

Next, letting \(N' = \{m_1, m_3\}\), note that \(P^*_{|N' \cup \mu^*(N')}\) is
Let $\pi(m_1) = m_3$, $\pi(m_2) = m_1$, $\pi(w_1) = w_3$ and $\pi(w_2) = w_1$. Let $P'$ be the preference profile in the permuted problem. Note that $P' = P|_{N \cup \mu(N)}$. Since $\pi_\mu = \mu^*_N$, by anonymity of $\varphi$, we have $\mu^*_{N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}}$.

Finally, letting $N' = \{m_2, m_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

$$
\begin{array}{cccc}
P^*_{m_2} & P^*_{m_3} & P^*_{w_2} & P^*_{w_3} \\
w_3 & w_3 & m_2 & m_2 \\
w_2 & m_3 & w_2 & m_3 \\
m_2 & w_2 & m_3 & w_3 \\
\end{array}
$$

Let $\pi(m_1) = m_2$, $\pi(m_2) = m_3$, $\pi(w_1) = w_2$ and $\pi(w_2) = w_3$. Let $P'$ be the preference profile in the permuted problem. Note that $P' = P|_{N \cup \mu(N)}$. Since $\pi_\mu = \mu^*_N$, by anonymity of $\varphi$, we have $\mu^*_{N'} \in \varphi(N' \cup \mu^*(N'), P^*_{|N \cup \mu^*(N')}}$.

Hence, by CCON we have $\mu^* \in \varphi(M^*, W^*, P^*)$, a contradiction.

**Proposition 3.3** If a solution $\varphi$ satisfies IR, PO, GF, CON and CCON, then $\varphi(p) \subseteq S(p)$ for each problem $p \in P$. 

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**Proof.** Let \( \varphi \) be a solution satisfying IR, PO, GF, CON and CCON. Let \( p = (A, P) \) be a problem. Let \( \mu \in \varphi(p) \). By consistency of \( \varphi \), we have \( \mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)}) \) for each \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \). By Proposition 4.2, \( \mu|_N \in S(N \cup \mu(N), P|_{N \cup \mu(N)}) \). Since \( S \) satisfies CCON, we have \( \mu \in S(p) \). ■

To show that no proper subsolution of the core satisfies consistency in many-to-one matching problems, Toda (2006) uses a bracing lemma. The lemma below is its adaptation for one-to-one matching problems.

**Lemma 3.1 (Bracing Lemma)** For each \( p = (A, P) \in P \) and each \( \mu \in S(p) \), there exists \( \bar{p} \) such that \( S(\bar{p}) = \{\bar{\mu}\} \), \( \bar{p} \) is the reduced problem of \( \bar{p} \) at \( \bar{\mu} \) and \( \bar{\mu}|_{M \cup W} = \mu \).

**Proposition 3.4** No proper subsolution of the core satisfies CON.

**Proof.** Let \( \varphi \) be a subsolution of the core satisfying CONS. Let \( p = (A, P) \) be a problem. Let \( \mu \in S(p) \). Then by Lemma 3.1, there exists \( \bar{p} \) such that \( S(\bar{p}) = \{\bar{\mu}\} \), \( \bar{p} \) is reduced problem of \( \bar{p} \) at \( \bar{\mu} \) and \( \bar{\mu}|_{M \cup W} = \mu \). Since \( \varphi \) is a subsolution of the core we have \( \varphi(\bar{p}) = S(\bar{p}) = \{\bar{\mu}\} \). Since \( \varphi \) satisfies CON, \( \mu \in \varphi(p) \). Hence, \( \varphi \equiv S \). ■

**Theorem 3.1** The core \( S \) is the unique solution satisfying IR, PO, GF, CON and CCON.
Proof. One can easily check that the core $S$ satisfies the above listed axioms. For the uniqueness, let $\varphi$ be a solution satisfying these five axioms. Then by Proposition 3.3, $\varphi(p) \subseteq S(p)$ for each $p \in P$. By Proposition 3.4, no proper subsolution of the core satisfies CON. Hence, $\varphi \equiv S$. ■

We also check the independence of the axioms. By doing that, we also show that dropping any one of the five axioms leads to the failure of Theorem 3.1.

- A solution which satisfies IR, PO, GF, CON, but not CCON.

For each $p \in P$, $\varphi(p) = \mathcal{IR}(p) \cap \mathcal{PO}(p)$.

- A solution which satisfies IR, PO, GF, CCON, but not CON.

Let $\varphi$ be defined as follows:

$$
\varphi(p) = \begin{cases} 
S(p) & \text{for any } p = (N \cup \mu(N), P_{(N,\mu(N))}) \text{ with } N = \{i,j\} \text{ and } \mu(i) \neq j \\
\mathcal{IR}(p) \cap \mathcal{PO}(p) & \text{otherwise.}
\end{cases}
$$

- A solution which satisfies IR, PO, CON, CCON, but not GF.

Let $\varphi$ be defined for each $p = (A, P)$ as $\varphi(p) = \{\mu \in \mathcal{IR}(p) \cap \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m\}$.\footnote{Refer to Lemma 7.1 in the Appendix I to see that $\varphi$ satisfies CCON.}

- A solution which satisfies IR, GF, CON, CCON, but not PO.
For each $p \in P$, $\varphi(p) = \mathcal{IR}(p)$.

- A solution which satisfies PO, GF, CON, CCON, but not IR.

Let $\varphi$ be defined for each $p = (A, P)$ as $\varphi(p) = \{\mu \in \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w)\}$.\(^{10}\)

### 3.5 Weak Preferences

In this section, we relax the assumption that each agent has a strict preference over his/her potential mate. Each agent $i \in A$ may be indifferent between matching with any two possible mates $j, k \in A(i)$; if it is the case, then we write $j \overset{I}{i} k$. For each agent $i \in A$, each $j, k \in A(i)$ we write $j \overset{R}{i} k$ if and only if either $j \overset{P}{i} k$ holds or $j \overset{I}{i} k$ holds. From this section on, each agent $i \in A$ has a complete and transitive preference relation over $A(i)$, denoted by $R_i$. Let $\mathcal{R}$ denote the set of all possible preference profiles $R \equiv (R_i)_A$. A problem $p$ is redefined as a pair $p = (A, R)$, where $A$ is a society, $R$ is the profile of their preferences over potential mates. Let $\mathbf{R}$ denote the set of all problems.

Let $p = (A, R) \in \mathbf{R}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for $p$ if for all $i \in A$, $\mu(i) \overset{R}{i} i$ or $\mu(i) = i$. Let $\mathcal{IR}(p)$ denote the set of all individually rational matchings. A pair of agents

\(^{10}\)Refer to Lemma 7.2 in the Appendix I to see that $\varphi$ satisfies CCON.
(i, j) blocks a matching $\mu \in \mathcal{M}(A)$ if $j R_i \mu(i)$ and $i R_j \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is stable for $p$ if it is individually rational for $p$ and there is no pair $(i, j)$ blocking $\mu$ at $p$. Let $S(p)$ denote the set of all stable matchings. Note that $S(p)$ is nonempty.\(^{11}\)

Given a problem $p = (A, R) \in \mathcal{R}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, $\mu$ Pareto dominates $\mu'$ if for all $i \in A$, $\mu(i) R_i \mu'(i)$ and for some $j \in A, \mu(j) R_j \mu'(j)$. A matching $\mu \in \mathcal{M}(A)$ is Pareto optimal for $p$ if there exists no matching $\mu' \in \mathcal{M}(A)$ which Pareto dominates $\mu$. Let $PO(p)$ denote the set of all Pareto optimal matchings. The other axioms defined in Section 2 are simply carried to the world of weak preferences just by replacing $P$ by $R$.

First note that the Pareto Stable rule, denoted by $PS$ and defined for each $p \in \mathcal{R}$ as $PS(p) = PO(p) \cap S(p)$, fails to satisfy converse consistency.\(^{12}\)

**Example 3.3** Let $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$. Let $R$ be

\(^{11}\)The stronger version of stability is defined as follows: A pair of agents $(i, j)$ weakly blocks a matching $\mu \in \mathcal{M}(A)$ if $j R_i \mu(i)$, $i R_j \mu(j)$ and for some $k \in \{i, j\}$, $R_k = P_k$. A matching $\mu \in \mathcal{M}(A)$ is strongly stable for $p$ if it is individually rational for $p$ and there is no pair $(i, j)$ weakly blocking $\mu$ at $p$. Note that the set of all strongly stable matchings may be empty for some problems.

\(^{12}\)When we have strict preferences $PS(p) = PO(p) \cap S(p)$. But when we allow indifferences, a stable matching may not be Pareto optimal. To see this, let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let $P$ be defined as follows:

\[
\begin{array}{cccc}
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
&w_1 & m_1 & w_2 \\
&m_2 & w_2 & m_2 \\
\end{array}
\]

Let $p = (M \cup W, P)$. Consider the matching $\mu$ where $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Note that $\mu \in S(p)$, but $\mu \notin PO(p)$.  

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defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>$R_{w_1}$</th>
<th>$R_{w_2}$</th>
<th>$R_{w_3}$</th>
<th>$R_{w_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$m_1m_2m_3m_4$</td>
<td>$m_1m_3m_4$</td>
<td>$m_1m_2m_4$</td>
<td>$m_2m_3$</td>
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<tr>
<td>$w_2$</td>
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<td>$m_2$</td>
<td>$m_3$</td>
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</tr>
<tr>
<td>$w_4$</td>
<td>$w_1$</td>
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<td>$w_4$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$m_4$</td>
<td></td>
</tr>
</tbody>
</table>

Let $p = (M \cup W, R)$. Consider the matching $\mu$ where $\mu(m_1) = w_1$, $\mu(m_2) = w_4$, $\mu(m_3) = w_2$, $\mu(m_4) = w_3$. Since $\mu$ is Pareto dominated by $\mu'$ where $\mu'(m_1) = w_1$, $\mu'(m_2) = w_3$, $\mu'(m_3) = w_4$, $\mu'(m_4) = w_2$, we have $\mu \notin \mathcal{PO}(p)$.

Hence, $\mu \notin \mathcal{PS}(p)$. But $\mu_{|N} \in \mathcal{PS}(N \cup \mu(N), R_{|N \cup \mu(N)})$ for each $N = \{i, j\} \subset A$ with $\mu(i) \neq j$.

Indeed, we have an impossibility result: there exists no solution satisfying Pareto optimality, anonymity and converse consistency.\textsuperscript{13}

\textbf{Theorem 3.2} No solution satisfies PO, AN and CCON.

\textbf{Proof.} Let $\varphi$ be a solution satisfying PO, AN and CCON. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let $R$ be defined as follows:

\textsuperscript{13}This result is a similar spirit of Proposition 4.2 in Özkal-Sanver (2010).
Let \( p = (M \cup W, R) \). Note that \( \mathcal{PO}(p) = \{\mu_1, \mu_2\} \) where \( \mu_1(m_1) = w_1, \mu_1(m_2) = w_2 \) and \( \mu_2(m_1) = w_2, \mu_2(m_2) = w_1 \). By Pareto optimality of \( \varphi \), we have \( \varphi(p) \subseteq \{\mu_1, \mu_2\} \).

Suppose \( \mu_1 \in \varphi(p) \). Introduce two agents \( m_3 \) and \( w_3 \) and extend the preferences of the agents of \( M \cup W \) to the larger set \( M \cup W \cup \{m_3, w_3\} \) of agents in the following way:

\[
\begin{array}{cccc|cccc}
R_{m_1} & R_{m_2} & R_{w_1} & R_{w_2} & R_{m_1}^* & R_{m_3}^* & R_{w_1}^* & R_{w_2}^* \\
\hline
w_1 & w_1 & m_1 & m_2 & m_1 & m_2 & m_1 & m_2 \\
w_2 & w_2 & w_1 & w_2 & w_1 & w_2 & w_1 & w_2 \\
m_1 & m_2 & w_1 & w_2 & w_1 & w_2 & w_1 & w_2 \\
\end{array}
\]

Let \( p^* = (M \cup W \cup \{m_3, w_3\}, R^*) \). Consider the matching \( \mu^* \) where \( \mu^*(i) = \mu_1(i) \) for all \( i \in M \cup W \) and \( \mu^*(m_3) = w_3 \). Since \( \mu^* \) is Pareto dominated by \( \mu' \) where \( \mu'(m_1) = w_3, \mu'(m_2) = w_1 \) and \( \mu'(m_3) = w_2 \), we have \( \mu^* \notin \varphi(p^*) \).

However, \( \mu^*_{|N} \in \varphi(N \cup \mu^*(N), R^*_{|N \cup \mu^*(N)}) \) for each \( N = \{i, j\} \subset A \) with \( \mu^*(i) \neq j \).
To see that, consider the following three subproblems: Letting \( N = \{m_1, m_2\} \), we have \((N \cup \mu^*(N), R^*_{(N \cup \mu^*(N))}) = p\) and \( \mu^*_N = \mu_1 \in \varphi(p) \).

Next, letting \( N = \{m_1, m_3\} \), we have \( \mathcal{PO}(N \cup \mu^*(N), R^*_{(N \cup \mu^*(N))}) = \{\tilde{\mu}\} \) where \( \tilde{\mu}(m_1) = w_1 \) and \( \tilde{\mu}(m_3) = w_3 \). Since \( \mu^*_N = \tilde{\mu} \), we have \( \mu^*_N \in \varphi(N \cup \mu^*(N), R^*_{(N \cup \mu^*(N))}) \).

Finally, letting \( N = \{m_2, m_3\} \), note that \( R^*_{(N \cup \mu^*(N))} \) is

\[
\begin{array}{cccc}
R^*_{m_2} & R^*_{m_3} & R^*_{w_2} & R^*_{w_3} \\
 w_2 & w_2 & m_2m_3 & m_2m_3 \\
 w_3 & w_3 & w_2 & w_3 \\
 m_2 & m_3 & & \\
\end{array}
\]

Let \( \pi(m_1) = m_2, \pi(m_2) = m_3, \pi(w_1) = w_2 \) and \( \pi(w_2) = w_3 \). Let \( R' \) be the preference profile in the permuted problem. Note that \( R' = R \). Since \( \pi_{|\mu} = \mu^*_N \), by anonymity of \( \varphi \), we have \( \mu^*_N \in \varphi(N \cup \mu^*(N), R^*_{(N \cup \mu^*(N))}) \).

Then by converse consistency \( \mu^* \in \varphi(p^*) \). Since \( \varphi \) satisfies Pareto optimality, we have \( \varphi(p) = \{\mu_2\} \). Next, introduce two agents \( m_4 \) and \( w_4 \) and extend the preferences of the agents of \( M \cup W \) to the larger set \( M \cup W \cup \{m_4, w_4\} \) of agents in the following way:
Let \( \tilde{\mu} = (M \cup W \cup \{m_4, w_4\}, \tilde{R}) \). Consider the matching \( \tilde{\mu} \) where \( \tilde{\mu}(i) = \mu_2(i) \) for all \( i \in M \cup W \) and \( \tilde{\mu}(m_4) = w_4 \). Since \( \tilde{\mu} \) is Pareto dominated by the matching \( \mu' \) where \( \mu'(m_1) = w_1, \mu'(m_2) = w_4 \) and \( \mu'(m_4) = w_2 \), we have \( \tilde{\mu} \not\in \varphi(\tilde{\mu}) \). However, \( \tilde{\mu}_{|N \cup \mu^*(N)} \in \varphi(\tilde{N} \cup \tilde{\mu}(N), \tilde{R}_{|N \cup \tilde{\mu}(N)}) \) for each \( N = \{i, j\} \subset A \) with \( \tilde{\mu}(i) \neq j \).

To see that, consider the following three subproblems: Letting \( N = \{m_1, m_2\} \), we have \( (N \cup \tilde{\mu}(N), \tilde{R}_{|N \cup \tilde{\mu}(N)}) = p \) and \( \tilde{\mu}_{|N \cup \tilde{\mu}(N)} = \mu_2 \in \varphi(p) \).

Next, letting \( N = \{m_2, m_4\} \), we have \( PO(N \cup \tilde{\mu}(N), \tilde{R}_{|N \cup \tilde{\mu}(N)}) = \{\overline{p}\} \) where \( \overline{p}(m_2) = w_1 \) and \( \overline{p}(m_4) = w_4 \). Since \( \tilde{\mu}_{|N} = \overline{p} \), we have \( \tilde{\mu}_{|N} \in \varphi(N \cup \tilde{\mu}(N), \tilde{R}_{|N \cup \tilde{\mu}(N)}) \).

Finally, letting \( N = \{m_1, m_4\} \), note that \( \tilde{R}_{|N \cup \tilde{\mu}(N)} \) is
Let $\pi(m_1) = m_4$, $\pi(m_2) = m_1$, $\pi(w_1) = w_2$ and $\pi(w_2) = w_4$. Let $R''$ be the preference profile in the permuted problem. Note that $R'' = R$. Since $\pi_{\mu} = \hat{\mu}_{|N}$, by anonymity of $\varphi$, we have $\hat{\mu}_{|N} \in \varphi(N \cup \hat{\mu}(N), \hat{R}_{|N \cup \hat{\mu}(N)})$. Then by converse consistency of $\varphi$, we have $\hat{\mu} \in \varphi(\hat{p})$, contradicting that $\varphi$ satisfies Pareto optimality, and completing the proof. 

By weakening the Pareto optimality axiom and adding a monotonicity condition, we end up with a characterization result.

Given a problem $p = (A, R) \in R$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, $\mu$ strongly Pareto dominates $\mu'$ if for all $i \in A$ we have $\mu(i) R_i \mu'(i)$ and for all $j \in A$ with $\mu(j) \neq \mu'(j)$, we have $\mu(j) P_j \mu'(j)$. A matching $\mu \in \mathcal{M}(A)$ is weakly Pareto optimal for $p$ if there exists no matching $\mu' \in \mathcal{M}(A)$ which strongly Pareto dominates $\mu$. Let $\mathcal{WPO}(p)$ denote the set of all weakly Pareto optimal matchings.

A weak Pareto optimal solution recommends at each problem among weakly Pareto optimal matchings:

**Weak Pareto optimality (WPO):** For each $p \in R$, $\varphi(p) \subseteq \mathcal{WPO}(p)$. 
Remark 3.2  Note that for each problem $p = (A, P)$, $\mathcal{PO}(p) = WPO(p)$. If a solution $\varphi$ satisfies IR, WPO, GF, CON and CCON then for each $p \in P$, $\varphi(p) = S(p)$.

Consider some problem $p = (A, R^*)$ and any matching $\mu \in \mathcal{M}(A)$. We define complete strict preference $P^\mu$ over $A$ as a monotonic stretching of $R^*$ for $\mu$ in the following way: For each $i \in A$, for any $j, k \in A(i)$; let $j P^\mu_i k$ if $j P^*_i k$ holds and let $\mu(i)P^\mu_i j$ for any $j \in A(i)$ such that $\mu(i)I^*_i j$. Using monotonic stretching we switch from weak preferences to some adequate form of strict preferences. Roughly speaking, monotonicity axiom allows us to check whether a matching $\mu$ is recommended or not for a problem $(A, R^*)$ by considering whether $\mu$ is recommended for $(A, P^\mu)$ where $P^\mu$ is monotonic stretching of $R^*$ for $\mu$. In other words, monotonicity of a solution $\varphi$ requires that for any matching $\mu$ and for any monotonic stretching $P^\mu$ of $R^*$, either $\mu$ must be recommended by $\varphi$ both for $(A, P^\mu)$ and $(A, R^*)$, or $\mu$ is not recommended by $\varphi$ for $(A, P^\mu)$ and $(A, R^*)$.

**Monotonicity(MON):** For each $p = (A, R^*) \in R$ and $\mu \in \mathcal{M}(A)$, and for any monotonic stretching $P^\mu$ of $R^*$, we have $\mu \in \varphi(A, P^\mu)$ if and only if $\mu \in \varphi(A, R^*)$.

**Proposition 3.5** If a solution $\varphi$ satisfies IR, WPO, GF, CCON and MON then for any $\mu \in \varphi(A, R)$, if $\mu|_N \in \varphi(N \cup \mu(N), R|_{N\cup\mu(N)})$ for any $N = \ldots$
If \( \{i, j\} \subseteq A \) with \( \mu(i) \neq j \), then we have \( \mu_{\mid N} \in \mathcal{S}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \).

**Proof.** Let \( \varphi \) be a solution satisfying IR, WPO, GF, CCON and MON. Let \( p = (A, R) \) be a problem. Take \( \mu \in \varphi(p) \) such that \( \mu_{\mid N} \in \varphi(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \) for all \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \), we want to show that \( \mu_{\mid N} \in \mathcal{S}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \). Suppose, by contrary, \( \mu_{\mid N} \notin \mathcal{S}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \) for some \( N = \{i, j\} \subseteq A \) with \( \mu(i) \neq j \). Let \( N = \{i, j\} \).

**Case 1**: First we consider the case that \( \mu(i) = i \) and \( \mu(j) = j \). Wlog, say \( i = m_1 \) and \( j = w_1 \). Then \( \mu(m_1) = m_1 \) and \( \mu(w_1) = w_1 \). Since \( \mu_{\mid N} \in \mathcal{T}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \) and \( \mu_{\mid N} \notin \mathcal{S}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \), \((m_1, w_1)\) forms a blocking pair. We have \( w_1 P_{m_1} m_1 \) and \( m_1 P_{w_1} w_1 \). Then by weak Pareto optimality of \( \varphi \), \( m_1 \) is matched with \( w_1 \) we have \( \mu_{\mid N} \notin \varphi(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \), a contradiction.

**Case 2**: Next we consider the case where only one agent is self matched. Let \( \mu(i) = i \) and \( \mu(j) \neq j \). Again, say \( i = m_1 \) and \( j = w_1 \). Then \( \mu(m_1) = m_1 \) and \( \mu(w_1) = m_2 \). Since \( \mu_{\mid N} \in \mathcal{T}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \) and \( \mu_{\mid N} \notin \mathcal{S}(N \cup \mu(N), R_{\mid N \cup \mu(N)}) \), there \((m_1, w_1)\) forms a blocking pair. Then, there are three possible preference profiles:
For the subcase 2(i) and 2(ii), let $P^\mu$ be a monotonic stretching of $R$ for $\mu$. Note that $\mu|_N \notin S(N \cup \mu(N), P^\mu_{|N \cup \mu(N)})$. By Theorem 3.1, $\mu|_N \notin \varphi(N \cup \mu(N), P^\mu_{|N \cup \mu(N)})$. By MON, $\mu|_N \notin \varphi(N \cup \mu(N), R_{|N \cup \mu(N)})$, a contradiction.

For the Subcase 2(iii), Note that $\mu|_N \notin S(N \cup \mu(N), R_{|N \cup \mu(N)})$. Since $R_{|N \cup \mu(N)}$ is strict, by Theorem 3.1; $\mu|_N \notin \varphi(N \cup \mu(N), R_{|N \cup \mu(N)})$, a contradiction.
Case 3: Finally, let $\mu(i) \neq i$ and $\mu(j) \neq j$. Again, say $i = m_1$ and $j = m_2$. Then $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. Since $\mu|_N \in IR(N \cup \mu(N), R_{|N\cup\mu(N)})$ and $\mu|_N \notin S(N \cup \mu(N), R_{|N\cup\mu(N)})$, assume $(m_1, w_2)$ forms a blocking pair. Then we have

\[
\begin{array}{|c|c|c|c|}
\hline
R_{m_1} & R_{m_2} & R_{w_1} & R_{w_2} \\
\hline
w_1 & & m_1 & \\
\hline
d & & m_2 & \\
\hline
\end{array}
\]

For any agent who is indifferent between the other agents apply the monotonic streching to the preferences profile say $p' = (A, P^u)$ and we are in the one of the nine cases of Proposition 3.2. For all these cases we know that $\mu|_N \notin S(N \cup \mu(N), P^u_{|N\cup\mu(N)})$. By Theorem 3.1, we have $\mu|_N \notin \varphi(N \cup \mu(N), P^u_{|N\cup\mu(N)})$. By MON, $\mu|_N \notin \varphi(N \cup \mu(N), R_{|N\cup\mu(N)})$, a contradiction. ■

**Proposition 3.6** If a solution $\varphi$ satisfies IR, WPO, GF, CON, CCON and MON then $\varphi(p) \subseteq S(p)$ for each problem $p \in R$.

**Proof.** Let $\varphi$ be a solution satisfying IR, WPO, GF, CON, CCON and MON. Let $p = (A, R)$ be a problem. Let $\mu \in \varphi(p)$. By consistency of $\varphi$, we have $\mu|_N \in \varphi(N \cup \mu(N), P^u_{|N\cup\mu(N)})$ for each $N = \{i, j\} \subseteq A$ with $\mu(i) \neq j$. By Proposition 3.5, $\mu|_N \in S(N \cup \mu(N), P^u_{|N\cup\mu(N)})$ for each $N = \{i, j\} \subseteq A$ with
Proposition 3.7  No proper subsolution of the core satisfies CON and MON.

Proof. Let \( \varphi \) be a subsolution of \( S \) satisfying both CON and MON, let 
\( p = (A, R) \) be a problem and let \( \mu \in S(p) \). Let \( P^\mu \) be a monotonic streching of \( R \) for \( \mu \). Since \( S \) satisfies MON, by MON, \( \mu \in S(A, P^\mu) \). By the same arguments as in the proof of Lemma 3.1, there exists a problem \( p' = (A', P') \) such that \( S(p') = \{\mu'\}, \mu'|_A = \mu, \) and \( (A, P^\mu) \) is a reduced problem of \( p' \) at \( \mu \). Since \( \varphi(p') = \{\mu'\} \) and \( \varphi \) satisfies CON, \( \mu \in \varphi(A, P^\mu) \). Since \( P^\mu \) is a monotonic stretching of \( R \) for \( \mu \), by MON we have \( \mu \in \varphi(A, R) \). This proves that \( \varphi \equiv S \).\(^{14}\)

Theorem 3.3  The core \( S \) is the unique solution satisfying IR, WPO, GF, CON, CCON and MON.

Proof. One can easily check that the core \( S \) satisfies the above listed axioms. For the uniqueness, let \( \varphi \) be a solution satisfying these six axioms. Then by Proposition 3.6, \( \varphi(p) \subseteq S(p) \) for each \( p \in R \). Also by Proposition 3.7, no proper subsolution of the core satisfies CON and MON. Hence, \( \varphi \equiv S \).

\(^{14}\)This proof is an adaptation of the proof of the Proposition 4.4 of Toda (2006).
Remark 3.3 Whether the core could be characterized by IR, WPO, GF, CON and CCON (relaxing the monotonicity axiom) is an open question worth to be investigated.

3.6 Concluding Remarks

We consider a marriage model where agents are allowed to stay selfmatched. First, we discuss how the definition of converse consistency can be adapted to this more general framework. Next, we show that the core is no more the unique solution satisfying individual rationality, Pareto optimality, anonymity, consistency, and converse consistency in this setting. We characterize the core by individual rationality, Pareto optimality, gender fairness, consistency and converse consistency. We also check the independence of these axioms.

Whether our characterization result can be extended to many-to one matching problems under suitable preference restrictions is on our agenda for further research. Finally, we consider what would happen if we further relax the condition that agents have strict preferences over their potential mates. We show that, if we allow indifferences on preferences, there exists no solution satisfying Pareto optimality, anonymity and converse consistency. Toda (2006) characterized the core (of many to one matching problems) by weak
unanimity, consistency, Maskin monotonicity and population monotonicity when indifferences are allowed. In this study, we characterized the core as the unique solution satisfying individual rationality, weak Pareto optimality, gender fairness, consistency, converse consistency and monotonicity.
4 A Maximal Conversely Consistent Subsolution of the Pareto Optimal Solution

4.1 Introduction

For many economic problems, in particular for matching problems consistency and converse consistency are two well-known axioms that are used for axiomatic analysis. In this study, we specially consider converse consistency for matching problems. We know that several solutions, including the Pareto-optimal solution in matching problems, fail to satisfy converse consistency. Thomson (1996) introduces the concept of maximal consistent subsolution of a solution in the general framework. By evaluating it, we maximally reduce the solution so that consistency is satisfied. Since consistency is preserved under unions, the union of consistent subsolutions can be described as the maximal consistent subsolution of a solution. However, union of two conversely consistent solutions may not be conversely consistent. So we cannot carry the concept of maximal consistent subsolution when we replace consistency by converse consistency. Therefore, for a solution we define the maximal converse consistent subsolution by using inclusion operator as the maximal converse consistent solution which is included in the solution. We aimed to compute maximal conversely consistent subsolution of the Pareto
optimal solution.

We study two-sided, one-to-one matching problems which is known as marriage problems in a restricted domain where there are equal numbers of agents from each side and being single is not allowed. We consider a well-known solution concept: the Pareto-optimal solution. Pareto-optimal solution in matching problems does not satisfy converse consistency. The main aim of this essay is to compute its maximal conversely consistent subsolution. Specifically, in matching problems, converse consistency is defined as follows: Consider some problem $p$ and some solution $\varphi$. Take any matching $\mu$. The requirement is the following: If the reduced matching of $\mu$ with respect to each subgroup of two matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to the subgroup of two matched pairs, then $\mu$ must be a matching recommended by $\varphi$ at the original problem $p$. That is the original definition of Sasaki and Toda (1992) where they show that the core is the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse consistency. Some versions of converse consistency are studied in matching problems. \cite{15} As far as we know, in the literature this is the first study on computing the maximal conversely consistent subsolution of a solution. However, there are a few papers

in the literature computing the minimal consistent extension of solutions\textsuperscript{16}, as well as the minimal conversely consistent extension of solutions\textsuperscript{17}.

To compute the maximal conversely consistent subsolution of the Pareto-optimal solution, we introduce the concept of serial men-ordering. For a fixed total order on the set of men it turns out that maximal converse consistent subsolution of the Pareto optimal solution is a correspondence which associates with each problem the set of all Pareto optimal matchings when the number of agents in the society larger than four, and the set of all stable matchings and the set of matchings which are chosen by serial men-ordering solution when there are four agents or less then four agents. So, maximal converse consistent subsolution of the Pareto optimal solution is not unique. For different orders on the set of men we will get different subsolutions. Also we show that this result is not true for general domains where there are unequal number of men and women and being single is allowed.

This essay proceeds as follows: Section 4.2 presents the basic notions and axioms. Section 4.3 introduces the serial men-ordering solution and gives our results. Finally, section 4.4 makes some closing remarks.

\textsuperscript{17}See Thomson (1996, 1999), and Özkal-Sanver (2012)
4.2 Model and Axioms

We consider matching problems with equal numbers of men and women. Let $M$ and $W$ be two disjoint universal sets. Let $M$ be a nonempty and finite subset of $M$. Similarly, let $W$ be a nonempty and finite subset of $W$. Let $A = M \cup W$ be the set of agents. We assume that $\#M = \#W$ where $\#$ denotes the cardinality of a set.

For each agent $i \in A$, the set of potential mates of $i$, denoted by $A(i)$, is defined as

$$A(i) = \begin{cases} W & \text{if } i \in M \\ M & \text{if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by $P_i$. Let $\mathcal{P}$ denote the set of all possible preference profiles $P = (P_i)_{i \in A}$.

Let $\succ$ be a total order on the set of men $M$. For any woman $w \in W$ we say that $P_w$ is coherent to $\succ$ if the following condition is satisfied: For all $m, m^0 \in M$, $mP_wm^0$ if and only if $m \succ m^0$.

Similarly, for a given total order $\succ$ on the set of women $W$, for any man $m \in M$, we say that $P_m$ is coherent to $\succ$ if the following condition is satisfied: For all $w, w^0 \in W$, $wP_mm^0$ if and only if $w \succ w^0$.

A matching is a function $\mu : A \rightarrow A$ such that for each $i \in A$, we have $\mu(i) \in A(i)$ and for each pair $\{j, k\} \subset A$, $\mu(j) = k$ implies $\mu(k) = j$. Agent $\mu(i)$ is the mate of agent $i$ under matching $\mu$. Throughout the essay,
we sometimes write \((m, w) \in \mu\) to denote \(\mu(m) = w\). Let \(\mathcal{M}(A)\) denote the set of all matchings for \(A\). A (matching) problem is a pair \(p = (A, P)\), where \(A\) is the set of agents and \(P\) is the preference profile. Let \(\mathbf{P}\) denote the set of all problems.

A matching \(\mu \in \mathcal{M}(A)\) is **individually rational** for \(p\) if for all \(i \in A\), \(\mu(i) P_i i\) or \(\mu(i) = i\). A pair of agents \(\{i, j\}\) blocks a matching \(\mu \in \mathcal{M}(A)\) if \(j P_i \mu(i)\) and \(i P_j \mu(j)\). A matching \(\mu \in \mathcal{M}(A)\) is **stable** for \(p\) if it is individually rational for \(p\) and there is no pair \(\{i, j\} \subset A\) blocking \(\mu\). Let \(S(p)\) denote the set of all stable matchings. Given a problem \(p = (A, P) \in \mathbf{P}\) and two matchings \(\mu, \mu' \in \mathcal{M}(A)\), \(\mu\) dominates \(\mu'\) if there exists a coalition \(K \subseteq A\) such that for all \(i \in K\), \(\mu(i) \in K\) and \(\mu(i) P_i \mu'(i)\). A matching \(\mu\) is **undominated** if there exists no matching \(\mu' \in \mathcal{M}(A)\) which dominates \(\mu\). The **core** of \(p\) is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor (1990)). Given a problem \(p = (A, P) \in \mathbf{P}\) and two matchings \(\mu, \mu' \in \mathcal{M}(A)\) with \(\mu' \neq \mu\), \(\mu\) **Pareto dominates** \(\mu'\) if for all \(i \in A\), \(\mu(i) P_i \mu'(i)\) whenever \(\mu(i) \neq \mu'(i)\).

A matching \(\mu \in \mathcal{M}(A)\) is **Pareto optimal** for \(p\) if there exists no matching \(\mu' \in \mathcal{M}(A)\) which Pareto dominates \(\mu\). Let \(PO(p)\) denote the set of all Pareto optimal matchings.

A **solution** is a correspondence \(\varphi\) that associates with each problem \(p = (A, P)\) a nonempty subset of \(\mathcal{M}(A)\). Let \(\Phi\) be the set of solutions. The
**core solution** is the correspondence $S$ that associates with each problem $p$ its set of stable matchings $S(p)$.

Given a problem $p = (A, P)$ and a subset of set of agents $A'$, a **reduced problem** of $p$ with respect to $A'$ is a problem where the preference profile $P$ is restricted to agents in $A'$. Formally; for all $p = (A, P) \in \mathcal{P}$ and all $A' \subseteq A$, $p' = (A', P|_{A'}) \in \mathcal{P}$ is the reduced problem of $p$ with respect to $A'$.

Write $\mu(N) = \{ \mu(i) : i \in N \}$. Given a matching $\mu \in \mathcal{M}(A)$ and a subset of agents $N \subseteq M \cup W$, a **reduced matching** of $\mu$ with respect to $N$ is a matching $\mu|_N : N \cup \mu(N) \rightarrow N \cup \mu(N)$ such that for all $i \in N \cup \mu(N)$, $\mu|_N(i) = \mu(i)$.

Now, we are ready to define our axioms on solutions:

The first axiom requires that at each problem the solution recommends among Pareto optimal matchings:

**Pareto optimality (PO):** For each $p \in \mathcal{P}, \varphi(p) \subseteq PO(p)$.

The second axiom requires that renaming men among men and renaming women among women do not change the result.

**Anonymity (AN):** For all $A = M \cup W$ and all $A' = M' \cup W'$ with $\#M = \#M'$ and $\#W = \#W'$, let $\pi : A \rightarrow A'$ be a bijection such that $\pi(M) = M'$ and $\pi(W) = W'$. For all $p = (A, P)$, let $P'$ be such that for all $i \in A$ and all $j, k \in A(i)$, $j P' k$ if and only if $\pi(j) P_{\pi(i)} \pi(k)$. And also for all $\mu \in \mathcal{M}(A)$ define $\pi_\mu \in \mathcal{M}(A')$ by setting for all $i \in A$, $\pi_\mu(i) = \pi(\mu(\pi^{-1}(i)))$. 

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If $\mu \in \varphi(A, P)$, then $\pi_\mu \in \varphi(A', P')$.

Now we define consistency axiom. Consider some problem $p$ and some solution $\varphi$. Take any matching $\mu$ recommended by $\varphi$ at $p$. If the reduced matching of $\mu$ with respect to each subgroup of matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to this subgroup of matched pairs, then we say that the solution $\varphi$ is consistent.

**Consistency:** For each $p = (A, P) \in \mathbf{P}$, each $\mu \in \varphi(p)$, and each $N \subseteq A$, we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$.

The next axiom is central to our analysis. Consider some problem $p$ and some solution $\varphi$. Take any matching $\mu$. The requirement is that if the reduced matching of $\mu$ with respect to each subgroup of two matched pairs is among the recommendations made by the solution $\varphi$ for the reduced problem of $p$ with respect to the subgroup of these two matched pairs, then $\mu$ must be a matching recommended by $\varphi$ at the original problem $p$.

**Converse Consistency:** For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each subset $\{i, j\} = N \subseteq A$ with $\mu(i) \neq j$, $\mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$, then $\mu \in \varphi(p)$.

Given a solution $\varphi$, the minimal converse consistent extension of $\varphi$, denoted by $MCCE_\varphi$, is defined by Thomson (1996) as follows: \footnote{Converse consistency is preserved under intersections.}
\[ MCCE_{\varphi} = \bigcap_{\psi \in \Psi} \psi \text{ where } \Psi = \{ \psi \in \Phi : \psi \supseteq \varphi, \psi \text{ is converse consistent} \} \]

Given a solution \( \varphi \) that contains at least one consistent subsolution, the maximal consistent subsolution of \( \varphi \), denoted by \( MXCS_{\varphi} \), is defined by Thomson (1996) as follows:

\[ MXCS_{\varphi} = \bigcup_{\psi \in \Psi} \psi \text{ where } \Psi = \{ \psi \in \Phi : \psi \subseteq \varphi, \psi \text{ is consistent} \} \]

By definition of consistency it is direct consequence that consistency is preserved under unions. So the above definition is well defined. However, union of two conversely consistent solutions may not be conversely consistent. In the below example we show that converse consistency is not preserved under unions.

**Example 4.1** It is a well-known result that for any problem \( p \), \( S(p) \) is conversely consistent. It is shown by Özkal-Sanver (2013) for any problem \( p \), \( MB(p) \)\(^{19}\) is conversely consistent. But \( S \cup MB \) is not conversely consistent. To see that consider the problem \( p = (M \cup W, P) \) where \( M = \{m_1, m_2, m_3, m_4\}, W = \{w_1, w_2, w_3, w_4\} \) and \( P \) as follows:

---

\(^{19}\)Men barter-proof solution \( MB \) is defined by Özkal-Sanver (2012) in the following way: A matching is men-barter-proof whenever there is no such a pair of men who benefit from switching their mates among themselves. More formally, a matching \( \mu \in \mathcal{M}(A) \) is **men-barterproof** for \( p = (A, P) \in \mathcal{P} \), if there exists no pair of men \( \{m, m'\} \subset M \) bartering at \( \mu \). Let \( MB(p) \) denote the set of the men-barterproof matchings for \( p \). The **men-barterproof solution** is the correspondence \( MB \) that associates with each problem \( p \) the set of men-barterproof matchings \( MB(p) \).
Let \( \mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\} \). Since \( (m_3, w_4) \) is a blocking pair and \( (m_1, m_2) \) barters we have \( \mu \notin (S \cup MB)(p) \). But any two pair restriction of \( \mu \) is in \( S \cup MB \). Hence \( S \cup MB \) is not conversely consistent.

Therefore we can not carry the concept of maximal consistent subsolution when the point of departure is converse consistency. For a solution \( \varphi \), we define the maximal converse consistent subsolution of \( \varphi \), denoted by \( MXCCS_\varphi \), as follows: For each \( p = (A, P) \in P \), we have \( MXCCS_\varphi(p) \subsetneq \varphi(p) \) and there is no conversely consistent solution \( \psi \) with \( MXCCS_\varphi(p) \subsetneq \psi(p) \subsetneq \varphi(p) \).

As we stated before the main aim of this essay is to compute its maximal conversely consistent subsolution. It is a well-known result that the Pareto-optimal solution fails to satisfy converse consistency. To see that consider the following example:

**Example 4.2** The Pareto-optimal solution does not satisfy converse consis-
Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let $P$ be defined as follows:

$$
\begin{array}{cccccccc}
  & P_{m_1} & P_{m_2} & P_{m_3} & P_{m_4} & P_{w_1} & P_{w_2} & P_{w_3} & P_{w_4} \\
  w_1 & w_2 & w_3 & w_4 & m_4 & m_1 & m_2 & m_3 \\
  w_2 & w_3 & w_4 & w_1 & m_3 & m_4 & m_1 & m_2 \\
  w_3 & w_4 & w_1 & w_2 & m_2 & m_3 & m_4 & m_1 \\
  w_4 & w_1 & w_2 & w_3 & m_1 & m_2 & m_3 & m_4 \\
\end{array}
$$

Let $p = (M \cup W, P)$. Consider the matching where $\mu(m_1) = w_1$, $\mu(m_2) = w_4$, $\mu(m_3) = w_2$, $\mu(m_4) = w_3$. There are six different subproblems with $\#N = 2$. Let $N_1 = \{m_1, m_2\}$, $N_2 = \{m_1, m_3\}$, $N_3 = \{m_4, m_2\}$, $N_4 = \{m_2, m_3\}$, $N_5 = \{m_2, m_4\}$, $N_6 = \{m_3, m_4\}$. One can easily check that $\mu_{|N_i} \in PO(N_i \cup \mu(N_i), P_{N_i \cup \mu^*(N_i)})$ for all $i = 1, 2, ..., 6$. However, $\mu \notin PO(p)$.\(^{20}\)

4.3 Maximal Conversely Consistent Subsolution of the Pareto-Optimal Solution

The maximal conversely consistent subsolution of the Pareto optimal solution whenever it is anonymous is a rule which associates with each problem the set of all Pareto optimal matchings when there are more than four agents and

\(^{20}\mu^* = \{(m_1, w_1), (m_2, w_3), (m_3, w_4), (m_4, w_2)\}$ Pareto dominates $\mu$. 

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all stable matchings otherwise. We state this in the following proposition

**Proposition 4.1** The maximal conversely consistent and anonymous sub-solution of the Pareto optimal solution is

\[ MXCCSP_O(M \cup W, P) = \begin{cases} \mathcal{S}(M \cup W, P) & \text{if } \#M = \#W = 2 \\ \mathcal{P}_O(M \cup W, P) & \text{otherwise} \end{cases} \]

**Proof.** Sasaki and Toda (1992), Lemma 2 states that if a solution \( \varphi \) satisfies PO, CCON and AN then \( \varphi(M, W, P) \subseteq \mathcal{S}(M \cup W, P) \) for all \((M, W, P)\) with \( \#M = \#W = 2 \)

Before to proceed we define the *serial men ordering solution* based on the names (or indexes) of men in the society. Let \( p \equiv (M \cup W, P) \) be a problem and \( \succ \) be a fixed order on the set of men \( M \). *Serial men ordering solution* \( \text{SMO}_\succ \) with respect to \( \succ \) is defined in the following way: Let \( \mu \in \mathcal{M}(M \cup W) \) be any matching for this society and \( \succ_\mu \) be an ordering on the set of women \( W \) defined as \( w_i \succ_\mu w_j \) if and only if \( \mu(w_j) \succ \mu(w_i) \). Let \( m \in M \) be any agent in \( M \) we write \( t(P_m; W) \) for the top-ranked agent in \( W \) with respect to the preference profile \( P_m \) of agent \( m \), hence \( t(P_m; W)P_mw \) for all \( w \in W \). Similarly, \( t(P_m; W) \) is defined for any \( w \in W \). Then we look at the top-ranked agents and define \( s_w \), as the cardinality of of the agents who ranked them as the best. That is \( s_w = \# \{m : t(P_m; W) = w \} \) for any \( w \in W \). Similarly, \( s_m \) is defined for any \( m \in M \). Let \( \arg \max_{i \in M \cup W} s_i \) = \{ \( i \in M \cup W \) | for all \( i \in M \cup W \), \( s_j \leq s_i \} \). We consider any agent \( i^* \in \arg \max_{i \in M \cup W} s_i \) if \( P_{i^*} \) is coherent to order \( \succ \)
or if $P_t^*$ is coherent to order $\succ_\mu$ then $\mu \in SMO_\succ$.  

**Remark 4.1** $SMO_\succ$ does not satisfy anonymity. To see that consider the problem $p = (A, P)$ where $A = \{m_1, m_2, w_1, w_2\}$ and $P$ as follows:

$$
\begin{array}{ccc}
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_1 & w_1 & m_1 & m_1 \\
w_2 & w_2 & m_2 & m_2 \\
m_1 & m_2 & w_1 & w_2 \\
\end{array}
$$

Assume that $m_1 \succ m_2$. Then $SMO_\succ(p) = \{\mu_1, \mu_2\}$ where $\mu_1(m_1) = w_1$, $\mu_1(m_2) = w_2$, $\mu_2(m_1) = w_2$ and $\mu_2(m_2) = w_1$. Let $\pi(m_1) = m_2$, $\pi(m_2) = m_1$, $\pi(w_1) = w_1$ and $\pi(w_2) = w_2$. Let $P'$ be the preference profile in the permuted problem. Then $P'$ is the following preference profile:

$$
\begin{array}{ccc}
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_1 & w_1 & m_2 & m_2 \\
w_2 & w_2 & m_1 & m_1 \\
m_1 & m_2 & w_1 & w_2 \\
\end{array}
$$

Note that $\pi_{\mu_2} = \{(m_1, w_1), (m_2, w_2)\}$. Since $m_1 \succ m_2$ we have $w_2 \succ_{\pi_{\mu_2}} w_1$. Note that $t(P_{m_1}; W) = w_1$, $t(P_{m_2}; W) = w_1$, $t(P_{w_1}; W) = m_2$ and $t(P_{w_2}; W) = m_1$.

\[21\text{Note that } SMO_\succ \text{ is not well defined solution. This solution is useful and meaningful for the societies where there are two men and two women.}\]
\( t(P_{w_2}, W) = m_2 \) then \( \arg \max_{i \in M \cup W} x_i = \{w_1, m_2\} \) but \( P_{w_1} \) is not coherent to \( \succ \) and \( P_{m_2} \) is not coherent to \( \succ_{\mu_2} \). Hence, \( \pi_{\mu_2} \notin SMO(\succ, A) \).

**Theorem 4.1** A maximal conversely consistent subsolution of the Pareto optimal solution for a fixed order \( \succ \) on \( M \) is

\[
MXCCS_{PO}(M \cup W, P) = \begin{cases} 
S(M \cup W, P) \cup SMO_{\succ} (M \cup W, P) & \text{if } \#M = \#W = 2 \\
PO(M \cup W, P) & \text{otherwise}
\end{cases}
\]

**Proof.** First we will show that this solution is converse consistent. Assume \( MXCCS_{PO} \) is not converse consistent. Then there is a problem \( p = (M \cup W, P) \) and a matching \( \mu \in M(M \cup W) \) such that \( \mu|_N \in MXCCS_{PO}(N \cup \mu(N), P|_{N \cup \mu(N)}) \) for each \( N = \{i, j\} \subset M \cup W \) with \( \mu(i) \neq j \) but \( \mu \notin MXCCS_{PO}(p) \), which means \( \mu \notin PO(M \cup W, P) \). Then there exists \( \mu' \in M(A) \) Pareto dominating \( \mu \). Let \( \succ \) be a fixed order on the set of men \( M \). There exists at least an agent \( i \in A \) such that \( \mu'(i) \succ_i \mu(i) \). Let \( m \) be the first agent with respect to \( \succ \) who is strictly better off. Let \( \mu(m) = w \) and \( \mu'(w) = m' \). Let \( \mu(m') = w' \). Let \( N = \{m, m'\} \). Then \( P|_{N \cup \mu(N)} \) is the following preference profile

<table>
<thead>
<tr>
<th>( P_m )</th>
<th>( P_{m'} )</th>
<th>( P_w )</th>
<th>( P_{w'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>( m' )</td>
<td>( w' )</td>
<td>( m )</td>
</tr>
</tbody>
</table>

Then there are four possibilities for the other agents:
First note that $\mu_{|N} \notin S(N \cup \mu(N), P_{|N \cup \mu(N)})$. Also, $\mu_{|N} \notin SMO_\succ(N \cup \mu(N), P_{|N \cup \mu(N)})$. Since $m \succ m'$ then we have $w \succ w$. Note that $t(P_m; W) = w$ and $t(P_{m'}; W) = w$, $t(P_w; W) = m'$ and $t(P_{w'}; W) = m'$, then $\arg \max s_i = \{w, m'\}$ but $P_w$ is not coherent to $\succ$ and $P_{m'}$ is not coherent to $\succ$. Hence $\mu_{|N} \notin MXCCS_{\succ}(N \cup \mu(N), P_{|N \cup \mu(N)})$.

<table>
<thead>
<tr>
<th>$P_m$</th>
<th>$P_{m'}$</th>
<th>$P_w$</th>
<th>$P_{w'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$w$</td>
<td>$m'$</td>
<td>$m'$</td>
</tr>
<tr>
<td>$w'$</td>
<td>$w'$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
</tbody>
</table>

First note that $\mu_{|N} \notin S(N \cup \mu(N), P_{|N \cup \mu(N)})$. Also, $\mu_{|N} \notin SMO_\succ(N \cup \mu(N), P_{|N \cup \mu(N)})$. Since $m \succ m'$ then we have $w \succ w$. Note that $t(P_m; W) = w$ and $t(P_{m'}; W) = w$, then $\arg \max s_i = \{w\}$ but $P_w$ is not coherent to $\succ$. Hence $\mu_{|N} \notin MXCCS_{\succ}(N \cup \mu(N), P_{|N \cup \mu(N)})$. 

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First note that \( \mu|_{N} \notin S(N \cup \mu(N), P|_{N \cup \mu(N)}) \). Also, \( \mu|_{N} \notin SMO_{\succ}(N \cup \mu(N), P|_{N \cup \mu(N)}) \). Since \( m \succ m' \) then we have \( w' \succ_{\mu} w \). Note that \( t(P_{w}; W) = m' \) and \( t(P_{w'}; W) = m' \), then \( \arg \max_{i \in N \cup \mu(N)} \) \( s_i = \{m'\} \) but \( P_{m'} \) is not coherent to \( \succ_{\mu} \). Hence \( \mu|_{N} \notin MXCCSP_{\mathcal{O}}(N \cup \mu(N), P|_{N \cup \mu(N)}) \).

In this case \( \mu|_{N} \notin \mathcal{O}(N \cup \mu(N), P|_{N \cup \mu(N)}) \).

Now, we have to show \( MXCCSP_{\mathcal{O}} \) is maximal. Assume there is a conversely consistent solution \( \psi \) such that \( MXCCSP_{\mathcal{O}} \subset \psi \subset \mathcal{O} \). Then there is a problem \( p = (M \cup W, P) \) with \( \#M = \#W = 2 \) and a Pareto optimal matching \( \mu \in \psi(p) \) but \( \mu \notin MXCCSP_{\mathcal{O}}(p) \). Let \( M = \{m_{1}, m_{2}\} \) and \( W = \{w_{1}, w_{2}\} \). And assume that \( m_{1} \succ m_{2} \). Then we have one of the six
cases of the following preference profiles:\footnote{For the other ten cases $MXCCS_{PO}$ chooses all the Pareto optimal matchings.}

\begin{center}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_1 & w_1 & m_2 & m_1 \\
\hline
w_2 & w_2 & m_1 & m_2 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_1 & w_1 & m_2 & m_2 \\
\hline
w_2 & w_2 & m_1 & m_1 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\end{center}

\begin{center}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_2 & w_1 & m_2 & m_2 \\
\hline
w_1 & w_2 & m_1 & m_1 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_2 & w_2 & m_1 & m_2 \\
\hline
w_1 & w_1 & m_2 & m_1 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\end{center}

\begin{center}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_1 & w_2 & m_2 & m_2 \\
\hline
w_2 & w_1 & m_1 & m_1 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\begin{array}{|cc|cc|}
\hline
P_{m_1} & P_{m_2} & P_{w_1} & P_{w_2} \\
\hline
w_2 & w_2 & m_2 & m_2 \\
\hline
w_1 & w_1 & m_1 & m_1 \\
\hline
m_1 & m_2 & w_1 & w_2 \\
\hline
\end{array}
\end{center}

Assume we are in the first case. For this preference profile $\text{MXCCS}_{PO}(p) = \{(m_1, w_2), (m_2, w_1)\}$ then we have $\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)$. We introduce four new agents $m_3,m_4,w_3$ and $w_4$ to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3,m_4,w_3,w_4\}$ of agents in
the following way:

\[
\begin{array}{cccc}
<table>
<thead>
<tr>
<th>P_{m_1}^*</th>
<th>P_{m_2}^*</th>
<th>P_{m_3}^*</th>
<th>P_{m_4}^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>w_3</td>
<td>w_1</td>
<td>w_4</td>
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<tr>
<td>w_1</td>
<td>w_4</td>
<td>w_3</td>
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<tr>
<td>w_2</td>
<td>w_2</td>
<td>w_1</td>
<td>w_1</td>
</tr>
<tr>
<td>w_4</td>
<td>w_3</td>
<td>w_2</td>
<td>w_3</td>
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</tbody>
</table>
\end{array}
\]

\[
\begin{array}{cccc}
<table>
<thead>
<tr>
<th>P_{w_1}^*</th>
<th>P_{w_2}^*</th>
<th>P_{w_3}^*</th>
<th>P_{w_4}^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>m_2</td>
<td>m_4</td>
<td>m_1</td>
<td>m_3</td>
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<tr>
<td>m_3</td>
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<td>m_1</td>
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<tr>
<td>m_4</td>
<td>m_3</td>
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</tbody>
</table>
\end{array}
\]

Let \( \mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\} \). Since \( \mu^* \) is Pareto dominated by \( \{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\} \), we have \( \mu^* \notin \psi(M^*, W^*, P^*) \). However, \( \mu^{*}_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}) \) for each \( N' = \{i, j\} \subset M^* \cup W^* \).

To see that, consider the following subproblems: Letting \( N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\} \), we have \( P_{|N'\cup\mu^*(N')}}^* = P_{|N'\cup\mu(N')}^* \) and \( \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}} \).

Next, letting \( N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\} \), note that \( P_{|N'\cup\mu^*(N')}}^* \) is
Since $m_1 > m_3$ then we have $w_3 \succ w_1$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg\max s_i = \{w_3\}$ and $P_{w_3}$ is coherent to $\succ$. Then $\mu^*_{|N'} \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu^*_{|N'} \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

<table>
<thead>
<tr>
<th>$P^*_{m_1}$</th>
<th>$P^*_{m_4}$</th>
<th>$P^*_{w_1}$</th>
<th>$P^*_{w_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$w_4$</td>
<td>$m_1$</td>
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<td>$m_4$</td>
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<td>$m_1$</td>
<td>$m_4$</td>
<td>$w_1$</td>
<td>$w_4$</td>
</tr>
</tbody>
</table>

Then $PO(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

<table>
<thead>
<tr>
<th>$P^*_{m_2}$</th>
<th>$P^*_{m_3}$</th>
<th>$P^*_{w_2}$</th>
<th>$P^*_{w_3}$</th>
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</thead>
<tbody>
<tr>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$m_2$</td>
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<td>$w_3$</td>
<td>$w_2$</td>
<td>$m_3$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$w_2$</td>
<td>$w_3$</td>
</tr>
</tbody>
</table>

Then $PO(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is
\[
\begin{array}{cccc}
P_{m_2}^* & P_{m_4}^* & P_{w_2}^* & P_{w_4}^* \\
w_4 & w_2 & m_4 & m_4 \\
w_2 & w_4 & m_2 & m_2 \\
m_2 & m_4 & w_2 & w_4 \\
\end{array}
\]

Since \(m_4 \succ m_2\) then we have \(w_2 \succ_{\mu^*} w_4\). Note that \(t(P_{w_2}; M) = m_4\) and \(t(P_{w_4}; M) = m_4\), then \(\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_4\}\) and \(P_{m_4}\) is coherent to \(\succ_{\mu^*}\) . Then \(\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}\). Hence \(\mu_{|N'}^* \in MXCCS_P(\psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}\).

Finally, letting \(N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}\), note that \(P_{|N' \cup \mu^*(N')}^*\) is

\[
\begin{array}{cccc}
P_{m_3}^* & P_{m_4}^* & P_{w_3}^* & P_{w_4}^* \\
w_4 & w_4 & m_3 & m_3 \\
w_3 & w_3 & m_4 & m_4 \\
m_3 & m_4 & w_3 & w_4 \\
\end{array}
\]

Since \(m_3 \succ m_4\) then we have \(w_4 \succ_{\mu^*} w_3\). Note that \(t(P_{m_3}; W) = w_4\) and \(t(P_{m_4}; W) = w_4, t(P_{w_3}; M) = m_3\) and \(t(P_{w_4}; M) = m_3\), then \(\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_3, w_4\}\) and \(P_{w_4}\) is coherent to \(\succ\) and \(P_{m_3}\) is coherent to \(\succ_{\mu^*}\) . Then \(\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}\). Hence \(\mu_{|N'}^* \in MXCCS_P(\psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}\).

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Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Similar arguments work for the other five cases. 23

Remark 4.2 This solution defined in the above theorem does not satisfy converse consistency when some agents are allowed to be single. Assume to get a contradiction, in this more general domain $MXCCS_{PO}$ satisfies converse consistency. Then consider the following problem $p = (M \cup W, P)$ where $M \cup W = \{m_1, m_2, m_3, w_1, w_2, w_3\}$ and $P$ as follows:

<table>
<thead>
<tr>
<th>$P_w^1$</th>
<th>$P_w^2$</th>
<th>$P_w^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_2$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$w_1$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$m_2$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$m_1$</td>
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<td>$m_2$</td>
<td>$m_3$</td>
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</tr>
<tr>
<td>$w_1$</td>
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</tr>
<tr>
<td>$w_2$</td>
<td>$m_3$</td>
<td>$w_3$</td>
</tr>
</tbody>
</table>

with an order $m_1 \succ m_2 \succ m_3$.

Let $\mu^* = \{(m_1), (m_2, w_1), (m_3, w_2), (w_3)\}$. Since $\mu^*$ is Pareto dominated by $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$, we have $\mu^* \notin MXCCS_{PO}(M, W, P)$. However, $\mu^*_{|N'} \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ for each $N' = \{i, j\} \subset M \cup W$.

23For details, see the appendix II.
To see that, consider the following subproblems: Letting \( N' \cup \mu^*(N') = \{m_1, m_2, w_1\} \), note that \( P_{|N' \cup \mu^*(N')|} \) is

\[
\begin{array}{ccc}
P_{m_1} & P_{m_2} & P_{w_1} \\
w_1 & w_1 & m_1 \\
m_1 & m_2 & m_2 \\
w_1 & \\
\end{array}
\]

Note that \( t(P_{m_1}; W) = w_1 \) and \( t(P_{m_2}; W) = w_1 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\} \) and \( P_{w_1} \) is coherent to \( \succ \). Then \( \mu^*_{|N'|} \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')|}) \).

Hence \( \mu^*_{|N'|} \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')|}). \)

Next letting \( N' \cup \mu^*(N') = \{m_1, m_3, w_2\} \), note that \( P_{|N' \cup \mu^*(N')|} \) is

\[
\begin{array}{ccc}
P_{m_1} & P_{m_3} & P_{w_2} \\
w_2 & w_2 & m_1 \\
m_1 & m_3 & m_3 \\
w_2 & \\
\end{array}
\]

Note that \( t(P_{m_1}; W) = w_2 \) and \( t(P_{m_3}; W) = w_2 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\} \) and \( P_{w_2} \) is coherent to \( \succ \). Then \( \mu^*_{|N'|} \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')|}) \).

Hence \( \mu^*_{|N'|} \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')|}). \)

Next letting \( N' \cup \mu^*(N') = \{m_1, w_3\} \), note that \( P_{|N' \cup \mu^*(N')|} \) is
Then $PO(N' \cup \mu^*(N'), P_{[N' \cup \mu^*(N')]}) = \mu^*_N \in MXCCSP_O(N' \cup \mu^*(N'), P_{[N' \cup \mu^*(N')]})$.

Next letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_2\}$, note that $P_{[N' \cup \mu^*(N')]}$ is

\[
\begin{array}{cccc}
  P_{m_1} & P_{w_3} & P_{m_2} & P_{m_3} & P_{w_1} & P_{w_2} \\
  m_1 & w_3 & w_2 & w_2 & m_3 & m_2 \\
  w_3 & m_1 & w_1 & m_2 & m_2 & m_3 \\
  w_2 & m_3 & m_3 & m_2 & m_2 & w_2 \\
\end{array}
\]

Note that $t(P_{m_2}; W) = w_2$ and $t(P_{m_3}; W) = w_2$, then $\arg \max s_i = \{w_2\}$ and $P_{w_2}$ is coherent to $\succ$. Then $\mu^*_N \in SMO_{\succ}(N' \cup \mu^*(N'), P_{[N' \cup \mu^*(N')]})$.

Hence $\mu^*_N \in MXCCSP_O(N' \cup \mu^*(N'), P_{[N' \cup \mu^*(N')]})$.

Next letting $N' \cup \mu^*(N') = \{m_2, w_1, w_3\}$, note that $P_{[N' \cup \mu^*(N')]}$ is

\[
\begin{array}{ccc}
  P_{w_3} & P_{w_1} & P_{w_2} \\
  m_2 & m_2 & m_2 \\
  w_1 & w_1 & w_3 \\
  m_2 & m_2 & w_3 \\
\end{array}
\]
Note that \( t(P_{w_1}; M) = m_2 \) and \( t(P_{w_3}; M) = m_2 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_2\} \) and \( P_{m_2} \) is coherent to \( \succ_{\mu^*} \) since \( w_3 \succ_{\mu^*} w_1 \). Then \( \mu^*_W \in SMO_x(N' \cup \mu^*(N'), P_{N' \cup \mu^*(N')}) \).

Finally, letting \( N' \cup \mu^*(N') = \{m_2, w_2, w_3\} \), note that \( P_{N' \cup \mu^*(N')} \) is

\[
\begin{array}{ccc}
P_{m_3} & P_{w_2} & P_{w_3} \\
\hline
w_3 & m_3 & m_3 \\
w_2 & w_1 & w_3 \\
m_2 & & \\
\end{array}
\]

Note that \( t(P_{w_2}; M) = m_3 \) and \( t(P_{w_3}; M) = m_3 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{m_3\} \) and \( P_{m_3} \) is coherent to \( \succ_{\mu^*} \) since \( w_3 \succ_{\mu^*} w_2 \). Then \( \mu^*_W \in SMO_x(N' \cup \mu^*(N'), P_{N' \cup \mu^*(N')}) \). Hence \( \mu^*_W \in MXCCSP_D(N' \cup \mu^*(N'), P_{N' \cup \mu^*(N')}) \).

Hence, by CCON we have \( \mu^* \in MXCCSP_D(M, W, P) \), which gives a contradiction.

### 4.4 Concluding Remarks

In this essay, we study two sided one-to-one matching problems where there are equal numbers of agents from each sides. We compute a maximal conversely consistent subsolution of the Pareto optimal solution as a correspondence which associates with each problem the set consisting of all the Pareto optimal matchings if there are at least three men and three women, all stable
matchings and pareto optimal matchings which are chosen by serial men-
ordering solution for a fixed order on the set of men. if there are two men
and two women Therefore, it turns out that maximal conversely consistent
subsolution of Pareto optimal rule is not unique. We also show that failure
of this result on general domains.
5 Consistent Enlargements of the Core in Roommate Problems

Duygu Nizamoğulları and İpek Özkal-Sanver

5.1 Introduction

The consistency axiom is introduced to the matching theory literature by Sasaki and Toda (1992) who characterize the core of marriage (two sided, one-to-one matching) problems as the unique correspondence which satisfies Pareto optimality, anonymity, consistency and converse consistency. Their analysis is followed by Sasaki (1995) and Toda (2005) who give two different characterizations of the core of assignment problems with indivisible goods and money, each of which uses consistency as one of the axioms. In a similar way, Toda (2006) uses consistency in characterizing the core of many-to-one matching problems. All of these characterizations are in environments which ensure the nonemptiness of the core. We consider roommate (one-sided matching) problems where, as Gale and Shapley (1962) show, the core may be empty. There is a huge literature considering core extensions which are solutions coinciding with the core whenever it is non-empty. Given the critical role of consistency in characterizing the core of various types of match-
ing problems, Özkal-Sanver (2010) aimed at characterizing the class of consistent core extensions. Nevertheless, she ended up with an impossibility: There exists no refinement of a core extension which is consistent, a fortiori no consistent core extension. This result covers several solution concepts of the literature. For instance, absorbing matchings (Inarra et al., 2008a), maximum stable matchings (Tan 1990), almost stable matchings (Abraham et al. 2006), P-Stable matchings (Inarra et al., 2008b), stochastically stable matchings (Klaus et al., 2008) are all core extensions. On the other hand, a solution which picks the median stable matchings (Serhuraman and Teo 2001, Klaus and Klijn 2009) whenever the core is nonempty, is not a core extension, but a pseudo refinement of the core.

Thomson (1990) and Thomson (1996) introduce the concept of minimal consistent extension of well-defined solution in the general framework. Our aim is to compute consistent enlargements of the core. By computing it, we evaluate the extent to which the core would have to be expanded in order to be well-defined and consistent. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. We also show that for any fixed order on the set of agents the solution which picks all stable matchings and the serial dictatorship matching with respect to this order is a minimal consistent enlargement of the core. Since for different orders there may be different
enlargements, minimal consistent core enlargement is not unique.

Section 5.2 presents the model. Section 5.3 introduces the axioms we consider. Section 5.4 states our results.

5.2 Model

We consider roommate problems. We write $A$ for the universal set of agents. We assume $A$ to be countably infinite. Let $A$ be a nonempty and finite subset of $A$. For each agent $i \in A$ the set of potential mates of $i$ equals $A$. The preference of $i \in A$ over $A$ is a complete, transitive and antisymmetric binary relation $P_i$. Let $P$ denote the set of all possible preference profiles $P \equiv (P_i)_{i \in A}$.

A matching for a set of agents $A$ is a function $\mu : A \rightarrow A$ such that for all $j, k \in A$, $\mu(j) = k$ implies $\mu(k) = j$. Here, $\mu(i)$ is the mate of agent $i$ at $\mu$. We say that agent $i$ is self-matched at $\mu$ if $\mu(i) = i$. Let $M(A)$ denote the set of all matchings for $A$.

A (roommate) problem is a pair $p \equiv (A, P)$, where $A$ is an arbitrary set of agents and $P$ is the profile of their preferences over potential mates in $A$. Let $P$ denote the set of all problems.

A matching $\mu \in M(A)$ is individually rational for $p$ if and only if for all $i \in A$, $\mu(i) P_i i$ or $\mu(i) = i$. Let $I(p)$ denote the set of all such matchings.
A pair of agents \((i, j)\) with \(i \neq j\), blocks a matching \(\mu \in \mathcal{M}(A)\) if and only if \(j \ P_i \mu(i) \) and \(i \ P_j \mu(j)\). A matching \(\mu \in \mathcal{M}(A)\) is stable for \(p\) if and only if it is individually rational for \(p\) and there is no pair \((i, j)\) blocking \(\mu\). Let \(S(p)\) denote the set of all such matchings. A matching \(\mu \in \mathcal{M}(A)\) is in the core of \(p\) if there is no \(A' \subseteq A\) such that \(\bigcup_{i \in A'} \{\mu(i)\} = A'\) and \(\mu' \ P_i \mu\) for all \(i \in A'\). The core of \(p\) is the set \(\mathcal{CO}(p)\) of all matchings which are in the core of \(p\). As in marriage model, the core of \(p\) equals the set of stable matchings at \(p\). As Gale and Shapley (1962) show the core may be empty.

A matching \(\mu \in \mathcal{M}(A)\) Pareto dominates a matching \(\mu' \in \mathcal{M}(A)\) if and only if for all \(i \in A\), \(\mu(i) \ R_i \mu'(i)\) and for some \(j \in A\), \(\mu(j) \ P_j \mu'(j)\). A matching \(\mu \in \mathcal{M}(A)\) is Pareto optimal for \(p\) if and only if there is no \(\mu' \in \mathcal{M}(A)\) Pareto dominating \(\mu\). Let \(\mathcal{PO}(p)\) denote the set of all such matchings.

A solution is a correspondence \(\varphi\) that associates with each problem \(p \equiv (A, P) \in \mathcal{P}\) a subset of \(\mathcal{M}(A)\).\(^{24}\) A well-defined solution is a correspondence \(\varphi\) that associates with each problem \(p \equiv (A, P) \in \mathcal{P}\) a non-empty subset of \(\mathcal{M}(A)\). A core extension is a solution \(\varphi\) such that for any \(p \in \mathcal{P}\) with \(S(p) \neq \emptyset\), \(\varphi(p) = S(p)\).

Now, we define axioms for solutions:

First, at each problem the solution only recommends Pareto optimal matchings:

---

\(^{24}\)For some problems \(p \equiv (A, P) \in \mathcal{P}\), \(\varphi(p)\) can be empty set.
Pareto Optimality [PO]: For all \( p \in \mathbf{P} \), \( \varphi(p) \subseteq \mathcal{PO}(p) \).

To introduce the other axioms, we need some further definitions. Given a problem \( p \equiv (A, P) \) and a subset of agents \( A' \) of the original set of agents \( A \), for each agent \( i \in A' \), define a preference relation over \( A' \), denoted by \( P'_i \), such that for all \( i, j, k \in A' \), \( j P'_i k \) if and only if \( j P_i k \). Then, set \( P|_{A'} \equiv (P'_i)_{i \in A'} \). For all \( p \equiv (A, P) \in \mathbf{P} \) and all proper subsets \( A' \subset A \), \( p' \equiv (A', P|_{A'}) \in \mathbf{P} \) is the reduced problem of \( p \) with respect to \( A' \). A matched pair under \( \mu \) is simply \((i, \mu(i))\), and \( \mu(i) \) may be equal to \( i \). Writing \( \mu(A') = \{ \mu(i) : i \in A' \} \), next define \( \mu|_{A'} \) by \( \mu|_{A'} : A' \cup \mu(A') \rightarrow A' \cup \mu(A') \) such that \( \mu|_{A'}(i) = \mu(i) \) for all \( i \in A' \cup \mu(A') \).

Consider some problem \( p \) and some solution \( \varphi \). Let \( \mu \) be a matching recommended by \( \varphi \) at \( p \). We require that the restriction of \( \mu \) to each subgroup of matched pairs is among the recommendations made by the solution \( \varphi \) for the reduced problem of \( p \) with respect to this subgroup:

**Consistency:** For all \( p \equiv (A, P) \in \mathbf{P} \) with \( \varphi(p) \neq \emptyset \) and all \( \mu \in \varphi(p) \), \( \mu|_{A'} \in \varphi(A' \cup \mu(A')) \), \( P|_{A' \cup \mu(A')} \) for all proper subsets \( A' \subset A \).

**Weak Consistency:** For all \( p \equiv (A, P) \in \mathbf{P} \) with \( \varphi(p) \neq \emptyset \) and all \( \mu \in \varphi(p) \), then either \( \varphi(A' \cup \mu(A')) \), \( P|_{A' \cup \mu(A')} \) = \( \emptyset \) or \( \mu|_{A'} \in \varphi(A' \cup \mu(A')) \), \( P|_{A' \cup \mu(A')} \) for all proper subsets \( A' \subset A \).

\( ^{25} \)Note that, \( (A' \cup \mu(A'), P|_{A' \cup \mu(A')}) \) is simply the reduced problem of \( p \) with respect to \( A' \cup \mu(A') \).

\( ^{26} \)Note that, consistency implies weak consistency. But converse is not true. To see that
Now we define consistent enlargements and minimal consistent enlargements of a solution.

Given a solution $\varphi$, its **enlargement** $E_{\varphi}$ is a well-defined solution such that for any $p \in P$ we have $\varphi(p) \subseteq E_{\varphi}(p)$. Its **consistent enlargement** $CE_{\varphi}$ is an enlargement which satisfies consistency. Its **minimal consistent enlargement** $MCE_{\varphi}$ is an enlargement such that there is no consistent enlargement $\psi$ of $\varphi$ with $\varphi(p) \subseteq \psi(p) \subseteq MCE_{\varphi}(p)$ for any $p \in P$.

In particular, a **consistent core enlargement** $CE_S$ is a well-defined consistent solution which picks at leasts for some problem more than the core whenever it is non-empty and a **minimal consistent core enlargement** $MCE_S$ is a minimal consistent well-defined solution with this property.

### 5.3 Results

We define the sub operation $\lor$ on the set of solutions as a solution $\varphi_1 \lor \varphi_2$ that satisfies the following: If $\mu \in (\varphi_1 \lor \varphi_2)$ then either $\mu \in \varphi_1(p)$ or $\mu \in \varphi_2(p)$ for any problem $p = (A, P) \in P$. The sub operator $\lor$ is closed for consistent solutions. We state this result in the following proposition.

**Proposition 5.1** For any two consistent solutions $\varphi_1$ and $\varphi_2$, the solution

\[
\varphi(A, P) = \begin{cases} 
PO(A, P) & \text{if } |A| > 4 \\
\emptyset & \text{otherwise.}
\end{cases}
\]

$\varphi$ satisfies weak consistency but it is not consistent.
\( \varphi_1 \lor \varphi_2 \) is consistent.

**Proof.** Let \( \varphi_1 \) and \( \varphi_2 \) be two solutions that satisfy consistency and let \( p = (A, P) \) be a problem with \((\varphi_1 \lor \varphi_2)(A, P) \neq \emptyset\). Take any \( \mu \in (\varphi_1 \lor \varphi_2)(A, P) \).

Then either \( \mu \in \varphi_1(A, P) \) or \( \mu \in \varphi_2(A, P) \).

If \( \mu \in \varphi_1(A, P) \) then by consistency of \( \varphi_i \) we have \( \mu_{|A} \in \varphi_i(A' \cup \mu(A'), P_{A' \cup \mu(A')}) \) for all proper subsets \( A' \subset A \) for any \( i \in \{1, 2\} \).

Hence, \( \mu_{|A'} \in (\varphi_1 \lor \varphi_2)(A' \cup \mu(A'), P_{A' \cup \mu(A')}) \) for all proper subsets \( A' \subset A \), which means \( \varphi_1 \lor \varphi_2 \) is a consistent solution. ■

**Remark 5.1** For any two weakly consistent solutions \( \varphi_1 \) and \( \varphi_2 \), the solution \( \varphi_1 \lor \varphi_2 \) may not be weakly consistent. To show this, in example 5.1 we consider the following weakly consistent (indeed, well-defined and consistent) solution: the serial dictatorship solution based on the names (or indexes) of agents.

We define serial dictatorship solution as follows: For any problem \( p = (A, P) \) let the agent with the lowest index in \( A \) match with his best agent in \( A \), if any. Remove the agents from the set \( A \) and let the agent with the lowest index in the remaining set match with his best agent in this set etc.

More formally, fix a total order \( \succ \) of agents in \( A \). Take any arbitrary finite set \( A = \{a_1, \ldots, a_{|A|}\} \) where, without loss of generality, \( a_k \succ a_l \) holds for all \( k, l \in \{1, \ldots, |A|\} \) with \( k < l \). For all \( p = (A, P) \), for all non-empty \( A' \subset A \), and for all \( a_n \in A \), we write \( t(P_{a_n}; A') \) for the top-ranked agent.
in \( A' \) with respect to the preference \( P_{an} \) of agent \( a_n \), hence \( t(P_{an}; A') R_{an} \) 
\( a_l \) for all \( a_l \in A' \). The **serial dictatorship matching** \( \mu^\rightharpoonup \) (with respect to \( \succ \)) for \( p \equiv (A, P) \) is recursively defined as follows: \( \mu^\rightharpoonup(a_1) = t(P_{a_1}; A) \). For any \( n \in \{2, \ldots, |A|\} \), write \( A_n = A \setminus \bigcup_{k=1}^{k=n-1} \{a_k, \mu^\rightharpoonup(a_k)\} \). Note that either \( a_n \not\in A_n \), hence \( a_n = \mu^\rightharpoonup(a_k) \) for some \( k < n \) or \( a_n \in A_n \), in which case we set \( \mu^\rightharpoonup(a_n) = t_n(P_{a_n}; A_n) \). Given a fixed order \( \succ \) on \( A \), the rule which assigns to each problem its serial dictatorship matching is called the **serial dictatorship solution** (with respect to \( \succ \)), and denoted by \( SD_{\succ} \).

**Example 5.1** Let \( \varphi_1 \) and \( \varphi_2 \) be two solutions defined as follows:

\[
\varphi_1(A, P) = \begin{cases} 
\PO(A, P) & \text{if } |A| > 4 \\
\emptyset & \text{otherwise.}
\end{cases}
\]

\[
\varphi_2(A, P) = \bigcup_{\succ} SD_{\succ}(A, P)
\]

Consider the problem \( p = (A, P) \), where \( A = \{a, b, c, d, e\} \) and \( P \) as follows:

<table>
<thead>
<tr>
<th>( P_a )</th>
<th>( P_b )</th>
<th>( P_c )</th>
<th>( P_d )</th>
<th>( P_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>( c )</td>
<td>( b )</td>
<td>( b )</td>
<td>( e )</td>
</tr>
<tr>
<td>( b )</td>
<td>( a )</td>
<td>( d )</td>
<td>( c )</td>
<td>( d )</td>
</tr>
<tr>
<td>( d )</td>
<td>( d )</td>
<td>( a )</td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td>( d )</td>
<td>( c )</td>
</tr>
<tr>
<td>( e )</td>
<td>( e )</td>
<td>( e )</td>
<td>( e )</td>
<td>( a )</td>
</tr>
</tbody>
</table>
Let \( \mu = \{(a, b), (c, d), (e)\} \). Clearly \( \mu \in (\varphi_1 \lor \varphi_2)(A, P) \). Then consider the subproblem \( p' = (A' \cup \mu(A'), P_{|A'\cup\mu(A')}) \) where \( A' = \{a, c\} \). Since \( \mu_{|A'} \notin SD_\varphi \) for any order \( \varphi \) we have \( \mu_{|A'} \notin (\varphi_1 \lor \varphi_2)(p') \). Hence \( \varphi_1 \lor \varphi_2 \) does not satisfy weak consistency.

**Proposition 5.2** Let \( \varphi_1 \) be a consistent solution and let \( \varphi = \varphi_1 \lor \varphi_2 \). The solution \( \varphi \) is a consistent enlargement of \( \varphi_1 \) if and only if \( \varphi_2 \) satisfies the followings:

**i)** For any problem \( p = (A, P) \) if \( \varphi_1(p) = \emptyset \) then \( \varphi_2(p) \neq \emptyset \)

**ii)** For any \( \mu \in \varphi(p) \), if or any proper subset \( A' \) of \( A \), \( \mu_{|A'} \notin \varphi_1(A', P_{|A'}) \) then \( \mu_{|A'} \in \varphi_2(A', P_{|A'}) \).

**Proof.** Let \( \varphi_1 \) be a consistent solution and let \( \varphi = \varphi_1 \lor \varphi_2 \).

First, assume that \( \varphi \) is consistent enlargement of \( \varphi_1 \). Take any \( p = (A, P) \). Suppose \( \varphi_1(p) = \emptyset \). Since \( \varphi \) is a well-defined solution this implies \( \varphi_2(p) \neq \emptyset \).

Take any \( \mu \in \varphi(p) \) and suppose \( \mu_{|A'} \notin \varphi_1(A', P_{|A'}) \) for some proper subset \( A' \) of \( A \). Since \( \varphi \) is consistent \( \mu_{|A'} \in \varphi(A', P_{|A'}) \). Hence, \( \mu_{|A'} \in \varphi_2(A', P_{|A'}) \).

Now, for the other direction assume that \( \varphi_2 \) satisfies (i) and (ii). By (i), for any problem \( p = (A, P) \) we have \( (\varphi_1 \lor \varphi_2)(p) \neq \emptyset \). Take any \( \mu \in \varphi(p) \).

Then we have either \( \mu \in \varphi_1(p) \) or \( \mu \in \varphi_2(p) \). By consistency of \( \varphi \) and by (ii), \( \varphi \) satisfies consistency. \( \blacksquare \)
Now, we are ready to prove our main result in which we compute a minimal consistent enlargement of the core.

**Proposition 5.3** For any fixed order \( \succ \) on the set of agents a minimal consistent enlargement of the core \( MCE_S \) is \( S \vee SD \).

**Proof.** Let \( \succ \) be a fixed order.

Note that, \( SD \) is well-defined and consistent solution.\(^2\) Since \( S \) satisfies consistency then \( S \vee SD \) will be a consistent well-defined solution by Proposition 4.3.

To show \( S \vee SD \) is minimal suppose for a contradiction \( S \vee SD \) is not minimal. Then there is a well-defined consistent solution \( \psi \) such that \( S \subsetneq \psi \subsetneq S \vee SD \). Hence, there is a problem \( p' = (A', P') \) such that \( \mu \in S \vee SD(p') \) but \( \mu \notin \psi(p') \). Note that if \( S(p) = \emptyset \) for any \( p = (A, P) \) then we have \( \psi(p) = SD(p) \). This means that \( S(p') \neq \emptyset \) and \( \mu \) is the serial dictatorship matching, that is \( \mu = \mu^\succ \) for the order \( \succ \).

Let \( A' = \{a_1, a_2, \ldots, a_n\} \) be the agents in \( A' \). Assume that \( a_1 \succ a_2 \succ \ldots \succ a_n \). We add three new agents \( x, y, z \) into the society with an order \( a_1 \succ a_2 \succ \ldots \succ a_n \succ x \succ y \succ z \). We extend the preferences of the agents of \( A' \) to the larger set \( A^* = A' \cup \{x, y, z\} \) of agents in a way that \( P^*_A = P' \) and new agents are unacceptable to the old agents and vice versa. That is:

\(^2\)Indeed, \( SD \) satisfies the property that is stated in Proposition 5.2.
Note that $S(A^*, P^*) = \emptyset$. Then $\psi(A^*, P^*) = SD_\succ (A^*, P^*)$ which means that the matching $\mu^* \in \psi(A^*, P^*)$ where $\mu^*(a_i) = \mu(a_i)$ for any $a_i \in A'$, $\mu^*(x) = y$ and $\mu^*(z) = z$. But $\mu_{A'}^* = \mu$ and $\mu \notin \psi(A', P')$, hence $\mu_{A'}^* \notin \psi(A' \cup \mu^*(A'), P_{|A' \cup \mu^*(A')})$. Since $\psi$ satisfies consistency this gives a contradiction.

**Remark 5.2** Since the solution $S \cup SD_\succ$ depends on the orders on the set of agents in the society, there is no unique minimal consistent core enlargement.

### 5.4 Concluding Remarks

Due to Özkal-Sanver (2010), we know that there exists no consistent core extension in roommate problems. Here, we study consistent enlargements of the core. As no core extension is consistent, a consistent enlargement of
the core must pick at leasts for some problem more than the core whenever it is not empty. For instance, the Pareto Optimal solution is a consistent enlargement of the core. We characterize the class of consistent enlargements of the core. Next, we compute minimal consistent core enlargements as a solution which picks all stable matchings and serial dictatorship matching for a fixed order on the set of agents.
6 Conclusion

Consistency and converse consistency are two well-known axioms that have recently played fundamental role in axiomatic analysis. These axioms impose invariance on the solution for the population changes. Consistency is a kind of independence of irrelevant alternatives axiom. Converse consistency is a kind of two-agent decentralization axiom. First we survey applications of consistency and converse consistency in the literature. These axioms have been applied to number of models such as bargaining, taxation, allocation problems, apportionment, matching theory. We shortly survey applications of these axioms for different economics problems. We present the results for matching problems in a detail way. There are several related open questions to be investigated further. In this thesis, we deal with some of them. First, we characterize the core of marriage problems in general domains. We extend the characterization result of Sasaki and Toda (1992) for general domains. That paper is the first study which introduces consistency and converse consistency to the matching literature. Then, we compute a maximal conversely consistent subsolution of the Pareto optimal solution in marriage problems. Finally, we study on consistent enlargements of the core in roommate problems. There are still open questions to be investigated. For instance, for many to one matching problems (college admission problems) characteriza-
tion of the core by using converse consistency is still open question worth to be investigated. There are some studies for college admission problems about these axioms. But in those studies, for instance Toda (2006) and Klaus and Klijn (2013), there are different definitions of reduced problem which lead to different definitions of consistency and converse consistency. Therefore, to study these axioms for college admission problems, first of all there should be a study about these different definitions. Then by deciding on a more reasonable and admissible version of the axiom for real life applications, similar analysis which are done throughout this thesis can be adapted to college admissions problems.
7 Appendices

Appendix I

Lemma 7.1 The solution $\varphi^*$ defined for each $p = (A, P)$ as $\varphi^*(p) = \{\mu \in \mathcal{TR}(p) \cap \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w) \text{ with } \mu(m) \neq m\}$ satisfies CCON.

Proof. Take any arbitrary problem $p = (A, P)$ and any matching $\mu \in \mathcal{M}(A)$ such that $\mu_{|N} \in \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$ for each $N = \{i, j\} \subset A$ with $\mu(i) \neq j$. We want to show that $\mu \in \varphi^*(p)$. Suppose, by contrary, $\mu \notin \varphi^*(p)$.

First we will show that $\mu \in IR(p)$. Suppose $\mu \notin \mathcal{TR}(p)$, then wlog we may assume that there exists $m$ such that $m P m \mu(m)$. Let $\mu(m) = w$. Taking some man $\hat{m} \in M\setminus\{m\}$, let $N = \{m, \hat{m}\}$. Either we have $w P_m m$ or $m P_w w$. In both cases, by individual rationality of $\varphi^*$, we have $\mu_{|N} \notin \varphi^*(N \cup \mu(N), P_{|N \cup \mu(N)})$, a contradiction.

Next we will show that $\mu \in \mathcal{PO}(p)$. Suppose $\mu \notin \mathcal{PO}(p)$, then there exists $\mu^* \in \mathcal{M}(A)$ Pareto dominating $\mu$. There exists at least an agent $i \in A$ such that $\mu^*(i) P_i \mu(i)$. Since $\mu \in \mathcal{TR}(p), \mu^*(i) \in A(i) \setminus \{i\}$, call it $j$, and we have $\mu^*(j) P_j \mu(j)$. Note that if $i \in M$, then $j \in W$ and if $j \in M$, then $i \in W$. So there exits $w$ such that $\mu^*(w) P_w \mu(w)$. There are two cases to consider. Either $\mu(w) = w$ or $\mu(w) \in M$.
Case 1: Suppose $\mu(w) = w$. Let $\mu^*(w) = m$. Either $\mu(m) = m$ and $\mu(m) \in W$.

Subcase 1.1: Let $\mu(m) = m$. Let $N = \{m, w\}$. Since $\mu \in IR(p)$, $w P_m m$ and $m P_w w$. Then, $\varphi^*(N \cup \mu(N), P_{[N \cup \mu]}(N)) = \{\tilde{\mu}\}$ where $\tilde{\mu}(m) = w$. So $\mu_{|N} \notin \varphi^*(N \cup \mu(N), P_{[N \cup \mu]}(N))$, a contradiction.

Subcase 1.2: Next suppose that $\mu(m) \neq m$, say $\mu(m) = w'$. As $\mu^*$ Pareto dominates $\mu$ and $\mu \in IR(p)$, we have $\mu^*(w') \neq w'$, call $\mu^*(w') = m'$. Let $N = \{m, w\}$. Recall that $P_{[N \cup \mu]}(N)$ is as follows:

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<th>$P_m$</th>
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Since $(m, w)$ blocks $\mu_{|N}$ and $\mu(m) \neq m$, we have $\mu_{|N} \notin \varphi^*(N \cup \mu(N), P_{[N \cup \mu]}(N))$, a contradiction.

Case 2: Suppose $\mu(w) \in M$. Let $\mu(w) = m''$. As $\mu^*$ Pareto dominates $\mu$ and $\mu \in IR(p)$, we have $\mu^*(w) \neq w$ and $\mu^*(m'') \neq m''$. Let $\mu^*(w) = m$ and $\mu^*(m'') = w''$. There are two subcases to consider. Either $\mu(w'') = w''$ or $\mu(w'') \in M$.

Subcase 2.1: First suppose that $\mu(w'') = w''$. Let $N = \{m'', w''\}$. As $\mu^*$ Pareto dominates $\mu$ and $\mu \in TR(p)$, $P_{[N \cup \mu]}(N)$ is as follows:
Since \((m'', w'')\) blocks \(\mu_N\) and \(\mu(m'') \neq m''\), we have \(\mu_N \notin \varphi^*(N \cup \mu(N), P_{N\cup\mu(N)})\), a contradiction.

**Subcase 2.2:** Next, suppose that \(\mu(w'') \in M\). Let \(\mu(w'') = m''\).\(^{28}\) Let \(N = \{m'', m'''\}\) Recall that \(P_{N\cup\mu(N)}\) is as follows:

\[
\begin{array}{ccc}
P_{m''} & P_w & P_{w''} \\
\hline
w'' & m'' & m'' \\
w & w & w'' \\
m'' & m''' & w''
\end{array}
\]

Regardless of the preferences of \(m'''\) and \(w\), we have \(\mu_N \notin \varphi^*(N \cup \mu(N), P_{N\cup\mu(N)})\), since \((m'', w'')\) is a blocking pair with \(\mu(m'') \neq m''\), a contradiction.

Since \(\mu \in TR(p) \cap PO(p)\), but \(\mu \notin \varphi^*(p)\), there exists a pair \((m^*, w^*)\) such that \(\mu(m^*) \neq m^*\) blocking \(\mu\). Let \(N = \{m^*, w^*\}\). Either \(\mu(w^*) = w^*\) or \(\mu(w^*) \in M\). In both cases, we have \(\mu_N \notin \varphi^*(N \cup \mu(N), P_{N\cup\mu(N)})\), since \((m^*, w^*)\) blocks \(\mu_N\) at the reduced problem, completing the proof.  

\(^{28}\)If we are in four agent set up, subcase 2.2 does not hold.
Lemma 7.2 The solution \( \varphi \) be defined for each \( p = (A, P) \) as \( \varphi(p) = \{ \mu \in \mathcal{PO}(p) \text{ such that there is no blocking pair } (m, w) \} \) satisfies CCON.

Proof. Take any arbitrary problem \( p = (A, P) \) and any matching \( \mu \in \mathcal{M}(A) \) such that \( \mu|_N \in \varphi(N \cup \mu(N), P_{[N \cup \mu(N)]}) \) for each \( N = \{i, j\} \subset A \) with \( \mu(i) \neq j \). We want to show that \( \mu \in \varphi(p) \). Suppose, by contrary, \( \mu \notin \varphi(p) \).

Next we will show that \( \mu \in \mathcal{PO}(p) \). Suppose \( \mu \notin \mathcal{PO}(p) \), then there exists \( \mu^* \in \mathcal{M}(A) \) Pareto dominating \( \mu \). There exists at least an agent \( i \in A \) such that \( \mu^*(i) P_i \mu(i) \). Wlog assume that \( i = m \). There are two cases to consider.

Either \( \mu^*(m) = m \) or \( \mu^*(m) \in W \).

Case 1: Suppose \( \mu^*(m) = m \). Let \( \mu(m) = w \). Either \( \mu^*(w) = w \) or \( \mu^*(w) \in M \).

Subcase 1.1: Let \( \mu^*(w) = w \). Let \( N = \{m, w\} \). Since \( \varphi \) satisfies PO we have \( \varphi(N \cup \mu(N), P_{[N \cup \mu(N)]}) = \{\bar{\mu}\} \) where \( \bar{\mu}(m) = m \) and \( \bar{\mu}(w) = w \). So \( \mu|_N \notin \varphi(N \cup \mu(N), P_{[N \cup \mu(N)]}) \), a contradiction.

Subcase 1.2: Next suppose that \( \mu^*(w) \neq w \), say \( \mu^*(w) = m' \). As \( \mu^* \) Pareto dominates \( \mu \), we have \( \mu^*(m') P_{m'} \mu(m') \), call \( \mu^*(w') = m' \). Let \( N = \{m, m'\} \). Recall that \( P_{[N \cup \mu(N)]} \) is as follows:
Since \((m', w)\) blocks \(\mu|_N\), we have \(\mu|_N \notin \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})\), a contradiction.

**Case 2:** Suppose \(\mu^*(m) \in W\). Let \(\mu^*(m) = w'\). As \(\mu^*\) Pareto dominates \(\mu\) we have \(\mu^*(w)P_\sigma \mu(w)\). When we take \(N = \{m, w'\}\), \((m', w)\) blocks \(\mu|_N\), we have \(\mu|_N \notin \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})\), a contradiction.

Since \(\mu \in \mathcal{PO}(p)\), but \(\mu \notin \varphi(p)\), there exists a pair \((m^*, w^*)\) blocking \(\mu\). Let \(N = \{m^*, w^*\}\). We have \(\mu|_N \notin \varphi^*(N \cup \mu(N), P|_{N \cup \mu(N)})\), since \((m^*, w^*)\) blocks \(\mu|_N\) at the reduced problem, completing the proof. ■

**Appendix II**

**Continuation of Proof of Theorem 4.1:**

**Proof.** Assume we are in the second case. For this preference profile \(MXCCS_{\mathcal{PO}}(p) = \{(m_1, w_2), (m_2, w_1)\}\) then we have \(\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)\). We introduce four new agents \(m_3, m_4, w_3\) and \(w_4\) to the society with an order \(m_1 > m_3 > m_4 > m_2\). We extend the preferences of the agents of \(M \cup W\) to the larger set \(M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}\) of agents in the following way:
Let \( \mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\} \). Since \( \mu^* \) is Pareto dominated by \( \{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\} \), we have \( \mu^* \notin \psi(M^*, W^*, P^*) \). However, \( \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}) \) for each \( N' = \{i, j\} \subset M^* \cup W^* \).

To see that, consider the following subproblems: Letting \( N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\} \), we have \( P^*_{|N' \cup \mu^*(N')} = P^*_{|N' \cup \mu^*(N')} \) and \( \mu^*_{|N'} = \mu^*_{|N} \in \psi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')}) \).

Next, letting \( N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\} \), note that \( P^*_{|N' \cup \mu^*(N')} \) is

Since \( m_1 \succ m_3 \) then we have \( w_3 \succ^* w_1 \). Note that \( t(P_{m_1}; W) = w_3 \) and \( t(P_{m_2}; W) = w_3 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\} \) and \( P_{w_3} \) is coherent to \( \succ^* \).
Then $\mu_{N'}^* \in SMO_\sim (N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}})$. Hence $\mu_{N'}^* \in MXCSC_{\mathcal{P}O}(N' \cup 
abla)$ which means $\mu_{N'}^* \in \psi(N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}}).

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P_{\{N' \cup \mu^*(N')\}}^*$ is

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Then $\mathcal{P}O(N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}}) = \mu_{N'}^* \in \psi(N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P_{\{N' \cup \mu^*(N')\}}^*$ is

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Then $\mathcal{P}O(N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}}) = \mu_{N'}^* \in \psi(N' \cup \mu^*(N'), P_{\{N' \cup \mu^*(N')\}})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P_{\{N' \cup \mu^*(N')\}}^*$ is

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Since $m_4 \succ m_2$ then we have $w_2 \succ_{\mu^*} w_4$. Note that $t(P_{m_2}; W) = w_2$ and $t(P_{m_4}; M) = w_2$, then $\arg \max_{i \in N \cup \mu^*(N')} s_i = \{w_2\}$ and $P_{w_2}$ is coherent to $\succ$. Then $\mu^*_N \in SMO_{\succ}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$. Hence $\mu^*_N \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$ which means $\mu^*_N \in \psi(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P^*_{N \cup \mu^*(N')}$ is

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Since $m_3 \succ m_4$ then we have $w_4 \succ_{\mu^*} w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg \max_{i \in N \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and $P_{w_4}$ is coherent to $\succ$ and $P_{m_3}$ is coherent to $\succ_{\mu^*}$. Then $\mu^*_N \in SMO_{\succ}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$. Hence $\mu^*_N \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$ which means $\mu^*_N \in \psi(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the third case. For this preference profile $MXCCS_{PO}(p) = \{(m_1, w_2), (m_2, w_1)\}$ then we have $\{(m_1, w_1), (m_2, w_2)\} \in \psi(p)$. We introduce four new agents $m_3, m_4, w_3$ and $w_4$ to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of
$M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

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Let $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Since $\mu^*$ is Pareto dominated by $\{(m_1, w_3), (m_2, w_1), (m_3, w_4), (m_4, w_2)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P^*_{|N' \cup \mu^*(N')})$ for each $N' = \{i, j\} \subset M^* \cup W^*$.

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P^*_{|N' \cup \mu^*(N')} = P_{|N \cup \mu(N)}$ and $\mu^*_{|N'} = \mu_{|N} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_1, w_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

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</table>
Since $m_1 \succ m_3$ then we have $w_3 \succ_{\mu^*} w_1$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg\max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and $P_{w_3}$ is coherent to $\succ$. Then $\mu^*_{|N'} \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu^*_{|N'} \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$ which means $\mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}).$

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_1, w_4\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

\[
\begin{array}{cccc}
P^*_{m_1} & P^*_{m_4} & P^*_{w_1} & P^*_{w_4} \\
 w_1 & w_4 & m_1 & m_4 \\
w_4 & w_1 & m_4 & m_1 \\
m_1 & m_4 & w_1 & w_4 \\
\end{array}
\]

Then $PO(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_2, w_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

\[
\begin{array}{cccc}
P^*_{m_2} & P^*_{m_3} & P^*_{w_2} & P^*_{w_3} \\
w_2 & w_3 & m_2 & m_3 \\
w_3 & w_2 & m_3 & m_2 \\
m_2 & m_3 & m_2 & w_3 \\
\end{array}
\]

Then $PO(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_2, w_4\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is
Since \( m_4 \succ m_2 \) then we have \( w_2 \succ \mu^* w_4 \). Note that \( t(P_{m_2}; W) = w_2 \) and \( t(P_{m_4}; M) = w_2 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_2\} \) and \( P_{w_2} \) is coherent to \( \succ \).

Then \( \mu^*_{|N'} \in SMO_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \). Hence \( \mu^*_{|N'} \in MXCCSP_{\psi}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \) which means \( \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \).

Finally, letting \( N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\} \), note that \( P^*_{|N' \cup \mu^*(N')} \) is

\[
\begin{array}{cccc}
P^*_{m_3} & P^*_{m_4} & P^*_{w_3} & P^*_{w_4} \\
w_4 & w_4 & m_3 & m_3 \\
w_3 & w_3 & m_4 & m_4 \\
m_3 & m_4 & w_3 & w_4 \\
\end{array}
\]

Since \( m_3 \succ m_4 \) then we have \( w_4 \succ \mu^* w_3 \). Note that \( t(P_{m_3}; W) = w_4 \) and \( t(P_{m_4}; W) = w_4 \), \( t(P_{w_3}; M) = m_3 \) and \( t(P_{w_4}; M) = m_3 \), then \( \arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\} \) and \( P_{w_4} \) is coherent to \( \succ \) and \( P_{m_3} \) is coherent to \( \succ \mu^* \). Then \( \mu^*_{|N'} \in SMO_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \). Hence \( \mu^*_{|N'} \in MXCCSP_{\psi}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \) which means \( \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) \).
Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the fourth case. For this preference profile $\text{MXCCSP}_\mu(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p)$. We introduce four new agents $m_3, m_4, w_3$ and $w_4$ to the society with an order $m_1 > m_3 > m_4 > m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

<table>
<thead>
<tr>
<th>$P^*_m$</th>
<th>$P^*_w$</th>
<th>$P^*_m$</th>
<th>$P^*_w$</th>
<th>$P^*_m$</th>
<th>$P^*_w$</th>
<th>$P^*_m$</th>
<th>$P^*_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_4$</td>
<td>$w_1$</td>
<td>$m_4$</td>
<td>$m_2$</td>
<td>$m_1$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_1$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$m_1$</td>
<td>$m_3$</td>
<td>$m_3$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$m_2$</td>
<td>$m_1$</td>
<td>$m_4$</td>
<td>$m_4$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$m_3$</td>
<td>$m_4$</td>
<td>$m_2$</td>
<td>$m_1$</td>
</tr>
</tbody>
</table>

Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since $\mu^*$ is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu^*_N \in \psi(N' \cup \mu^*(N'), P^*_{\mu^*(N')})$ for each $N' = \{i, j\} \subset M^* \cup W^*$.

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P^*_{N \cup \mu^*(N')} = P^*_{\mu^*(N)}$ and $\mu^*_N = \mu_N \in \psi(N' \cup \mu^*(N'), P^*_{\mu^*(N')})$. 113
Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2, w_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

<table>
<thead>
<tr>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_3$</td>
<td>$w_3$</td>
<td>$m_3$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_2$</td>
<td>$m_1$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$m_3$</td>
<td>$w_2$</td>
<td>$w_3$</td>
</tr>
</tbody>
</table>

Since $m_1 \succ m_3$ then we have $w_3 \succ w_2$. Note that $t(P_{m_1}; W) = w_3$ and $t(P_{m_2}; W) = w_3$, then $\arg\max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and $P_{w_3}$ is coherent to $\succ$.

Then $\mu^*_{|N'} \in SM\cap_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')})$. Hence $\mu^*_{|N'} \in M \cap_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}).$

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_2, w_4\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is

<table>
<thead>
<tr>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
<th>$P^*_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_2$</td>
<td>$w_4$</td>
<td>$m_1$</td>
<td>$m_4$</td>
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<tr>
<td>$w_4$</td>
<td>$w_2$</td>
<td>$m_4$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$m_4$</td>
<td>$w_2$</td>
<td>$w_4$</td>
</tr>
</tbody>
</table>

Then $P\cap_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}).$

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P^*_{|N' \cup \mu^*(N')}$ is
\[
\begin{array}{c|c|c|c|c|c|c}
\hline
P_{m_2}^* & P_{m_3}^* & P_{w_1}^* & P_{w_3}^* \\
\hline
w_1 & w_3 & m_2 & m_3 \\
\hline
w_3 & w_1 & m_3 & m_2 \\
\hline
m_2 & m_3 & w_1 & w_3 \\
\hline
\end{array}
\]

Then \( P \mathcal{O}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}) = \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} \).  

Next, letting \( N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\} \), note that \( P_{|N' \cup \mu^*(N')}^* \) is

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
P_{m_2}^* & P_{m_4}^* & P_{w_1}^* & P_{w_4}^* \\
\hline
w_1 & w_1 & m_4 & m_4 \\
\hline
w_4 & w_4 & m_2 & m_4 \\
\hline
m_2 & m_4 & w_1 & w_4 \\
\hline
\end{array}
\]

Since \( m_4 \succ m_2 \) then we have \( w_1 \succ_{\mu^*} w_4 \). Note that \( t(P_{m_2}; W) = w_1 \) and \( t(P_{m_4}; M) = w_1 \), then \( \arg \max_i s_i = \{w_1\} \) and \( P_{w_1} \) is coherent to \( \succ \).

Then \( \mu_{|N'}^* \in SMO_{\succeq}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} \). Hence \( \mu_{|N'}^* \in MXCCS_{PO}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} \) which means \( \mu_{|N'}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} \).

Finally, letting \( N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\} \), note that \( P_{|N' \cup \mu^*(N')}^* \) is

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
P_{m_3}^* & P_{m_4}^* & P_{w_3}^* & P_{w_4}^* \\
\hline
w_4 & w_4 & m_3 & m_3 \\
\hline
w_3 & w_3 & m_4 & m_4 \\
\hline
m_3 & m_4 & w_3 & w_4 \\
\hline
\end{array}
\]
Since $m_3 > m_4$ then we have $w_4 \succeq w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, then $\arg\max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and $P_{w_4}$ is coherent to $\succ$ and $P_{m_3}$ is coherent to $\succ \mu^*$. Then $\mu^*_N \in SMO_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^N)$. Hence $\mu^*_N \in MXCCS_{\rho_0}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^N)$ which means $\mu^*_N \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}^N)$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the fifth case. For this preference profile $MXCCS_{\rho_0}(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p)$. We introduce four new agents $m_3, m_4, w_3$ and $w_4$ to the society with an order $m_1 > m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

<table>
<thead>
<tr>
<th>$P_{m_1}^*$</th>
<th>$P_{m_2}^*$</th>
<th>$P_{m_3}^*$</th>
<th>$P_{m_4}^*$</th>
<th>$P_{w_1}^*$</th>
<th>$P_{w_2}^*$</th>
<th>$P_{w_3}^*$</th>
<th>$P_{w_4}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_4$</td>
<td>$w_1$</td>
<td>$m_4$</td>
<td>$m_2$</td>
<td>$m_1$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$w_1$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$m_3$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$m_1$</td>
<td>$m_1$</td>
<td>$m_4$</td>
<td>$m_4$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$m_3$</td>
<td>$m_4$</td>
<td>$m_2$</td>
<td>$m_1$</td>
</tr>
</tbody>
</table>

Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since $\mu^*$ is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin$
ψ(M*, W*, P*). However, µ* N' ∈ ψ(N' ∪ µ* (N'), P* N'∪µ*(N')) for each N' = {i, j} ⊂ M* ∪ W*

To see that, consider the following subproblems: Letting N' ∪ µ* (N') = {m1, m2, w1, w2}, we have P* N'∪µ*(N') = P|N∪µ(N) and µ* N' = µ|N ∈ ψ(N' ∪ µ* (N'), P* N'∪µ*(N')).

Next, letting N' ∪ µ* (N') = {m1, m3, w2, w3}, note that P* N'∪µ*(N') is

<table>
<thead>
<tr>
<th>P* m1</th>
<th>P* m3</th>
<th>P* w2</th>
<th>P* w3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w3</td>
<td>w3</td>
<td>m3</td>
<td>m1</td>
</tr>
<tr>
<td>w2</td>
<td>w2</td>
<td>m1</td>
<td>m3</td>
</tr>
<tr>
<td>m1</td>
<td>m3</td>
<td>w2</td>
<td>w3</td>
</tr>
</tbody>
</table>

Since m1 > m3 then we have w3 > µ* w2. Note that t(Pm1; W) = w3 and t(Pm2; W) = w3, then arg max s_i = {w3} and Pw3 is coherent to > . Then µ* N' ∈ SMO > (N' ∪ µ* (N'), P|N∪µ*(N')). Hence µ* N' ∈ MXCCSPO (N' ∪ µ* (N'), P|N∪µ*(N')) which means µ* N' ∈ ψ(N' ∪ µ* (N'), P|N∪µ*(N')).

Next, letting N' ∪ µ* (N') = {m1, m4, w2, w4}, note that P* N'∪µ*(N') is

<table>
<thead>
<tr>
<th>P* m1</th>
<th>P* m4</th>
<th>P* w2</th>
<th>P* w4</th>
</tr>
</thead>
<tbody>
<tr>
<td>w2</td>
<td>w4</td>
<td>m1</td>
<td>m4</td>
</tr>
<tr>
<td>w4</td>
<td>w2</td>
<td>m4</td>
<td>m1</td>
</tr>
<tr>
<td>m1</td>
<td>m4</td>
<td>w2</td>
<td>w4</td>
</tr>
</tbody>
</table>
Then $\mathcal{P}\mathcal{O}(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star}) = \mu_{|N'}^{\star} \in \psi(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P_{|N'\cup\mu^*(N')}^{\star}$ is

<table>
<thead>
<tr>
<th>$P_{m_2}^{\star}$</th>
<th>$P_{m_3}^{\star}$</th>
<th>$P_{w_1}^{\star}$</th>
<th>$P_{w_3}^{\star}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$w_3$</td>
<td>$m_2$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$w_1$</td>
<td>$m_3$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$w_1$</td>
<td>$w_3$</td>
</tr>
</tbody>
</table>

Then $\mathcal{P}\mathcal{O}(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star}) = \mu_{|N'}^{\star} \in \psi(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\}$, note that $P_{|N'\cup\mu^*(N')}^{\star}$ is

<table>
<thead>
<tr>
<th>$P_{m_2}^{\star}$</th>
<th>$P_{m_4}^{\star}$</th>
<th>$P_{w_1}^{\star}$</th>
<th>$P_{w_4}^{\star}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$w_1$</td>
<td>$m_4$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$m_2$</td>
<td>$m_4$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$m_4$</td>
<td>$w_1$</td>
<td>$w_4$</td>
</tr>
</tbody>
</table>

Since $m_4 \succ m_2$ then we have $w_1 \succ \mu^* w_4$. Note that $t(P_{m_2}; W) = w_1$ and $t(P_{m_4}; M) = w_1$, then $\arg \max \ s_i = \{w_1\}$ and $P_{w_1}$ is coherent to $\succ$.

Then $\mu_{|N'}^{\star} \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star})$. Hence $\mu_{|N'}^{\star} \in MXCCSP_{PO}(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star})$ which means $\mu_{|N'}^{\star} \in \psi(N' \cup \mu^*(N'), P_{|N'\cup\mu^*(N')}^{\star})$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N'\cup\mu^*(N')}^{\star}$ is
Since $m_3 \succ m_4$ then we have $w_4 \succ_\mu^* w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$. $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and $P_{w_4}$ is coherent to $\succ$ and $P_{m_3}$ is coherent to $\succ_{\mu^*}$. Then $\mu_{|N'}^* \in SMO_{\succ}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}).$ Hence $\mu_{|N'}^* \in MXCCSP_{\psi}(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}); P_{|N' \cup \mu^*(N')})$ which means $\mu^*_{|N'} \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')}).$

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.

Assume we are in the sixth case. For this preference profile $MXCCSP_{\psi}(p) = \{(m_1, w_1), (m_2, w_2)\}$ then we have $\{(m_1, w_2), (m_2, w_1)\} \in \psi(p).$ We introduce four new agents $m_3, m_4, w_3$ and $w_4$ to the society with an order $m_1 \succ m_3 \succ m_4 \succ m_2$. We extend the preferences of the agents of $M \cup W$ to the larger set $M^* \cup W^* = M \cup W \cup \{m_3, m_4, w_3, w_4\}$ of agents in the following way:

<table>
<thead>
<tr>
<th>$P_{m_3}^*$</th>
<th>$P_{m_4}^*$</th>
<th>$P_{w_3}^*$</th>
<th>$P_{w_4}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$m_3$</td>
<td>$m_3$</td>
</tr>
<tr>
<td>$w_3$</td>
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<td>$m_4$</td>
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<tr>
<td>$m_3$</td>
<td>$m_4$</td>
<td>$w_3$</td>
<td>$w_4$</td>
</tr>
</tbody>
</table>
Let $\mu^* = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$. Since $\mu^*$ is Pareto dominated by $\{(m_1, w_3), (m_2, w_2), (m_3, w_4), (m_4, w_1)\}$, we have $\mu^* \notin \psi(M^*, W^*, P^*)$. However, $\mu^*_{|\tilde{\mu}'} \in \psi(N' \cup \mu^*(N'), P^*_{|\tilde{\mu}^\prime \cup \mu^*(N')})$ for each $N' = \{i, j\} \subset M^* \cup W^*$.

To see that, consider the following subproblems: Letting $N' \cup \mu^*(N') = \{m_1, m_2, w_1, w_2\}$, we have $P^*_{|\tilde{\mu}^\prime \cup \mu^*(N')}$, and $\mu^*_{|\tilde{\mu}'} \in \psi(N' \cup \mu^*(N'), P^*_{|\tilde{\mu}^\prime \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_3, w_2, w_3\}$, note that $P^*_{|\tilde{\mu}^\prime \cup \mu^*(N')}$ is coherent to $\succ$.

Since $m_1 \succ m_3$ then $w_3 \succ_{\mu^*} w_2$. Note that $t(P^*_{m_1}; W) = w_3$ and $t(P^*_{m_2}; W) = w_3$, then $\arg\max_{i \in N' \cup \mu^*(N')} s_i = \{w_3\}$ and $P^*_{w_3}$ is coherent to $\succ$. 

\[
\begin{array}{cccccc}
P^*_{m_1} & P^*_{m_2} & P^*_{m_3} & P^*_{m_4} & P^*_{w_1} & P^*_{w_2} & P^*_{w_3} & P^*_{w_4} \\
\hline
w_3 & w_2 & w_4 & w_1 & m_4 & m_2 & m_1 & m_3 \\
w_2 & w_1 & w_3 & w_4 & m_2 & m_3 & m_3 & m_2 \\
w_1 & w_3 & w_2 & w_3 & m_1 & m_1 & m_4 & m_4 \\
w_4 & w_4 & w_1 & w_2 & m_3 & m_4 & m_2 & m_1 \\
\end{array}
\]
Then $\mu^*_{N'} \in SMO_\cap(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$. Hence $\mu^*_{N'} \in MXCS_\cap(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$ which means $\mu^*_{N'} \in \psi(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_1, m_4, w_2, w_4\}$, note that $P^*_{N \cup \mu^*(N')}$ is

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Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')}) = \mu^*_{N'} \in \psi(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_3, w_1, w_3\}$, note that $P^*_{N \cup \mu^*(N')}$ is

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Then $\mathcal{PO}(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')}) = \mu^*_{N'} \in \psi(N' \cup \mu^*(N'), P_{N \cup \mu^*(N')})$.

Next, letting $N' \cup \mu^*(N') = \{m_2, m_4, w_1, w_4\}$, note that $P^*_{N \cup \mu^*(N')}$ is

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Since $m_4 \succ m_2$ then we have $w_1 \succ \mu^*, w_4$. Note that $t(P_{m_2}; W) = w_1$ and $t(P_{m_4}; M) = w_1$, then $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_1\}$ and $P_{w_1}$ is coherent to $\succ$. Then $\mu_{[N']}^* \in SMO_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$. Hence $\mu_{[N']}^* \in MXCCSP_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$ which means $\mu_{[N']}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$.

Finally, letting $N' \cup \mu^*(N') = \{m_3, m_4, w_3, w_4\}$, note that $P_{|N' \cup \mu^*(N')}^*$ is

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Since $m_3 \succ m_4$ then we have $w_4 \succ \mu^*, w_3$. Note that $t(P_{m_3}; W) = w_4$ and $t(P_{m_4}; W) = w_4$, $t(P_{w_3}; M) = m_3$ and $t(P_{w_4}; M) = m_3$, $\arg \max_{i \in N' \cup \mu^*(N')} s_i = \{w_4, m_3\}$ and $P_{w_4}$ is coherent to $\succ$ and $P_{m_3}$ is coherent to $\succ \mu^*$. Then $\mu_{[N']}^* \in SMO_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$. Hence $\mu_{[N']}^* \in MXCCSP_\succ(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$ which means $\mu_{[N']}^* \in \psi(N' \cup \mu^*(N'), P_{|N' \cup \mu^*(N')} )$.

Hence, by CCON we have $\mu^* \in \psi(M^*, W^*, P^*)$, which gives a contradiction.
8 References


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